Dedicated to the memory of Lawrence Alan Shepp (1936-2013)
Stochastic Process. Appl. Vol. 150, 2022, (1015–1036)
Research Report No. 5, 2019, *Probab. Statist. Group Manchester* (24 pp)

Sticky Bessel Diffusions

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We consider a Bessel process X of dimension $\delta \in (0,2)$ having 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu \in (0,\infty)$. We show that (i) the process X can be characterised through its square $Y = X^2$ as a unique weak solution to the SDE system

$$dY_t = \delta\, I(Y_t > 0)\, dt + 2\sqrt{Y_t}\, dB_t$$
$$I(Y_t = 0)\, dt = \tfrac{1}{2\mu}\, d\ell_t^0(Y)$$

where B is a standard Brownian motion and $\ell^0(Y)$ is a diffusion local time process of Y at 0, and (ii) the transition density function of X can be expressed in the closed form as a convolution integral involving a Mittag-Leffler function and a modified Bessel function of the second kind. Appearance of the Mittag-Leffler function is novel in this context. We determine a (sticky) boundary condition at zero that characterises the transition density function of X as a unique solution to the Kolmogorov backward/forward equation of X. We also show that the convolution integral can be characterised as a unique solution to the generalised Abel equation of the second kind. Letting $\mu \downarrow 0$ (absorption) and $\mu \uparrow \infty$ (instantaneous reflection) the closed-form expression for the transition density function of X reduces to the ones found by Feller [5] and Molchanov [13] respectively.

1. Introduction

This paper is motivated by the question of David Roodman (personal communication) as to whether it is possible to find a closed-form expression for the transition density function of the Bessel process X having dimension $\delta \in (0,2)$ and 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu \in (0,\infty)$. The purpose of the paper is to provide an answer to this question.

Recall that the infinitesimal generator $\mathbb{L}_X$ of X acts in $(0,\infty)$ by the rule

$$\mathbb{L}_X = \frac{\delta - 1}{2x}\,\partial_x + \frac{1}{2}\,\partial_{xx} \tag{1.1}$$

where δ can be any real number representing a dimension of X (cf. [12]). We assume throughout that $0 < \delta < 2$ in which case 0 is a *regular* (i.e. *non-singular*) boundary point. Other cases are well analysed/understood (if $\delta \le 0$ then 0 is an *exit* boundary point, and if $\delta \ge 2$

Mathematics Subject Classification 2010. Primary 60J60, 60J65, 60H20. Secondary 35C15, 35K20, 35K67.
Key words and phrases: Slowly reflecting (sticky) boundary behaviour, sticky (Feller) boundary condition, Brownian motion, Bessel process, Feller branching diffusion, stochastic differential equation, transition probability density function, Kolmogorov backward/forward equation, diffusion local time, time change, scale function, speed measure, Mittag-Leffler function, modified Bessel function, generalised Abel equation.

then 0 is an *entrance* boundary point) see e.g. [1, p. 134] for closed-form expressions of the transition density functions in these cases. The boundary classification throughout the paper refers to the one due to Feller [6] (see e.g. [1, pp. 14-17] for a modern exposition).

In his paper [5, p. 180] Feller found a closed-form expression for the transition density function of X when 0 is an absorbing/killing boundary point (meaning that the motion of X terminates after 0 is reached). In fact Feller dealt with his branching diffusion process instead, however, it is well known that the two processes stand in one-to-one correspondence obtained by a simple change of time and scale (see e.g. [11, p. 357]). Feller's expression includes a modified Bessel function of the first kind having a positive index. Subsequently Molchanov [13, p. 312] found a closed-form expression for the transition density function of X when 0 is an instantaneously reflecting boundary point (meaning that X spends no time at 0 with a strictly positive Lebesgue measure). Molchanov's expression includes a modified Bessel function of the first kind having a negative index. A linear combination of of the modified Bessel functions of the first kind with positive and negative indices produces a modified Bessel function of the second kind. It might therefore be natural to conjecture that the latter function alone can be used to form a closed expression for the transition density function of X when 0 is a slowly reflecting (sticky) boundary point. We will see below however that this is not possible in general and that yet another special function is needed to accomplish this goal.

The paper is organised as follows. In Section 2 we use the Itô-McKean construction of sticky diffusions (cf. [9, Section 10]) and show that X can be characterised through its square $Y = X^2$ as a unique weak solution to the SDE system

$$dY_t = \delta\, I(Y_t > 0)\, dt + 2\sqrt{Y_t}\, dB_t \tag{1.2}$$

$$I(Y_t = 0)\, dt = \tfrac{1}{2\mu}\, d\ell_t^0(Y) \tag{1.3}$$

where B is a standard Brownian motion and $\ell^0(Y)$ is a diffusion local time process of Y at 0. In Section 3 we use the Green function of X (i.e. the Laplace transform of its transition density function in the time domain) to determine a (sticky) boundary condition at zero that characterises the transition density function of X as a unique solution to the Kolmogorov backward/forward equation of X. In Section 4 we calculate the limit at zero of the ratio between the first derivative of an increasing fundamental solution (eigenfunction) to the killed generator equation of X with respect to its scale function and the increasing fundamental solution itself. This limit coincides with the speed measure of X evaluated at the singleton consisting of zero and is expressed in terms of the dimension δ of X alone when the eigenvalue equals one. Making use of this expression in Section 5 we determine the Laplace transform of the transition density function of X in the time domain (i.e. the Green function of X). In Section 6 we apply Laplace inversion and show that the transition density function of X can be expressed in the closed form as a convolution integral involving a Mittag-Leffler function and a modified Bessel function of the second kind. Appearance of the Mittag-Leffler function is novel in this context (see [8, Appendix] for a review of the main properties of these functions including their Laplace transforms). In Section 7 we show that the convolution integral can be characterised as a unique solution to the generalised Abel equation of the second kind (see [14, pp. 141-142, 531, 548] for more details on this equation). We finally verify in Section 8 that letting $\mu \downarrow 0$ (absorption) and $\mu \uparrow \infty$ (instantaneous reflection) the closed-form expression for the transition density function of X reduces to the ones found by Feller [5] and Molchanov [13] respectively.

2. Stochastic differential equations

In this section we use the Itô-McKean construction of sticky diffusions (cf. [9, Section 10]) and show that the Bessel process X of dimension $\delta \in (0,2)$ having 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu \in (0,\infty)$ can be characterised through its square $Y = X^2$ as a unique weak solution to the SDE system

$$dY_t = \delta\, I(Y_t > 0)\, dt + 2\sqrt{Y_t}\, dB_t \tag{2.1}$$

$$I(Y_t = 0)\, dt = \tfrac{1}{2\mu}\, d\ell_t^0(Y) \tag{2.2}$$

where B is a standard Brownian motion and $\ell^0(Y)$ is a diffusion local time process of Y at 0 defined in (2.13) below. We assume that X starts at some x in $[0,\infty)$ so that Y in (2.1)+(2.2) starts at $y = x^2$ in $[0,\infty)$. The stochastic integral with respect to B in (2.1) is understood in Itô's sense. We refer to [3] for standard definitions of the week/strong solutions to SDEs including their uniqueness that we will use throughout.

1. Recall that the infinitesimal generator $\mathbb{L}_X$ of X acts in $(0,\infty)$ by the rule

$$\mathbb{L}_X = \frac{\delta - 1}{2x}\, \partial_x + \frac{1}{2}\, \partial_{xx} \tag{2.3}$$

and the stickiness of X at 0 is characterised by the fact that the domain $D(\mathbb{L}_X)$ of the infinitesimal generator $\mathbb{L}_X$ of X consists of functions $f \in C^2((0,\infty)) \cap C_b([0,\infty))$ such that $\mathbb{L}_X f \in C_b([0,\infty))$ and the following (sticky) boundary condition at 0 holds

$$\frac{df}{ds}(0+) = m(\{0\})\, \mathbb{L}_X f(0) \tag{2.4}$$

(cf. [15, p. 310]) where s is the scale function of X given by

$$s(x) = \frac{x^{-2\nu}}{-2\nu} \tag{2.5}$$

for $x \in [0,\infty)$, and m is the speed measure of X given by

$$m(dx) = 2x^{2\nu+1} dx \tag{2.6}$$

on $(0,\infty)$. The number $\nu := \delta/2 - 1$ in (2.5) and (2.6) is referred to as the index of X. Note that $\delta \in (0,2)$ if and only if $\nu \in (-1,0)$. The stickiness of X at 0 is measured by

$$m(\{0\}) = 1/\mu \tag{2.7}$$

where $\mu \in (0,\infty)$ is a given and fixed number. Letting $\mu \downarrow 0$ and $\mu \uparrow \infty$ we obtain the boundary behaviour of infinite stickiness (absorption) and zero stickiness (instantaneous reflection) of X at 0 respectively. Recall also that the diffusion local time process $\ell^x(X)$ of X at x in $[0,\infty)$ is defined by

$$\ell_t^x(X) = \lim_{\varepsilon \downarrow 0} \frac{1}{m([x-\varepsilon, x+\varepsilon])} \int_0^t I(x-\varepsilon \le X_s \le x+\varepsilon)\, ds \tag{2.8}$$

for $t \geq 0$. It is well known that the limit in (2.8) exists almost surely and that we have

$$\int_0^t f(X_s)\,ds = \int_{[0,\infty)} f(x)\,\ell_t^x(X)\,m(dx) \tag{2.9}$$

for all (bounded) measurable functions $f : [0,\infty) \to I\!R$ and all $t \geq 0$ (cf. [10, pp. 174-175]). Moreover, the mapping $(t,x) \mapsto \ell_t^x(X)$ is continuous on $I\!R_+ \times [0,\infty)$ almost surely (this can be derived by noting that $X_t = s^{-1}(s(X_t)) = s^{-1}(M_t) = s^{-1}(B_{T_t})$ where $T_t = A_t^{-1}$ for $A_t = (1/2)\int_0^\infty \ell_t^x(B)\,m_M(dx)$ so that $\ell_t^x(X) = \ell_{T_t}^{s(x)}(B)$ for $t \geq 0$ and $x \in [0,\infty)$ showing that the claim reduces to the case of standard Brownian motion resolved in [16]).

