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Best Constants in Kahane-Khintchine Inequalities for Complex Steinhaus Functions

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Let $\{\varphi_k\}_{k\geq 1}$ be a sequence of independent random variables uniformly distributed on $[0, 2\pi[$, and let $\|\cdot\|_\psi$ denote the Orlicz norm induced by the function $\psi(x) = \exp(|x|^2) - 1$. Then:

$$\Big\| \sum_{k=1}^n z_k e^{i\varphi_k} \Big\|_\psi \leq \sqrt{2} \Big(\sum_{k=1}^n |z_k|^2 \Big)^{1/2}$$

for all $z_1, \ldots, z_n \in \mathbf{C}$ and all $n \geq 1$. The constant $\sqrt{2}$ is shown to be the best possible. The method of proof relies upon a combinatorial argument, Taylor expansion, and the central limit theorem. The result is additionally strengthened by showing that the underlying functions are Schur-concave. The proof of this fact uses a result on multinomial distribution of Rinott, and Schur's proposition on sum of convex functions. The estimates obtained throughout are shown to be the best possible. The result extends and generalizes to provide similar inequalities and estimates for other Orlicz norms.

1. Introduction

Let $\{\varepsilon_i\}_{i\geq 1}$ be a Bernoulli sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $\|\cdot\|_\psi$ denote the Orlicz norm induced by the function $\psi(x) = \exp(|x|^2) - 1$. Thus, whenever X is a random variable defined on $(\Omega, \mathcal{F}, P)$, we have:

$$\|X\|_\psi = \inf \big\{ C > 0 \mid E\psi(|X|/C) \leq 1 \big\}$$

with convention $\inf \emptyset = +\infty$. Then:

$$\Big\| \sum_{i=1}^n a_i \varepsilon_i \Big\|_\psi \leq \sqrt{\frac{8}{3}} \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \tag{1.1}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ and all $n \geq 1$. The constant $\sqrt{8/3}$ is the best possible (see [13]). The inequality (1.1) is called *Kahane-Khintchine's inequality* in (exponential) Orlicz space, and is known to have a number of applications.

The present paper is motivated by the following observations. Consider random variables ε_i in (1.1). Recall that $P\{\varepsilon_i = \pm 1\} = 1/2$. Thus ε_i may be interpreted as uniformly distributed on the unit sphere $S_1 = \{-1, +1\}$ in $\mathbf{R}$. Now let $\{\sigma_k\}_{k\geq 1}$ be a sequence of independent random variables uniformly distributed on the unit sphere S_2 in $\mathbf{R}^2$. Then the problem appears worthy of consideration: Does the analogue of (1.1) remain valid, and what is the best possible constant in this case? It is the purpose of the paper to present solution for this problem.

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In this context we find it convenient to replace $\mathbf{R}^2$ with the set of complex numbers $\mathbf{C}$. Then $\sigma_k = e^{i\varphi_k}$ for $k \geq 1$, where φ_k's are independent and uniformly distributed on $[0, 2\pi[$. The analogue of (1.1) may be stated as follows:

$$\Big\| \sum_{k=1}^n z_k e^{i\varphi_k} \Big\|_\psi \leq C \Big(\sum_{k=1}^n |z_k|^2 \Big)^{1/2} \tag{1.2}$$

where $z_1, \ldots, z_n \in \mathbf{C}$ and $n \geq 1$. In order to prove (1.2) recall that (1.1) generalizes to the form:

$$\Big\| \sum_{i=1}^n X_i \Big\|_\psi \leq \sqrt{\frac{8}{3}} \Big(\sum_{i=1}^n \|X_i\|_\infty^2 \Big)^{1/2} \tag{1.3}$$

where $\{X_i\}_{i \geq 1}$ is any sequence of independent symmetric a.s. bounded random variables (see [13]). Notice that $z_k e^{i\varphi_k} \sim |z_k| e^{i\varphi_k}$, thus there is no restriction in (1.2) to assume that $z_k \in \mathbf{R}_+$ for all $k \geq 1$. Combining this fact with (1.3), we obtain by triangle inequality:

$$\Big\| \sum_{k=1}^n z_k e^{i\varphi_k} \Big\|_\psi = \Big\| \sum_{k=1}^n z_k \cos\varphi_k + i \sum_{k=1}^n z_k \sin\varphi_k \Big\|_\psi \leq 2\sqrt{\frac{8}{3}} \Big(\sum_{k=1}^n |z_k|^2 \Big)^{1/2}$$

for all $z_1, \ldots, z_n \in C$ and all $n \geq 1$. Thus (1.2) is valid with $C = 2\sqrt{8/3} = 3.26\ldots$ However, it is clear that this constant is far from being the best possible in (1.2).