2. The problem whether a sticky Bessel process X of dimension $\delta \in (0,2)$ arising from (2.3)+(2.4) above can be obtained from an SDE system driven by a standard Brownian motion to our knowledge has not been considered in the literature. Moreover, the SDE construction of X when $\delta = 1$ presented in [3, Theorem 5] does not extend to $\delta \in (0,1) \cup (1,2)$. Finally, it is known that the process X with $m(\{0\}) = 0$ is not a semimartingale when $\delta \in (0,1)$. For these reasons we choose to construct/characterise X through an SDE system for its square $Y = X^2$ using the arguments which apply simultaneously for all $\delta \in (0,2)$.

3. Recall that the infinitesimal generator $I\!L_Y$ of Y acts in $(0,\infty)$ by the rule

$$I\!L_Y = \delta\,\partial_y + 2y\,\partial_{yy} \tag{2.10}$$

and the stickiness of Y at 0 is characterised analogously to (2.4) above, where the scale function s_Y of Y is given by

$$s_Y(y) = \frac{y^{-\nu}}{-\nu} \tag{2.11}$$

for $y \in [0,\infty)$, and the speed measure m_Y of Y is given by

$$m_Y(dy) = \frac{y^\nu}{2}\,dy \tag{2.12}$$

on $(0,\infty)$. The diffusion local time process $\ell^y(Y)$ of Y at y in $[0,\infty)$ is defined analogously to (2.8) above and the analogue of the identity (2.9) including the joint continuity of $\ell(Y)$ holds for Y too. In particular, from the latter we see that

$$\ell_t^0(Y) = \lim_{\varepsilon \downarrow 0} \frac{1}{m_Y((0,\varepsilon])} \int_0^t I(0 < Y_s \leq \varepsilon)\,ds \tag{2.13}$$

and the identity (2.2) is satisfied with

$$m_Y(\{0\}) = 1/2\mu\,. \tag{2.14}$$

This shows that 0 is a slowly reflecting (sticky) boundary point for Y solving (2.1)+(2.2) with a stickiness parameter $1/2\mu \in (0,\infty)$. We will return to this point following the proof of the main result in this section that can be stated as follows.

Theorem 1. *The system* (2.1)+(2.2) *has a unique weak solution.*

Proof. 1. We show that the system (2.1)+(2.2) has a weak solution. For this, let B^1 be a standard Brownian motion defined on a probability space $(\Omega^1, \mathcal{F}^1, \mathsf{P}^1)$ and let $I\!F^1$ be the natural filtration of B^1 . Consider the stochastic differential equation

$$dZ_t = \delta\, dt + 2\sqrt{Z_t}\, dB_t^1 \tag{2.15}$$

with $Z_0 = z$ in $[0,\infty)$. Then it is well known (see e.g. [15, p. 439]) that the stochastic diffrential equation (2.15) has a unique strong (non-negative) solution Z which moreover has 0 as an instantaneously reflecting boundary point due to $\delta \in (0,2)$ (see e.g. [15, p. 442]). This means that $s_Z = s_Y$ on $[0,\infty)$ and $m_Z = m_Y$ on $(0,\infty)$ with $m_Z(\{0\}) = 0$ in the notation of (2.11) and (2.12) above. Following [9, p. 186 & pp. 200-201] consider the additive functional

$$A_t = t + \tfrac{1}{2\mu}\, \ell_t^0(Z) \tag{2.16}$$

where $\ell_t^0(Z) = \lim_{\varepsilon\downarrow 0} (1/m_Z((0,\varepsilon]))\int_0^t I(0<Z_s\le\varepsilon)\, ds$ is the diffusion local time of Z at 0 for $t \ge 0$. Note that $t \mapsto A_t$ is continuous and strictly increasing with $A_t \uparrow \infty$ as $t \uparrow \infty$ so that its (proper) inverse $t \mapsto T_t$ obtained by

$$T_t = A_t^{-1} \tag{2.17}$$

is well defined (finite) for all $t \ge 0$ and satisfies the same properties itself. Moreover, since $A = (A_t)_{t\ge 0}$ is adapted to $I\!\!F^1$ it follows that each T_t is a stopping time with respect to $I\!\!F^1$, so that $T = (T_t)_{t\ge 0}$ defines a time change with respect to $I\!\!F^1$. The fact that $t \mapsto T_t$ is continuous and strictly increasing with $T_t < \infty$ for $t \ge 0$ (or equivalently $A_t \uparrow \infty$ as $t \uparrow \infty$) ensures that standard time change transformations are applicable to continuous semimartingales and their stochastic integrals without extra conditions on their sample paths (see e.g. [15, pp. 7–9 & pp. 179–181]) and they will be used in the sequel with no explicit mention.

Consider the time-changed process

$$Y_t = Z_{T_t} \tag{2.18}$$

for $t \ge 0$. From (2.15) we see that

$$Y_t = z + \delta T_t + 2\int_0^{T_t} \sqrt{Z_s}\, dB_s^1 = z + \delta T_t + 2\int_0^t \sqrt{Z_{T_s}}\, dB_{T_s}^1 \tag{2.19}$$

for $t \ge 0$. Since B^1 is a continuous martingale with respect to $I\!\!F^1$ it follows that $B_T^1 = (B_{T_t}^1)_{t\ge 0}$ is a continuous martingale with respect to $I\!\!F_T^1 = (\mathcal{F}_{T_t}^1)_{t\ge 0}$. Moreover, we have

$$\begin{aligned} \langle B_T^1, B_T^1\rangle_t = T_t &= \int_0^{T_t} I(Z_s>0)\big(ds + \tfrac{1}{2\mu} d\ell_s^0(Z)\big) = \int_0^{T_t} I(Z_s>0)\, dA_s \\ &= \int_0^t I(Z_{T_s}>0)\, dA_{T_s} = \int_0^t I(Y_s>0)\, ds \end{aligned} \tag{2.20}$$

for $t \ge 0$. Note similarly that

$$B_{T_t}^1 = \int_0^{T_t} I(Z_s>0)\, dB_s^1 = \int_0^t I(Z_{T_s}>0)\, dB_{T_s}^1 = \int_0^t I(Y_s>0)\, dB_{T_s}^1 \tag{2.21}$$

for $t \ge 0$.

The identities (2.20) and (2.21) motivate to use a variant of Doob's martingale representation theorem in order to achieve (2.1). For this, take another Brownian motion B^0 defined on a probability space $(\Omega^0, \mathcal{F}^0, \mathsf{P}^0)$ and let $\mathbb{F}^0$ denote the natural filtration of B^0. Set $\Omega = \Omega^1 \times \Omega^0$, $\mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^0$, $\mathbb{F} = \mathbb{F}^1_T \times \mathbb{F}^0$ and $\mathsf{P} = \mathsf{P}^1 \times \mathsf{P}^0$. Then $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ is a filtered probability space. Extend all random variables X^1 and X^0 defined on Ω^1 and Ω^0 respectively to Ω by setting $X^1(\omega) := X^1(\omega_1)$ and $X^0(\omega) := X^0(\omega_0)$ for $\omega = (\omega_1, \omega_0) \in \Omega$. Then it is easily seen that B^1_T and B^0 remain (continuous) martingales with respect to $\mathbb{F}$ and B^0 remains a standard Brownian motion on $(\Omega, \mathcal{F}, \mathsf{P})$ as well (note that B^1_T and B^0 are independent). It follows therefore that the process B defined by

$$B_t = \int_0^t I(Y_s > 0)\, dB^1_{T_s} + \int_0^t I(Y_s = 0)\, dB^0_s \tag{2.22}$$

for $t \geq 0$ is a continuous martingale with respect to $\mathbb{F}$. From (2.20)-(2.22) we see that

$$\langle B, B \rangle_t = \int_0^t I(Y_s > 0)\, ds + \int_0^t I(Y_s = 0)\, ds = t \tag{2.23}$$

for all $t \geq 0$ and hence by Lévy's characterisation theorem it follows that B is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathsf{P})$.

Making use of (2.20)-(2.22) in (2.19) we see that

$$Y_t = z + \delta \int_0^t I(Y_s > 0)\, ds + 2 \int_0^t \sqrt{Y_s}\, dB_s \tag{2.24}$$

for $t \geq 0$ which shows that Y and B solve (2.1) with $Y_0 = y$ in $[0, \infty)$ if z is taken to be y. Moreover, we have

$$\begin{aligned} \int_0^t I(Y_s = 0)\, ds &= \int_0^t I(Z_{T_s} = 0)\, dA_{T_s} = \int_0^{T_t} I(Z_s = 0)\, dA_s \\ &= \int_0^{T_t} I(Z_s = 0) \big(ds + \tfrac{1}{2\mu} d\ell^0_s(Z) \big) = \tfrac{1}{2\mu} \ell^0_{T_t}(Z) \\ &= \frac{1}{2\mu} \lim_{\varepsilon \downarrow 0} \frac{1}{m_Z((0, \varepsilon])} \int_0^{T_t} I(0 < Z_s \leq \varepsilon)\, ds \\ &= \frac{1}{2\mu} \lim_{\varepsilon \downarrow 0} \frac{1}{m_Z((0, \varepsilon])} \int_0^t I(0 < Z_{T_s} \leq \varepsilon)\, dT_s \\ &= \frac{1}{2\mu} \lim_{\varepsilon \downarrow 0} \frac{1}{m_Y((0, \varepsilon])} \int_0^t I(0 < Y_s \leq \varepsilon)\, ds = \tfrac{1}{2\mu} \ell^0_t(Y) \end{aligned} \tag{2.25}$$

for $t \geq 0$ where in the penultimate equality we use (2.20) above. From (2.25) we see that Y satisfies (2.2) and this completes the proof of weak existence.

2. We show that uniqueness in law holds for the system (2.1)+(2.2). For this, we will undo the time change from the previous part of the proof starting with the notation afresh. Suppose that Y and B solve (2.1) subject to (2.2). As part of this hypothesis, we know that Y and B are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$, both Y and B are $\mathbb{F}$-adapted,

and B is not only a standard Brownian motion with respect to P but also a martingale with respect to $\mathbb{F}$. Consider the additive functional

$$T_t = \int_0^t I(Y_s > 0)\, ds \tag{2.26}$$

for $t \geq 0$ and note that $T_t \uparrow T_\infty$ as $t \uparrow \infty$ where $T_\infty \in (0, \infty]$. Since $t \mapsto T_t$ is increasing and continuous it follows that its (right) inverse $t \mapsto A_t$ defined by

$$A_t = \inf \{ s \geq 0 \mid T_s > t \} \tag{2.27}$$

is finite for all $t \in [0, T_\infty)$. Note that $t \mapsto A_t$ is increasing and right-continuous on $[0, T_\infty)$. Moreover, since $T = (T_t)_{t \geq 0}$ is adapted to $\mathbb{F}$ it follows that each A_t is a stopping time with respect to $\mathbb{F}$, so that A_t defines a time change with respect to $\mathbb{F}$ for $t \in [0, T_\infty)$.

Consider the time-changed process

$$Z_t = Y_{A_t} \tag{2.28}$$

for $t \geq 0$. From (2.1) we see that

$$Y_t = y + \delta \int_0^t I(Y_s > 0)\, ds + 2 \int_0^t \sqrt{Y_s}\, I(Y_s > 0)\, dB_s = y + \delta T_t + 2 \int_0^t \sqrt{Y_s}\, dM_s \tag{2.29}$$

where $M = (M_t)_{t \geq 0}$ is a continuous martingale with respect to $\mathbb{F}$ given by

$$M_t = \int_0^t I(Y_s > 0)\, dB_s \tag{2.30}$$

for $t \geq 0$. Note that $t \mapsto \langle M, M \rangle_t = \int_0^t I(Y_s > 0)\, ds = T_t$ is constant on each $[A_{s-}, A_s]$ and therefore the same is true for $t \mapsto M_t$ whenever $s > 0$ is given and fixed. It follows therefore that M_{A_t} is a continuous martingale with respect to $\mathcal{F}_{A_t}$ and we have

$$\langle M_A, M_A \rangle_t = \langle M, M \rangle_{A_t} = T_{A_t} = t \tag{2.31}$$

for $t \in [0, T_\infty)$. Using Lévy's characterisation theorem we can therefore conclude that $W_t := M_{A_t}$ is a standard Brownian motion for $t \in [0, T_\infty)$. Moreover, using that $t \mapsto M_t$ is constant on each $[A_{s-}, A_s]$ for $s > 0$, we conclude from (2.29) that

$$\begin{aligned} Z_t &= y + \delta T_{A_t} + 2 \int_0^{A_t} \sqrt{Y_s}\, dM_s \\ &= y + \delta t + 2 \int_0^t \sqrt{Y_{A_s}}\, dM_{A_s} = y + \delta t + 2 \int_0^t \sqrt{Z_s}\, dW_s \end{aligned} \tag{2.32}$$

for $t \in [0, T_\infty)$. Recalling that the stochastic differential equation (2.15) has a unique strong solution, this shows that Z_t for $t \in [0, T_\infty)$ is a squared Bessel process of dimension $\delta \in (0, 2)$ having 0 as an instantaneously reflecting boundary point. Moreover, using (2.2) we see that

$$t = T_t + \int_0^t I(Y_s = 0)\, ds = T_t + \tfrac{1}{2\mu}\, \ell_t^0(Y) \tag{2.33}$$

from where we find that

$$A_t = T_{A_t} + \tfrac{1}{2\mu}\,\ell^0_{A_t}(Y) = t + \tfrac{1}{2\mu}\,\ell^0_{A_t}(Y) \tag{2.34}$$

for $t \in [0, T_\infty)$. Since $t \mapsto T_t$ is constant on each $[A_{s-}, A_s]$ for $s > 0$ we see from (2.13) that

$$\begin{aligned} \ell^0_{A_t}(Y) &= \lim_{\varepsilon \downarrow 0} \frac{1}{m_Y((0,\varepsilon])} \int_0^{A_t} I(0 < Y_s \le \varepsilon)\, ds = \lim_{\varepsilon \downarrow 0} \frac{1}{m_Y((0,\varepsilon])} \int_0^{A_t} I(0 < Y_s \le \varepsilon)\, dT_s \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{m_Y((0,\varepsilon])} \int_0^{t} I(0 < Y_{A_s} \le \varepsilon)\, dT_{A_s} = \lim_{\varepsilon \downarrow 0} \frac{1}{m_Z((0,\varepsilon])} \int_0^{t} I(0 < Z_s \le \varepsilon)\, ds \\ &= \ell^0_t(Z) \end{aligned} \tag{2.35}$$

for $t \in [0, T_\infty)$. Inserting (2.35) back into (2.34) we obtain

$$A_t = t + \tfrac{1}{2\mu}\,\ell^0_t(Z) \tag{2.36}$$

for $t \in [0, T_\infty)$. Letting $t \uparrow T_\infty$ and using that the diffusion local time process $\ell^0(Z)$ of the squared Bessel process Z solving (2.32) is finite at every finite time, we see that $A_{T_\infty} < \infty$ while by (2.26)+(2.27) we see that $A_{T_\infty} = \infty$ whenever $T_\infty < \infty$. This shows that $T_\infty = \infty$ almost surely and consequently the process Z solves (2.32) for all $t \ge 0$. From (2.36) we see that $t \mapsto A_t$ is strictly increasing (and continuous) and hence

$$T_t = A_t^{-1} \tag{2.37}$$

is the proper inverse for $t \ge 0$ (implying also that $t \mapsto T_t$ is strictly increasing and continuous). It follows in particular that $A_{T_t} = t$ so that

$$Y_t = Y_{A_{T_t}} = Z_{T_t} \tag{2.38}$$

for $t \ge 0$. Since Z is a unique strong solution to the stochastic differential equation (2.32), we see from (2.36)-(2.38) that Y is a well-determined measurable functional of the standard Brownian motion W . This shows that the law of Y solving (2.1)+(2.2) is uniquely determined and the proof of weak uniqueness is complete. □

Remark 2. From (2.16)-(2.18) in the proof above we know that the process Y from Theorem 1 is Markov (cf. [17]). Moreover, recalling that the process Z solving (2.15) is Feller (see e.g. [15, p. 440]) we know that Y is strong Markov (cf. [17]). This fact combined with continuity of Y , and the fact established in the proof above that Y coincides with Z when away from 0 while $t \mapsto T_t = \int_0^t I(Y_s > 0)\, ds$ is strictly increasing on $I\!R_+$, show that Y is a regular diffusion process (in the sense of Itô and McKean [10]).

Setting $X := \sqrt{Y}$ we obtain a regular diffusion process with the scale function (2.5) and the speed measure (2.6). Thus X is a Bessel process of dimension $\delta \in (0, 2)$. To show that 0 is a slowly reflecting (sticky) boundary point for X we first verify the following fact.

Proposition 3. *We have*

$$\ell^0_t(X) = \tfrac{1}{2}\,\ell^0_t(Y) \tag{2.39}$$

for all $t \ge 0$.

Proof. Using the identity $X = \sqrt{Y}$ we find that

$$\ell_t^0(X) = \lim_{\varepsilon\downarrow 0} \frac{1}{m((0,\varepsilon])} \int_0^t I(0<X_s\le\varepsilon)\,ds = \lim_{\varepsilon\downarrow 0} \frac{1}{m((0,\varepsilon])} \int_0^t I(0<Y_s\le\varepsilon^2)\,ds \tag{2.40}$$
$$= \frac{1}{2} \lim_{\varepsilon\downarrow 0} \frac{1}{m_Y((0,\varepsilon^2])} \int_0^t I(0<Y_s\le\varepsilon^2)\,ds = \tfrac{1}{2}\,\ell_t^0(Y)$$

for all $t \ge 0$ as claimed, where in the penultimate inequality we use (2.6) and (2.12) to calculate $m((0,\varepsilon]) = \varepsilon^{2(\nu+1)}/(\nu+1)$ and $m_Y((0,\varepsilon]) = \varepsilon^{\nu+1}/2(\nu+1)$ so that $m((0,\varepsilon]) = 2m_Y((0,\varepsilon^2])$ for $\varepsilon > 0$ as applied, and the proof is complete. $\square$

Making use of (2.39) at the right-hand side of (2.2) we find that

$$\int_0^t I(X_s=0)\,ds = \int_0^t I(Y_s=0)\,ds = \tfrac{1}{2\mu}\,\ell_t^0(Y) = \tfrac{1}{\mu}\,\ell_t^0(X) \tag{2.41}$$

for all $t \ge 0$. This shows that (2.7) is satisfied and confirms that 0 is a slowly reflecting (sticky) boundary point for X as claimed.

Remark 4. In relation to the definition (2.16) in the proof of Theorem 1 it is instructive to observe that the semimartingale local time $L^0(Z)$ of the process Z is identically equal to zero. Indeed, noting from (2.15) that $d\langle Z,Z\rangle_t = 4Z_t dt$ we see that

$$0 \le L_t^0(Z) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t I(0\le Z_s\le\varepsilon)\,d\langle Z,Z\rangle_s = \lim_{\varepsilon\downarrow 0} \frac{4}{\varepsilon} \int_0^t I(0\le Z_s\le\varepsilon)\,Z_s\,ds \tag{2.42}$$
$$\le 4 \lim_{\varepsilon\downarrow 0} \int_0^t I(0\le Z_s\le\varepsilon)\,ds = 4\int_0^t I(Z_s=0)\,ds = 0$$

for all $t \ge 0$, where in the last equality we use the fact that 0 is an instantaneously reflecting boundary point for Z, implying the claim.