Our main aim in this paper is to present a method of proof which establishes (1.2) with the best possible constant C . We find it useful here to clarify the main points of the approach. For this consider (1.1) with given and fixed $a_1, \ldots, a_n \in \mathbf{R}$ and $n \geq 1$. Denote $S_n = \sum_{i=1}^n a_i \varepsilon_i$, and throughout assume that $\sum_{i=1}^n |a_i|^2 = 1$. Note that (1.1) follows as soon as we obtain:

$$E \exp\Big(\frac{|S_n|^2}{C^2} \Big) \leq 2 \tag{1.4}$$

with $C = \sqrt{8/3}$. Thus the problem reduces to estimate the left side of (1.4) in an optimal way. It turns out that the best estimate is as follows:

$$E \exp\Big(\frac{|S_n|^2}{C^2} \Big) \leq E \exp\Big(\frac{|Z|^2}{C^2} \Big) = \frac{C}{\sqrt{C^2 - 2}} \tag{1.5}$$

which is valid for all $C > \sqrt{2}$ and where $Z \sim N(0,1)$ is standard Gaussian variable. Identifying $C/\sqrt{C^2 - 2} = 2$, one gets $C = \sqrt{8/3}$ and completes the proof of (1.1).

Probably the best understanding of (1.5) may be obtained through the concept of *Schur-convexity* in the theory of majorization, which we find instructive here to explain in more detail. It should be noted that in this process we also clarify the reason for which (1.5) serves the best possible constant in (1.1). First note that expanding the integrands in (1.5) into Taylor series, it suffices to show:

$$E|S_n|^{2k} \leq E|Z|^{2k} \tag{1.6}$$

for all $k \geq 1$. This inequality follows by the central limit theorem from an intermediate fact which is by itself of theoretical interest:

(1.7) The map $(x_1, \ldots, x_n) \stackrel{\Phi}{\longmapsto} E\Big| \sum_{i=1}^n \sqrt{x_i}\, \varepsilon_i \Big|^{2k}$ is Schur-concave on $\mathbf{R}_+^n$.

Roughly speaking, this means that $\Phi(x)$ dominates $\Phi(y)$ whenever the components of vector x

are “less spread out” or “more nearly equal” than the components of vector y . More precisely, if $z_1^* \geq \ldots \geq z_n^*$ denote the components of vector z in decreasing order, then (1.7) means:

$$\sum_{i=1}^{k} x_i^* \leq \sum_{i=1}^{k} y_i^* \quad (k=1,\ldots,n-1) \;\;\&\;\; \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \;\Rightarrow\; \Phi(x) \geq \Phi(y) \tag{1.7'}$$

whenever $x, y \in \mathbf{R}_+^n$. In particular, we obtain:

$$E\Big|\sum_{i=1}^{n} a_i\varepsilon_i\Big|^{2k} \leq E\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i\Big|^{2k} \leq E\Big|\frac{1}{\sqrt{n+1}}\sum_{i=1}^{n+1}\varepsilon_i\Big|^{2k} \tag{1.8}$$

for all $k \geq 1$. Finally, passing to the limit in (1.8), we get (1.6) by the central limit theorem. Moreover, in this way we also see that inequality (1.6) eventually becomes an equality (with the choice $a_i = 1/\sqrt{n}$ for $i = 1,\ldots,n$) . The same fact carries over to (1.5), and explains how (1.5) serves the best possible constant in (1.1).

We remind that (1.6) is due to Khintchine [8], while (1.7) is due partially to Efron [2], and fully to Eaton [1]. It should be pointed out that two interesting generalizations of these results may be encountered, where $g(x) = |x|^{2k}$ for $k \geq 1$ is replaced by a wider class of functions (see [1]). For more information about the theory of majorization and its applications just indicated, we refer the reader to [11]. In particular, we find it instructive to remind on *Schur’s condition* which characterizes Schur’s convexity (concavity) in terms of first partial derivatives (see [11] p.57).