Remark 5. If $\delta \in (1,2)$ then a similar construction of X can be carried out directly if starting with a unique strong (non-negative) solution Z to the stochastic differential equation

$$dZ_t = \frac{\delta-1}{2Z_t}\,dt + dB_t^1 \tag{2.43}$$

in place of (2.15) above, and then proceeding as in the proof of Theorem 1 above (with X in place of Y throughout) upon setting

$$X_t = Z_{T_t} \tag{2.44}$$

where $T_t = A_t^{-1}$ and $A_t = t + (1/\mu)\,\ell_t^0(Z)$ for $t \ge 0$. Small modifications of the proof then show that the Bessel process X of dimension $\delta \in (1,2)$ having 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu \in (0,\infty)$ can be characterised as a unique weak solution to the SDE system

$$dX_t = \frac{\delta-1}{2X_t}\,I(X_t>0)\,dt + dB_t \tag{2.45}$$

$$I(X_t=0)\,dt = \tfrac{1}{\mu}\,d\ell_t^0(X) \tag{2.46}$$

where B is a standard Brownian motion and $\ell^0(X)$ is a diffusion local time process of X at 0 defined analogously to (2.13) above.

3. Sticky boundary condition

In this section we use the Green function of X (i.e. the Laplace transform of its transition density function in the time domain) to determine a (sticky) boundary condition at zero that characterises the transition density function of X as a unique solution to the Kolmogorov backward/forward equation of X.

1. Recall that a transition density function p of X with respect to the speed measure m of X is characterised by

$$\mathsf{P}_x(X_t \in A) = \int_A p(t;x,y)\, m(dy) \tag{3.1}$$

being valid for all measurable $A \subseteq [0,\infty)$ whenever $t > 0$ and $x \in [0,\infty)$ are given and fixed. It is well known that p can be chosen to be positive, jointly continuous in all the three variables, and symmetric in the two spatial variables (cf. [10, p. 149]).

2. In view of (2.6) we set $m(x) = 2x^{2\nu+1}$ for $x > 0$ and define

$$f(t;x,y) = p(t;x,y)m(y) \tag{3.2}$$

for $t > 0$ and $x, y \in (0,\infty)$. From (3.1) we see that f is a transition density function of X with respect to Lebesque measure on $(0,\infty)$ in the sense that

$$\mathsf{P}_x(X_t \in A) = \int_A f(t;x,y)\, dy \tag{3.3}$$

for all measurable $A \subseteq (0,\infty)$ whenever $t > 0$ and $x \in (0,\infty)$ are given and fixed.

Recalling (2.3) we know that f solves the Kolmogorov backward equation

$$f_t(t;x,y) = b(x)f_x(t;x,y) + \tfrac{1}{2}f_{xx}(t;x,y) \tag{3.4}$$

$$f(0+;x,y) = \delta_y(x) \ \text{ (weakly)} \tag{3.5}$$

for $t > 0$ and $x \in (0,\infty)$ where $b(x) = (\delta-1)/2x$ and $y \in (0,\infty)$ is given and fixed, and the Kolmogorov forward equation

$$f_t(t;x,y) = -(bf)_y(t;x,y) + \tfrac{1}{2}f_{yy}(t;x,y) \tag{3.6}$$

$$f(0+;x,y) = \delta_x(y) \ \text{ (weakly)} \tag{3.7}$$

for $t > 0$ and $y \in (0,\infty)$ where $b(y) = (\delta-1)/2y$ and $x \in (0,\infty)$ is given and fixed. In (3.5) and (3.7) above δ_z denotes a (formal) density function of the Dirac measure at z in $(0,\infty)$ and the weak convergence is understood to hold for the corresponding distribution functions.

The initial conditions (3.5) and (3.7) are insufficient to determine unique solutions to (3.4) and (3.6) respectively. In the reminder of this section we thus look for a slowly reflecting (sticky) boundary condition at zero (in the space domain) which when combined with a natural boundary condition at infinity (in the space domain) will accomplish this aim.

3. To exploit its symmetry in the two spatial variables we will perform our analysis in terms of the transition density function p satisfying (3.1) above. For this, recall that the Green function G of X is defined by

$$G(x,y) = \int_0^\infty e^{-\lambda t} p(t;x,y)\, dt \tag{3.8}$$

for $x, y \in [0,\infty)$ where $\lambda > 0$ is given and fixed. It is well known that

$$G(x,y) = \frac{\varphi(x)\,\psi(y)}{w} \quad \text{if } x \le y \text{ in } [0,\infty) \tag{3.9}$$
$$= \frac{\varphi(y)\,\psi(x)}{w} \quad \text{if } y \le x \text{ in } [0,\infty)$$

where the functions $\varphi : [0,\infty) \to I\!R$ and $\psi : [0,\infty) \to I\!R$ are uniquely determined (up to positive multiplicative constants) by

$$I\!L_X \varphi = \lambda\varphi \quad \text{on } (0,\infty) \tag{3.10}$$

$$x \mapsto \varphi(x) \quad \text{is increasing on } [0,\infty) \tag{3.11}$$

$$\frac{d\varphi}{ds}(0+) = m(\{0\})\, I\!L_X \varphi(0) \tag{3.12}$$

$$I\!L_X \psi = \lambda\psi \quad \text{on } (0,\infty) \tag{3.13}$$

$$x \mapsto \psi(x) \quad \text{is decreasing on } [0,\infty) \tag{3.14}$$

$$\psi(\infty-) = \frac{d\psi}{ds}(\infty-) = 0 \tag{3.15}$$

and the Wronskian w is a constant (not dependent on x) defined by

$$w = \frac{d\varphi}{ds}(x)\,\psi(x) - \varphi(x)\,\frac{d\psi}{ds}(x) \tag{3.16}$$

for $x \in [0,\infty)$. Note that each of G, φ, ψ, w depends on λ but we will omit this dependence from the notation for simplicity. The condition (3.12) reflects the fact that 0 is a slowly reflecting (sticky) boundary point for X, and the condition (3.15) reflects the fact that ∞ is a natural boundary point for X. Setting $\tau_c = \inf\{\, t \ge 0 \mid X_t = c \,\}$ it is well known that the following probabilistic representation of the functions φ and ψ is valid

$$\mathsf{E}_x\big(e^{-\lambda\tau_c}\big) = \frac{\varphi(x)}{\varphi(c)} \quad \text{if } x \le c \text{ in } [0,\infty) \tag{3.17}$$
$$= \frac{\psi(x)}{\psi(c)} \quad \text{if } c \le x \text{ in } [0,\infty)$$

where $\lambda > 0$ is given and fixed. From this representation it is easily seen that φ and ψ belong to the domain $D(I\!L_X)$ of the infinitesimal generator $I\!L_X$ of X as used in (3.12) and (3.15) above respectively. We will see below that the solutions φ and ψ to (3.10)-(3.12) and (3.13)-(3.15) can be expressed explicitly in terms of modified Bessel functions.

4. From (3.8) and (3.9) we see that

$$\int_0^\infty e^{-\lambda t}\, p(t;x,y)\, dt = \frac{\varphi(x)\,\psi(y)}{w} \tag{3.18}$$

for $x \le y$ in $[0,\infty)$ where $\lambda > 0$ is given and fixed. Differentiating with respect to s in (3.18), letting $x \downarrow 0$ and making use of (3.12) (with (2.7) above) and (3.10), we find that

$$\int_0^\infty e^{-\lambda t}\, \frac{\partial_2 p}{\partial s}(t;0+,y)\, dt = \frac{d\varphi}{ds}(0+)\,\frac{\psi(y)}{w} = m(\{0\})\, I\!L_X \varphi(0)\,\frac{\psi(y)}{w} = \frac{\lambda}{\mu}\,\frac{\varphi(0)\,\psi(y)}{w} \tag{3.19}$$

$$= \frac{\lambda}{\mu} \int_0^\infty e^{-\lambda t} p(t; 0+, y)\, dt$$

for all $y \in (0, \infty)$. We point out that ∂_2 in (3.19) refers to a change of the second argument of p, i.e. we have $(\partial_2 p/\partial s)(t; x, y) = \lim_{h \to 0}(p(t; x+h, y) - p(t; x, y))/(s(x+h) - s(x))$ with $(\partial_2 p/\partial s)(t; 0+, y) = \lim_{x \downarrow 0}(\partial_2 p/\partial s)(t; x, y)$ for $t > 0$ and $x, y \in (0, \infty)$. Similarly, we point out that ∂_3 refers to a change of the third argument of p, and $(\partial_3 p/\partial s)(t; x, y)$ with $(\partial_3 p/\partial s)(t; x, 0+)$ are defined analogously for $t > 0$ and $x, y \in (0, \infty)$. Note that $(\partial_2 p/\partial s)(t; x, y) = \partial_x p(t; x, y)/s'(x)$ and $(\partial_3 p/\partial s)(t; x, y) = \partial_y p(t; x, y)/s'(y)$ for $t > 0$ and $x, y \in (0, \infty)$. These identities can be used to calculate the limits when either x or y from $(0, \infty)$ tends to either 0 or ∞ as needed throughout.

Integrating by parts we find that

$$\lambda \int_0^\infty e^{-\lambda t} p(t; 0+, y)\, dt = \int_0^\infty e^{-\lambda t} p_t(t; 0+, y)\, dt \tag{3.20}$$

for $y \in (0, \infty)$. Inserting this expression back into (3.19) we obtain

$$\int_0^\infty e^{-\lambda t} \left[\frac{\partial_2 p}{\partial s}(t; 0+, y) - \frac{1}{\mu} p_t(t; 0+, y) \right] dt \tag{3.21}$$

for all $\lambda > 0$. By the uniqueness theorem for the Laplace transform we can conclude that the following *sticky boundary condition* is satisfied

$$\frac{\partial_2 p}{\partial s}(t; 0+, y) = \frac{1}{\mu} p_t(t; 0+, y) \tag{3.22}$$

for all $t > 0$ and all $y \in (0, \infty)$. Using exactly the same arguments, or exploiting the symmetry of p in the two spatial variables directly, we can conclude that the following *sticky boundary condition* is satisfied

$$\frac{\partial_3 p}{\partial s}(t; x, 0+) = \frac{1}{\mu} p_t(t; x, 0+) \tag{3.23}$$

for all $t > 0$ and all $x \in (0, \infty)$.

5. From (3.8) and (3.9) we see that

$$\int_0^\infty e^{-\lambda t} p(t; x, y)\, dt = \frac{\varphi(y)\, \psi(x)}{w} \tag{3.24}$$

for $y \le x$ in $[0, \infty)$ where $\lambda > 0$ is given and fixed. Letting $x \to \infty$ in (3.24) and using that the first limit value in (3.15) equals zero, we see that

$$\int_0^\infty e^{-\lambda t} p(t; \infty-, y)\, dt = 0 \tag{3.25}$$

for all $\lambda > 0$ and $y \in (0, \infty)$. Similarly, differentiating with respect to s in (3.24), letting $x \to \infty$ and using that the second limit value in (3.15) equals zero, we see that

$$\int_0^\infty e^{-\lambda t} \frac{\partial_2 p}{\partial s}(t; \infty-, y)\, dt = 0 \tag{3.26}$$

for all $\lambda > 0$ and $y \in (0,\infty)$. By the uniqueness theorem for the Laplace transform we can conclude from (3.25) and (3.26) that the following *normal boundary condition* is satisfied

$$p(t;\infty-,y) = \frac{\partial_2 p}{\partial s}(t;\infty-,y) = 0 \tag{3.27}$$

for all $t > 0$ and all $y \in (0,\infty)$. Using exactly the same arguments, or exploiting the symmetry of p in the two spatial variables directly, we can conclude that the following *normal boundary condition* is satisfied

$$p(t;x,\infty-) = \frac{\partial_3 p}{\partial s}(t;x,\infty-) = 0 \tag{3.28}$$

for all $t > 0$ and all $x \in (0,\infty)$.

6. Recalling from (3.2) that

$$p(t;x,y) = \frac{f(t;x,y)}{m(y)} \tag{3.29}$$

for $t > 0$ and $x, y \in (0,\infty)$ we see that the boundary conditions (3.22)+(3.23) and (3.27)+(3.28) expressed in terms of p translate to the corresponding boundary conditions expressed in terms of f. There exist a variety of sufficient conditions for justifying differentiation under the integral sign in (3.19) and integration by parts in (3.20) respectively. Reversing the order of the arguments above under these sufficient conditions then yields uniqueness of the solution to the initial value problems (3.4)+(3.5) and (3.6)+(3.7) under the boundary conditions (3.22)+(3.27) and (3.23)+(3.28) respectively. We will omit fuller details on the sufficient conditions and the uniqueness results in the present paper. Instead we will focus on finding a closed-form expression for the transition density function f of X and then verifying that this function solves the initial-boundary value problems (3.4)+(3.5) with (3.22)+(3.27) and (3.6)+(3.7) with (3.23)+(3.28) respectively.

4. Limit at zero

In this section we calculate the limit at zero of the ratio between the first derivative of an increasing fundamental solution (eigenfunction) to the killed generator equation of X with respect to its scale function and the increasing fundamental solution itself. This limit coincides with the speed measure of X evaluated at the singleton consisting of zero and is expressed in terms of the dimension δ of X alone when the eigenvalue equals one.

1. Consider the differential equation (3.10)+(3.13) rewritten as

$$\frac{1}{2}y''(x) + \frac{\delta-1}{2x}y'(x) - \lambda y(x) = 0 \tag{4.1}$$

for $x > 0$ where $\delta \in (0,2)$ so that $\nu := \delta/2-1 \in (-1,0)$. Recall that the *modified Bessel functions* of the *first* and *second kind* are respectively defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)} \tag{4.2}$$

$$K_\nu(x) = \tfrac{\pi}{2\sin(\nu\pi)}\big(I_{-\nu}(x) - I_\nu(x)\big) \tag{4.3}$$

for $x \in [0,\infty)$. It is well known that two linearly independent solutions to (4.1) are given by $x \mapsto x^{-\nu} I_\nu(x\sqrt{2\lambda})$ and $x \mapsto x^{-\nu} K_\nu(x\sqrt{2\lambda})$ where the first function is increasing and the second function is decreasing on $(0,\infty)$. Moreover, due to $K_\nu(x) \sim (\sqrt{\pi/2x})e^{-x}$ as $x \to \infty$ and $(x^{-\nu}K_\nu(x))' = -x^{-\nu}K_{\nu+1}(x)$ for $x > 0$ (see (4.8)+(4.9) below) it is easily seen using (2.5) that the second function satisfies (3.15). Since the two functions solve (4.1) it follows that $x \mapsto x^{-\nu} I_{-\nu}(x\sqrt{2\lambda})$ is a solution to (4.1) as well. Moreover, it is easily seen using (4.9) below that this function is increasing as well when $\nu \in (-1,0)$. Motivated by these facts we set

$$\varphi(x) := a\, x^{-\nu} I_\nu(x\sqrt{2\lambda}) + b\, x^{-\nu} I_{-\nu}(x\sqrt{2\lambda}) \tag{4.4}$$

$$\psi(x) := x^{-\nu} K_\nu(x\sqrt{2\lambda}) \tag{4.5}$$

and seek (positive) constants a and b such that φ satisfies (3.12) above. Once such constants a and b are found we will know that φ and ψ from (4.4) and (4.5) solve (3.10)-(3.12) and (3.13)-(3.15) respectively.

2. From (3.12) with (3.10) and (2.7) we see that

$$\frac{d\varphi}{ds}(0+) = m(\{0\})\mathbb{L}_X\varphi(0) = \frac{\lambda}{\mu}\varphi(0+) \tag{4.6}$$

which forms an equation for the constants a and b to be found. To determine the equation we calculate the limit appearing in (4.6) as follows.

Lemma 6. *We have*

$$\frac{(d\varphi/ds)(0+)}{\varphi(0+)} = \frac{\Gamma(\nu+1)}{\Gamma(-\nu)}\,\frac{2^{\nu+1}}{\lambda^\nu}\,\frac{b}{a}\,. \tag{4.7}$$

Proof. We will make use of the following well-known relations

$$\big(x^{-\nu} I_\nu(x)\big)' = x^{-\nu} I_{\nu+1}(x) \tag{4.8}$$

$$\big(x^{\nu} I_\nu(x)\big)' = x^{\nu} I_{\nu-1}(x) \tag{4.9}$$

$$I_\nu(x) \sim \tfrac{1}{\Gamma(\nu+1)}\big(\tfrac{x}{2}\big)^\nu \quad \text{as } x \downarrow 0 \tag{4.10}$$

for $x > 0$ (see e.g. [1, p. 638]). By (4.8) and (4.9) we have

$$\begin{aligned}
\varphi'(x) &= a\big(x^{-\nu} I_\nu(x\sqrt{2\lambda})\big)' + b\big(x^{-\nu} I_{-\nu}(x\sqrt{2\lambda})\big)' \\
&= \frac{a}{(\sqrt{2\lambda})^{-\nu}}\big((x\sqrt{2\lambda})^{-\nu} I_\nu(x\sqrt{2\lambda})\big)' + \frac{b}{(\sqrt{2\lambda})^{-\nu}}\big((x\sqrt{2\lambda})^{-\nu} I_{-\nu}(x\sqrt{2\lambda})\big)' \\
&= \frac{a}{(\sqrt{2\lambda})^{-\nu}}(x\sqrt{2\lambda})^{-\nu} I_{\nu+1}(x\sqrt{2\lambda})\sqrt{2\lambda} + \frac{b}{(\sqrt{2\lambda})^{-\nu}}(x\sqrt{2\lambda})^{-\nu} I_{-\nu-1}(x\sqrt{2\lambda})\sqrt{2\lambda} \\
&= a\sqrt{2\lambda}\, x^{-\nu} I_{\nu+1}(x\sqrt{2\lambda}) + b\sqrt{2\lambda}\, x^{-\nu} I_{-\nu-1}(x\sqrt{2\lambda})
\end{aligned} \tag{4.11}$$

for $x > 0$. Hence by (4.10) we find that

$$\varphi'(x) \sim a\sqrt{2\lambda}\, x^{-\nu} \frac{1}{\Gamma(\nu+2)} \Big(\frac{x\sqrt{2\lambda}}{2}\Big)^{\nu+1} + b\sqrt{2\lambda}\, x^{-\nu} \frac{1}{\Gamma(-\nu)} \Big(\frac{x\sqrt{2\lambda}}{2}\Big)^{-\nu-1} \tag{4.12}$$

as $x \downarrow 0$. Recalling from (2.5) that $s'(x) = x^{-2\nu-1}$ for $x > 0$ we see from (4.12) that

$$\frac{\varphi'(x)}{s'(x)} \sim \frac{2a}{\Gamma(\nu+2)} \Big(\frac{\sqrt{2\lambda}}{2}\Big)^{\nu+2} x^{2\nu+2} + \frac{2b}{\Gamma(-\nu)} \Big(\frac{\sqrt{2\lambda}}{2}\Big)^{-\nu} \tag{4.13}$$

as $x \downarrow 0$. Since $x^{2\nu+2} \to 0$ as $x \downarrow 0$ this shows that

$$\frac{\varphi'(0+)}{s'(0+)} = \frac{2b}{\Gamma(-\nu)} \Big(\frac{2}{\lambda}\Big)^{\nu/2}. \tag{4.14}$$

Similarly, by (4.10) we find that

$$\begin{aligned} \varphi(x) &\sim a\, x^{-\nu} \frac{1}{\Gamma(\nu+1)} \Big(\frac{x\sqrt{2\lambda}}{2}\Big)^{\nu} + b\, x^{-\nu} \frac{1}{\Gamma(-\nu+1)} \Big(\frac{x\sqrt{2\lambda}}{2}\Big)^{-\nu} \\ &= \frac{a}{\Gamma(\nu+1)} \Big(\frac{\lambda}{2}\Big)^{\nu/2} + \frac{b}{\Gamma(-\nu+1)} \Big(\frac{\lambda}{2}\Big)^{-\nu/2} x^{-2\nu} \end{aligned} \tag{4.15}$$

as $x \downarrow 0$. Since $x^{-2\nu} \to 0$ as $x \downarrow 0$ this shows that

$$\varphi(0+) = \frac{a}{\Gamma(\nu+1)} \Big(\frac{\lambda}{2}\Big)^{\nu/2}. \tag{4.16}$$

Combining (4.14) and (4.16) we obtain (4.7) as claimed. □

3. From (4.6) and (4.7) we see that

$$b = \frac{\lambda^{\nu+1}}{c}\, a \tag{4.17}$$

for $a > 0$ where c is a positive constant defined by

$$c = c(\mu, \nu) := 2^{\nu+1} \frac{\Gamma(\nu+1)}{\Gamma(-\nu)}\, \mu\,. \tag{4.18}$$

Inserting b from (4.17) into (4.4) above, we obtain all solutions φ to (3.10)-(3.12) above indexed by $a > 0$.