We turn back to the original problem of finding the optimal method for the proof of (1.2). To the best of our knowledge the theory of majorization described above does not support our needs explicitly for the optimal (1.2). In particular, putting $S_n = \sum_{k=1}^n z_k e^{i\varphi_k}$ for given $z_1,\ldots,z_n \in \mathbf{C}$ such that $\sum_{k=1}^n |z_k|^2 = 1$, nothing seems in general to be known about the analogue of (1.6) and (1.7) in this case, as well as about their generalizations to a wider class of functions as mentioned above. It should be noted that for $p \geq 1$ we have:

$$E\Big|\sum_{k=1}^{n} z_k e^{i\varphi_k}\Big|^{2p} = E\bigg(\Big(\sum_{k=1}^{n} z_k \cos\varphi_k\Big)^2 + \Big(\sum_{k=1}^{n} z_k \sin\varphi_k\Big)^2\bigg)^p$$

To conclude, we may notice that actually two separate problems have appeared. The first one is to prove (1.2) in an optimal way which will provide the best possible constant C . The second and more general one is to prove the analogue of (1.6) and (1.7) for a wider class of functions as indicated above which will include $g(x) = |x|^{2p}$ for $p \geq 1$. In this paper we find complete solution for the first problem, and partial solution for the second problem which covers our needs for the first problem. The approach makes no attempt to obtain a more general solution for the second problem. It is left as worthy of consideration.

2. Preliminary facts

In this section we introduce notation and collect facts needed for the main results in the next section. Throughout we denote $\psi(x) = \exp(|x|^2)-1$ and work with the Orlicz norms:

$$\|X\|_\psi = \inf\{\, C>0 \mid E\psi(|X|/C) \leq 1 \,\}$$

$$\|X\|_{T_\psi} = \inf\{\, C>0 \mid E\psi(|X|/C) \leq C \,\}$$

$$\|X\|_{\Upsilon_\psi} = E\psi(|X|)$$

where X is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Recall that $\|\cdot\|_\psi$ is called *the gauge norm*. We remark that the quantity $\|X\|_{T_\psi}$ emerged in the study [5]. Its interest relies upon the fact that for more general functions ψ , the map $\|\cdot\|_\psi$ need not be an Fréchet norm, but $\|\cdot\|_{T_\psi}$ is so (see [5] p.17,18). The quantity $\|X\|_{\Upsilon_\psi}$ is of an intermediate value for both $\|X\|_\psi$ and $\|X\|_{T_\psi}$. For more information about the Orlicz norm just introduced we refer to [5] and [13].

We turn to the concept of Schur-convexity in the theory of majorization. Let $z_1^* \geq \ldots \geq z_n^*$ denote the components of vector $z \in \mathbf{R}^n$ in decreasing order. Given $x, y \in \mathbf{R}^n$, we say that x is *majorized* by y and write $x \prec y$, if the conditions are fulfilled:

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i \quad \& \quad \sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^*$$

for all $k = 1, \ldots, n-1$. For instance, we have:

(2.1) $$\Big(\frac{1}{n}, \ldots, \frac{1}{n}\Big) \prec \Big(\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0\Big) \prec \big(\tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0\big) \prec (1, 0, \ldots, 0)$$

(2.2) $$\Big(\frac{1}{n}, \ldots, \frac{1}{n}\Big) \prec (a_1, \ldots, a_n) \prec (1, 0, \ldots, 0) \text{ whenever } a_i \geq 0 \text{ and } \textstyle\sum_{i=1}^n a_i = 1$$

(2.3) $$\big(n^{-1}\textstyle\sum_{i=1}^n x_i\big)(1, \ldots, 1) \prec (x_1, \ldots, x_n) \text{ whenever } x_i \geq 0 \ .$$

Let $D \subset \mathbf{R}^n$ be a set, and let $\Phi : D \to \mathbf{R}$ be a function. Then Φ is said to be *Schur-convex* on D , if $\Phi(x) \leq \Phi(y)$ whenever $x, y \in D$ and $x \prec y$. The map Φ is said to be *Schur-concave*, if $-\Phi$ is Schur-convex. It is easily verified that if Φ is Schur-convex, and D is symmetric, then Φ is symmetric as well. Moreover, if Φ is symmetric and convex, then Φ is Schur-convex.

The following well-known result combined with (2.1)-(2.3) may provide a lot of interesting inequalities (for proof see [11] p.64).

Proposition 2.1 (Schur; Hardy-Litlewood-Pólya)

If $I \subset \mathbf{R}$ is an interval, and $g : I \to \mathbf{R}$ is convex, then the function:

$$\Phi(x) = \sum_{i=1}^n g(x_i)$$

is Schur-convex on I^n .