5. Laplace transform

In this section we make use of the limit at zero from the previous section and explicitly determine the Laplace transform of the transition density function of X in the time domain (i.e. the Green function of X).

Recall that the Green function of X defined by (3.8) above can be expressed by (3.9) where φ and ψ are given by (4.4) and (4.5) with (4.17)+(4.18) respectively and w is defined

in (3.16) above. It is easily seen/verified that the Wronskian of $x \mapsto x^{-\nu} I_\nu(x\sqrt{2\lambda})$ and $x \mapsto x^{-\nu} K_\nu(x\sqrt{2\lambda})$ as well as the Wronskian of $x \mapsto x^{-\nu} I_{-\nu}(x\sqrt{2\lambda})$ and $x \mapsto x^{-\nu} K_\nu(x\sqrt{2\lambda})$ are both equal to 1 and hence from (4.4) and (4.5) we see that w from (3.16) is given by

$$w = a+b = a\Big(1+\frac{\lambda^{\nu+1}}{c}\Big) \tag{5.1}$$

upon recalling (4.17)+(4.18) for the second equality.

Corollary 7 (Laplace transform). *The Green function of X defined by* (3.8) *above can be explicitly expressed as*

$$\begin{aligned} G(x,y) &= (xy)^{-\nu}\Big[I_{-\nu}(x\sqrt{2\lambda})K_\nu(y\sqrt{2\lambda}) \\ &\qquad - c\,\tfrac{2\sin(\nu\pi)}{\pi}\Big(\frac{K_\nu(x\sqrt{2\lambda})K_\nu(y\sqrt{2\lambda})}{c+\lambda^{\nu+1}}\Big)\Big] \quad \textit{if } x \le y \textit{ in } [0,\infty) \\ &= (xy)^{-\nu}\Big[I_{-\nu}(y\sqrt{2\lambda})K_\nu(x\sqrt{2\lambda}) \\ &\qquad - c\,\tfrac{2\sin(\nu\pi)}{\pi}\Big(\frac{K_\nu(x\sqrt{2\lambda})K_\nu(y\sqrt{2\lambda})}{c+\lambda^{\nu+1}}\Big)\Big] \quad \textit{if } y \le x \textit{ in } [0,\infty)\,. \end{aligned} \tag{5.2}$$

Proof. Combining (3.9) with (4.4)+(4.5) and (4.17)+(4.18) above and making use of (5.1) above we find that

$$\begin{aligned} G(x,y) &= \frac{\varphi(x)\,\psi(y)}{w} = \frac{\big(a\,x^{-\nu}I_\nu(x\sqrt{2\lambda}) + b\,x^{-\nu}I_{-\nu}(x\sqrt{2\lambda})\big)\,y^{-\nu}K_\nu(y\sqrt{2\lambda})}{a+b} \\ &= (xy)^{-\nu}\,\frac{\big(I_\nu(x\sqrt{2\lambda}) + (\lambda^{\nu+1}/c)\,I_{-\nu}(x\sqrt{2\lambda})\big)\,K_\nu(y\sqrt{2\lambda})}{1+\lambda^{\nu+1}/c} \\ &= (xy)^{-\nu}\,\frac{\big(cI_\nu(x\sqrt{2\lambda}) + \lambda^{\nu+1}I_{-\nu}(x\sqrt{2\lambda})\big)\,K_\nu(y\sqrt{2\lambda})}{c+\lambda^{\nu+1}} \\ &= (xy)^{-\nu}\,\frac{\big[c\big(I_\nu(x\sqrt{2\lambda}) - I_{-\nu}(x\sqrt{2\lambda})\big) + (c+\lambda^{\nu+1})\,I_{-\nu}(x\sqrt{2\lambda})\big]\,K_\nu(y\sqrt{2\lambda})}{c+\lambda^{\nu+1}} \\ &= (xy)^{-\nu}\Big[I_{-\nu}(y\sqrt{2\lambda})K_\nu(x\sqrt{2\lambda}) - c\,\tfrac{2\sin(\nu\pi)}{\pi}\Big(\frac{K_\nu(x\sqrt{2\lambda})K_\nu(y\sqrt{2\lambda})}{c+\lambda^{\nu+1}}\Big)\Big] \end{aligned} \tag{5.3}$$

for $x \le y$ in $[0,\infty)$. This establishes the first identity in (5.2) above. The second identity in (5.2) then follows by the spatial symmetry in (3.9) and this completes the proof. □

6. Inverse Laplace transform

In this section we apply Laplace inversion to the Green function from the previous section and show that the transition density function of X can be expressed in the closed form as a convolution integral inolving a Mittag-Leffler function and a modified Bessel function of the second kind.

Recall that the *Mittag-Leffler function* is defined by

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)} \tag{6.1}$$

for $x \in I\!R$ where $\alpha > 0$ and $\beta > 0$ are given and fixed. Integrating term by term in (6.1), and summing up the resulting Laplace transforms of the power functions, one finds that

$$\int_0^{\infty} e^{-\lambda t}\, t^{\beta-1} E_{\alpha,\beta}(-ct^{\alpha})\, dt = \frac{\lambda^{\alpha-\beta}}{c+\lambda^{\alpha}} \tag{6.2}$$

for $\lambda > 0$ and $c > 0$ (cf. [8, p. 267]). Equipped with this identity we can now apply Laplace inversion to the Green function (5.2) as follows.

Theorem 8 (Inverse Laplace transform). *The transition density function of X defined by* (3.1) *above can be explicitly expressed as*

$$\begin{aligned} p(t;x,y) &= (xy)^{-\nu} \frac{1}{2t}\, e^{-(x^2+y^2)/2t} I_{-\nu}\Big(\frac{xy}{t}\Big) \\ &\quad + c\, \tfrac{2\sin(-\nu\pi)}{\pi}\, (xy)^{-\nu} \int_0^t (t-s)^{\nu} E_{\nu+1,\nu+1}\big(-c(t-s)^{\nu+1}\big) \frac{1}{2s}\, e^{-(x^2+y^2)/2s} K_{\nu}\Big(\frac{xy}{s}\Big)\, ds \end{aligned} \tag{6.3}$$

for $t > 0$ and $x, y \in [0,\infty)$ where $c = c(\mu,\nu)$ is given by (4.18) *above.*

Proof. Recognising the right-hand side of (6.2) as a building block in the right-hand side of (5.2) with $\alpha = \beta = \nu+1$, we aim to apply Laplace inversion to the products of $I_{-\nu}$ and K_{ν} as well as K_{ν} and K_{ν} that appear on the right-hand side of (5.2) as functions of λ as well as x and y respectively. For this, denoting by L^{-1} the inverse Laplace transform (with respect to λ), and recalling that $K_{\nu} = K_{-\nu}$, it is readily derived from (56) in [4, p. 284] that

$$L^{-1}\big[I_{-\nu}(\sqrt{a\lambda})\, K_{\nu}(\sqrt{b\lambda})\big](t) = \frac{1}{2t}\, e^{-(a+b)/4t} I_{-\nu}\Big(\frac{\sqrt{ab}}{2t}\Big) \tag{6.4}$$

for $t > 0$ and $a, b \in (0,\infty)$ given and fixed. Moreover, using the same arguments it is readily derived from (64) in [4, p. 285] that

$$L^{-1}\big[K_{\nu}(\sqrt{a\lambda})\, K_{\nu}(\sqrt{b\lambda})\big](t) = \frac{1}{2t}\, e^{-(a+b)/4t} K_{\nu}\Big(\frac{\sqrt{ab}}{2t}\Big) \tag{6.5}$$

for $t > 0$ and $a, b \in (0,\infty)$ given and fixed. Applying the inverse Laplace transform L^{-1} to the first term on the right-hand side of (5.2), and making use of (6.4) above with $a = 2x^2$ and $b = 2y^2$, we obtain the first term on the right-hand side of (6.3) as claimed. Moreover, applying the inverse Laplace transform L^{-1} to the second term on the right-hand side of (5.2), making use of (6.2) above with $\alpha = \beta = \nu+1$ and (6.5) above with $a = 2x^2$ and $b = 2y^2$, we obtain the second term on the right-hand side of (6.3) using the convolution theorem for the Laplace transform as claimed. This completes the proof. □

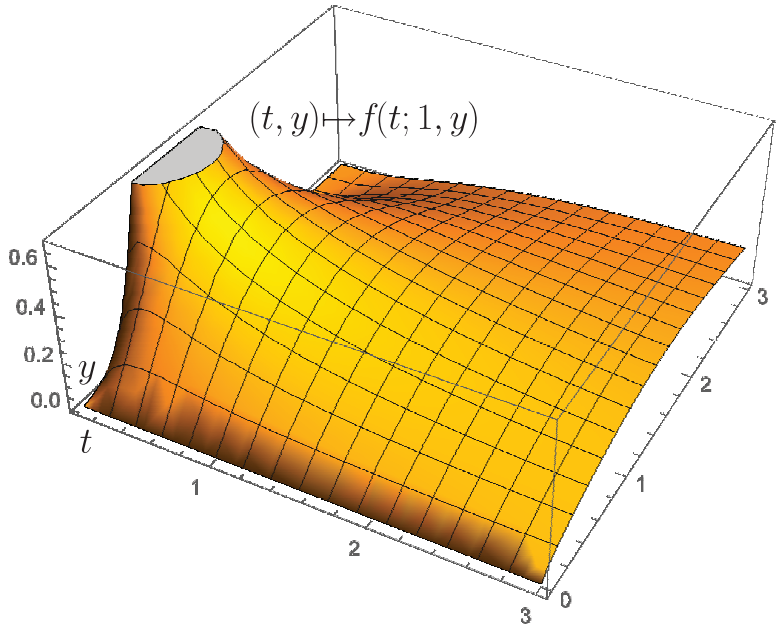

Figure 1. The transition density function $(t,y) \mapsto f(t;1,y)$ of the Bessel process X having dimension $\delta = 3/2$ (i.e. index $\nu = -1/4$) and 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu = 5$. Defined by (3.3) the function f can be expressed in a closed from using (6.3) combined with (3.2) and (2.6). Note that X starts at 1 so that $f(0;1,y) \to \infty$ as $y \to 1$. Note also that $f(t;1,y) \to 0$ as $y \downarrow 0$ for each $t > 0$ given and fixed (cf. Figure 2).

Remark 9 (Atoms at zero). Note that if either x or y in (6.3) equals 0 then the identity (6.3) is understood in the limiting sense. For example, fixing $x \in (0,\infty)$ and letting $y \downarrow 0$ in (6.3), it is easily seen using (4.10) and $K_\nu(x) \sim (\Gamma(-\nu)/2^{\nu+1})\, x^\nu$ as $x \downarrow 0$ when $\nu < 0$, that the following limiting identity holds

$$p(t;x,0) = c\,\tfrac{\sin(-\nu\pi)}{\pi}\,\tfrac{\Gamma(-\nu)}{2^{\nu+1}} \int_0^t (t-s)^\nu E_{\nu+1,\nu+1}\big(-c(t-s)^{\nu+1}\big) \frac{1}{s^{\nu+1}}\, e^{-x^2/2s}\, ds \tag{6.6}$$

for $t > 0$ and $x \in [0,\infty)$ where $c = c(\mu,\nu)$ is given by (4.18) above. In particular, using (3.1) with $A = \{0\}$ and (2.7) this yields

$$\begin{aligned} \mathsf{P}_x(X_t{=}0) &= p(t;x,0)\, m(\{0\}) = \tfrac{1}{\mu}\, p(t;x,0) \\ &= \tfrac{\sin(-\nu\pi)}{\pi}\, \Gamma(\nu{+}1) \int_0^t (t-s)^\nu E_{\nu+1,\nu+1}\big(-2^{\nu+1}\tfrac{\Gamma(\nu+1)}{\Gamma(-\nu)}\,\mu\,(t-s)^{\nu+1}\big) \frac{1}{s^{\nu+1}}\, e^{-x^2/2s}\, ds \end{aligned} \tag{6.7}$$

for $t > 0$ and $x \in [0,\infty)$. Moreover, from (3.1) and (3.2) we see that

$$\int_0^\infty f(t;x,y)\, dy + \tfrac{1}{\mu}\, p(t;x,0) = 1 \tag{6.8}$$

for $t > 0$ and $x \in [0,\infty)$. Since $\frac{1}{\mu}\, p(t;x,0) > 0$ this shows that $y \mapsto f(t;x,y)$ does not integrate to 1 over $(0,\infty)$ for $t > 0$ and $x \in [0,\infty)$ given and fixed. The fact that X_t takes

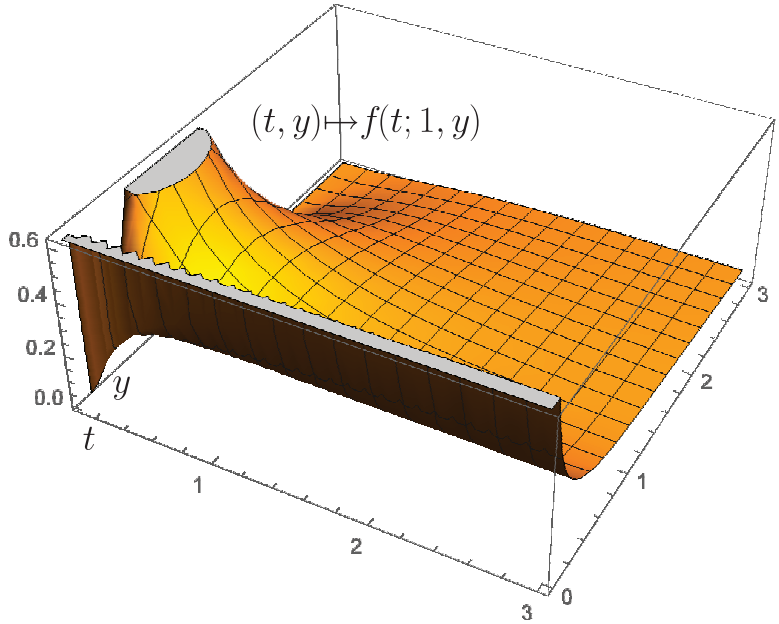

Figure 2. The transition density function $(t,y)\mapsto f(t;1,y)$ of the Bessel process X having dimension $\delta=1/5$ (i.e. index $\nu=-9/10$) and 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu=5$. Defined by (3.3) the function f can be expressed in a closed from using (6.3) combined with (3.2) and (2.6). Note that X starts at 1 so that $f(0;1,y)\to\infty$ as $y\to 1$. Note also that $f(t;1,y)\to\infty$ as $y\downarrow 0$ for each $t>0$ given and fixed (cf. Figure 1).

value 0 with a strictly positive probability expressed by (6.7) above for $t>0$ and $x\in[0,\infty)$ given and fixed is a consequence of the slowly reflecting (sticky) boundary behaviour of X at 0 . Plots of the transition density function f of X (with respect to Lebesque measure) are given in Figures 1 and 2. The corresponding plots of the transition density function p of X (with respect to its speed measure) look similarly in both cases to the plot of f in Figure 1 with one notable exception: unlike the values of $f(t;x,0+)$ from Figures 1 and 2 respectively, the values of $p(t;x,0+)$ are both strictly positive and finite for $t>0$ and $x\in[0,\infty)$ given and fixed (see Figure 3).

Remark 10. The function

$$e_{\alpha,\beta}(t;c):=t^{\beta-1}E_{\alpha,\beta}(-ct^{\alpha}) \tag{6.9}$$

stated following the exponential function under the integral sign in (6.2) above for $t\geq 0$ is sometimes refereed to as the *generalised Mittag-Leffler function* (see [7, p. 206] and the references therein). Note that the same function (with $\alpha=\beta=\nu+1$) appears as the function of $t-s$ in the convolution integral of (6.3) above as well as in similar other expressions derived throughout (such as (6.6), (6.7) above and (7.3), (7.7), (7.9) below).

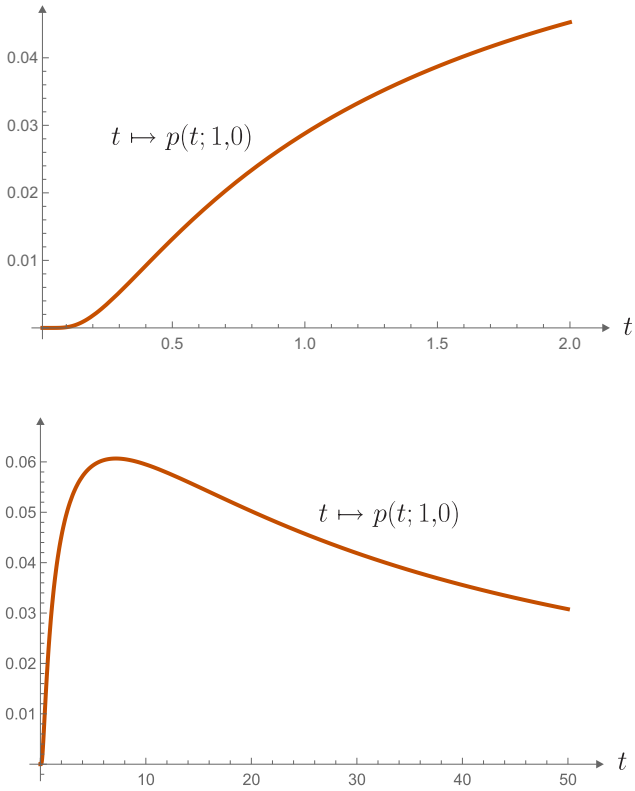

Figure 3. A zero section $t \mapsto p(t;1,0)$ of the transition density function p of the Bessel process X having dimension $\delta = 1/5$ (i.e. index $\nu = -9/10$) and 0 as a slowly reflecting (sticky) boundary point with a stickiness parameter $1/\mu = 5$ for small values of t (upper plot) and large values of t (lower plot). Recall from (6.7) that $\mathsf{P}_1(X_t{=}0) = (1/\mu)\,p(t;1,0) = 5\,p(t;1,0)$ for $t > 0$ in this case.

7. Generalised Abel equation

In this section we show that the convolution integral appearing in the transition density function of X from the previous section can be characterised as a unique solution to the generalised Abel equation of the second kind.

Recall that the transition density function of X when the boundary point 0 is either absorbing/killing or instantaneously reflecting are respectively given by

$$p_a(t;x,y) = (xy)^{-\nu}\,\frac{1}{2t}\,e^{-(x^2+y^2)/2t}\,I_{-\nu}\Big(\frac{xy}{t}\Big) \tag{7.1}$$

$$p_r(t;x,y) = (xy)^{-\nu}\,\frac{1}{2t}\,e^{-(x^2+y^2)/2t}\,I_{\nu}\Big(\frac{xy}{t}\Big) \tag{7.2}$$

for $t > 0$ and $x, y \in [0,\infty)$. Recall also from Section 1 above that (7.1) was derived by Feller [5, p. 180] and (7.2) was derived by Molchanov [13, p. 312]. On closer comparison we see that Feller's expression (7.1) appears as the first term on the right-hand side of the general transition density function (6.3). The second term on the right-hand side of (6.3) can therefore be viewed

as a 'correction' term from absorbing to sticky boundary behaviour. Note that (6.3) itself can be viewed as the convolution identity $p = p_a + e * (p_r - p_a)$ where $e(t) := c e_{\nu+1,\nu+1}(t;c)$ for $t \geq 0$ upon recalling (6.9) above. To describe such a correction term from instantaneously reflecting to sticky boundary behaviour we may subtract and add the I_ν term on the right-hand side of (6.3) upon recalling (4.3) above. This yields the following analogue of (6.3) above

$$
\begin{aligned}
(7.3) \qquad p(t;x,y) &= (xy)^{-\nu} \frac{1}{2t} e^{-(x^2+y^2)/2t} I_\nu\Big(\frac{xy}{t}\Big) \\
&- \tfrac{2\sin(-\nu\pi)}{\pi} (xy)^{-\nu} \Bigg[\frac{1}{2t} e^{-(x^2+y^2)/2t} K_\nu\Big(\frac{xy}{t}\Big) \\
&- c \int_0^t (t-s)^\nu E_{\nu+1,\nu+1}\big(-c(t-s)^{\nu+1}\big) \frac{1}{2s} e^{-(x^2+y^2)/2s} K_\nu\Big(\frac{xy}{s}\Big) ds \Bigg]
\end{aligned}
$$

for $t > 0$ and $x, y \in [0,\infty)$. On closer comparison we see that Molchanov's expression (7.2) appears as the first term on the right-hand side of the general transition density function (7.3). The second term on the right-hand side of (7.3) can therefore be viewed as a 'correction' term from instantaneously reflecting to sticky boundary behaviour. It turns out moreover that this correction term can be linked to a generalised Abel equation as follows.

Corollary 11 (Generalised Abel equation). *The transition density function of X defined by* (3.1) *above can be explicitly expressed as*

$$
(7.4) \qquad p(t;x,y) = p_r(t,x,y) - q(t;x,y)
$$

where p_r is given by (7.2) *above and q can be characterised as a unique solution (in the class of locally integrable functions) to the generalised Abel equation*

$$
(7.5) \qquad q(t;x,y) + \mu \frac{2^{\nu+1}}{\Gamma(-\nu)} \int_0^t \frac{q(s;x,y)}{(t-s)^{-\nu}} ds = Q(t;x,y)
$$

where the function Q is defined by

$$
(7.6) \qquad Q(t;x,y) = \tfrac{2\sin(-\nu\pi)}{\pi} (xy)^{-\nu} \frac{1}{2t} e^{-(x^2+y^2)/2t} K_\nu\Big(\frac{xy}{t}\Big)
$$

for $t > 0$ with $x, y \in (0,\infty)$ given and fixed.

Proof. Comparing (7.3) and (7.4) we see that

$$
(7.7) \qquad q(t) = Q(t) - c \int_0^t \frac{E_{\nu+1,\nu+1}(-c(t-s)^{\nu+1})}{(t-s)^{-\nu}} Q(s)\, ds
$$

for $t > 0$ upon removing the notational dependence on x and y throughout for simplicity. Recall that the generalised Abel equation

$$
(7.8) \qquad g(t) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds = G(t)
$$

has a unique solution (in the class of integrable functions) given by

$$g(t) = G(t) + \lambda \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}} G(s)\, ds \tag{7.9}$$

for $t \in (0,T)$ whenever $\lambda \in I\!R$, $\alpha > 0$ and G is an integrable function on $(0,T)$ with $T > 0$ given and fixed (cf. [2, p. 3]). From (7.7) and (7.9) we see that g coincides with q if we set $G = Q$, $\lambda = -c$ and $\alpha = \nu+1$. Recalling that $K_\nu(x) \sim (\sqrt{\pi/2x})e^{-x}$ as $x \to \infty$ we see that $t \mapsto G(t) = Q(t;x,y)$ is integrable on $(0,T)$ as needed. By (4.18) we see that $-\lambda/\Gamma(\alpha) = c/\Gamma(\nu+1) = (2^{\nu+1}/\Gamma(-\nu))\mu$ and this shows that (7.8) reduces to (7.5) as claimed completing the proof. □

8. Limiting cases

In this section we show that letting $\mu \downarrow 0$ (absorption) and $\mu \uparrow \infty$ (instantaneous reflection) the closed-form expression for the transition density function p of X from Theorem 8 and Corollary 11 reduces to the ones found by Feller [5] and Molchanov [13] respectively as discussed in Section 1 above. We also show that p reduces to a known closed-form expression when $\delta = 1$ i.e. $\nu = -1/2$ in which case X is a standard Brownian motion in $[0,\infty)$ having 0 as a slowly reflecting (sticky) boundary point.

1. Recall that the transition density function p of X (with respect to its speed measure) is given by (6.3) and (7.3) above where $c = c(\mu,\nu)$ is given by (4.18) above. To indicate the dependence of p on μ through c we will write p_μ in place of p when needed. Recall also that Feller's absorbing/killing transition density function p_a and Molchanov's instantaneously reflecting transition density function p_r are given by (7.1) and (7.2) above respectively.

Proposition 12. *We have*

$$\lim_{\mu\downarrow 0} p_\mu = p_a \quad \& \quad \lim_{\mu\uparrow\infty} p_\mu = p_r \tag{8.1}$$

pointwise on $(0,\infty)\times[0,\infty)^2$.

Proof. Recalling from (4.18) above that $c = c(\mu,\nu) = 2^{\nu+1}(\Gamma[\nu+1)/\Gamma(-\nu))\mu$, letting $\mu \downarrow 0$ i.e. $c \downarrow 0$ in (6.3), and using that $s \mapsto \mathsf{E}_{\nu+1,\nu+1}(-c(t-s)^{\nu+1})$ is uniformly bounded on $[0,t]$ over $c > 0$ while $s \mapsto K_\nu(xy/s)$ is bounded on $[0,t]$, we see by the dominated convergence theorem that the first equality in (8.1) holds as claimed. Similarly, letting $\mu \uparrow \infty$ i.e. $c \uparrow \infty$ in (7.3), using that $x\,\mathsf{E}_{\nu+1,\nu+1}(x) \to 0$ as $x \to -\infty$ and that $s \mapsto \mathsf{E}_{\nu+1,\nu+1}(-c(t-s)^{\nu+1})$ is uniformly bounded on $[0,t]$ over $c > 0$ while $s \mapsto K_\nu(xy/s)$ is bounded on $[0,t]$, we see by the dominated convergence theorem that the final term in (7.3) tends to zero. Adding the remaining two terms on the right-hand side of (7.3) and using (4.3) we obtain the second equality in (8.1) as claimed and the proof is complete. □

2. If $\delta = 1$ i.e. $\nu = -1/2$ then X is a standard Brownian motion in $[0,\infty)$ having 0 as a slowly reflecting (sticky) boundary point. A closed-form expression for the transition density function p of X (with respect to its speed measure) is known in this case and can be obtained

by letting the killing parameter γ tend to zero in the more general expression for p given in [1, Section 11, p. 125]. This yields the following closed-form expression

$$p(t;x,y) = \frac{1}{2\sqrt{2\pi t}}\Big[\exp\Big(-\frac{(x-y)^2}{2t}\Big) - \exp\Big(-\frac{(x+y)^2}{2t}\Big)\Big] + \mu\,\exp\Big(2\mu(x+y)+2\mu^2 t\Big)\,\mathrm{Erfc}\Big(\frac{x+y}{\sqrt{2t}}+\mu\sqrt{2t}\Big) \tag{8.2}$$

for $t>0$ and $x,y\in[0,\infty)$ where $\mathrm{Erfc}(z)=(2/\sqrt{\pi})\int_z^\infty e^{-w^2}dw$ for $z\in I\!R$. Recalling that $I_{1/2}(x)=\sqrt{2/\pi x}\,\mathrm{sh}(x)$ for $x\in I\!R$ we see that the first term on the right-hand side of (6.3) coincides with the first term on the right-hand side of (8.2). To show that the second term on the right-hand side of (6.3) can be expressed as the second term on the right-hand side of (8.2), we may first recall that $E_{1/2,1/2}(x)=x\,e^{x^2}\mathrm{Erfc}(-x)+1/\sqrt{\pi}$ and $K_{-1/2}(x)=\sqrt{\pi/2x}\,e^{-x}$ for $x\in I\!R$, and then make use of the Laplace transforms (10) (p. 177), (27) (p. 146), (1) (p. 137) and the inverse Laplace transforms (14) (p. 246), (3) (p. 245) in [4], combined with the convolution theorem for the Laplace transform. As this calculation is somewhat lengthy but still straightforward we will omit fuller details. This shows that (6.3) reduces to (8.2) when $\delta=1$ i.e. $\nu=-1/2$ while also remaining valid when $\delta\in(0,1)\cup(1,2)$ i.e. $\nu\in(-1,-1/2)\cup(-1/2,0)$ as established in this paper.

Acknowledgements. The author is grateful to David Roodman of Open Philanthropy for insightful discussions and useful comments.

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