The next fundamental theorem characterizes Schur-convexity (concavity) in terms of first partial derivatives (for proof see [11] p.57).

Theorem 2.2 (Schur-Ostrowski)

Let $I \subset \mathbf{R}$ be an interval, and let $\Phi : I^n \to \mathbf{R}$ be continuously differentiable. Then Φ is Schur-convex on I^n , if and only if the following two conditions are satisfied:

(2.4) Φ *is symmetric on* I^n

(2.5) $$(x_i - x_j)\Big(\frac{\partial \Phi}{\partial x_i}(x) - \frac{\partial \Phi}{\partial x_j}(x)\Big) \geq 0 \text{ for all } x \in I^n \text{ and all } i \neq j \ .$$

Moreover, whenever (2.4) is satisfied, (2.5) may be replaced by the condition:

(2.5') $$(x_1 - x_2)\Big(\frac{\partial \Phi}{\partial x_1}(x) - \frac{\partial \Phi}{\partial x_2}(x)\Big) \geq 0 \text{ for all } x \in I^n \ .$$

The same characterization remains valid for Schur-concave functions Φ *with inequalities (2.5) and (2.5') being reversed.*

The following result on multinomial distribution of Rinott is shown to be useful. It might be proved in a straightforward way by verifying Schur's condition (2.5') in Theorem 2.2 (see [15]).

(2.6) *Let* $X = (X_1, \ldots, X_n)$ *be a random vector from the multinomial distribution with parameters* $z = (z_1, \ldots, z_n) \in [0,1]^n$ *and* $p \ge 1$ *, where* $\sum_{i=1}^n z_i = 1$ *. In other words:*

$$P\{ X_1 = p_1, \ldots, X_n = p_n \} = \frac{p\,!}{p_1! \, \ldots \, p_n!} \, z_1^{p_1} \ldots z_n^{p_n}$$

for $p_1, \ldots, p_n \in \mathbf{Z}_+^n$ *with* $\sum_{i=1}^n p_i = p$ *. If* Ψ *is Schur-convex (-concave), then the function* $z \mapsto E_z \Psi(X)$ *is Schur-convex (-concave).*

We conclude with a few facts on the complex Steinhaus sequence of random variables. Let $\{\varphi_k\}_{k \ge 1}$ be a sequence of independent random variables uniformly distributed on $[0, 2\pi[$. Then $\{e^{i\varphi_k}\}_{k \ge 1}$ is a sequence of random variables uniformly distributed on the unit sphere S_2 in $\mathbf{C}$. We denote $\sigma_k = e^{i\varphi_k}$ for $k \ge 1$, and the sequence $\{\sigma_k\}_{k \ge 1}$ is called a *complex Steinhaus sequence*.

Note that $E \cos \varphi_1 = E \sin \varphi_1 = E \cos \varphi_i \, \sin \varphi_j = 0$ for $i \ne j$, and $E \cos^2 \varphi_1 = E \sin^2 \varphi_1 = 1/2$. Thus by the two-dimensional central limit theorem we have:

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{i\varphi_k} \overset{\sim}{\longrightarrow} Z_1 + i Z_2 \tag{2.7}$$

as $n \to \infty$, where $Z_1 \sim Z_2 \sim N(0, 1/2)$ are independent Gaussian variables with joint density:

$$f_{(Z_1, Z_2)}(x, y) = \frac{1}{\pi} \exp(-x^2 - y^2) \tag{2.8}$$

for $x + iy \in \mathbf{C}$. It should be recalled that $\mathbf{C}$ is topologically the same as $\mathbf{R}^2$. Thus week convergence in (2.7) coincides for both $\mathbf{C}$ and $\mathbf{R}^2$. From (2.8) we get:

$$\begin{aligned} E \exp\Big(|Z_1 + i Z_2|^2 / C^2 \Big) &= E \exp\Big(\big(Z_1^2 + Z_2^2\big) / C^2 \Big) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\Big(\big(x^2 + y^2\big) / C^2 \Big) \cdot \frac{1}{\pi} \exp(-x^2 - y^2) \; dx \, dy \\ &= \Big(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\Big(- \Big(1 - \frac{1}{C^2}\Big) x^2 \Big) \, dx \, \Big)^2 = \frac{1}{1 - \frac{1}{C^2}} = \frac{C^2}{C^2 - 1} \end{aligned} \tag{2.9}$$

for all $C > 1$. Identifying $C^2/C^2 - 1 = 2$, we obtain $C = \sqrt{2}$. Thus we have:

$$\| Z_1 + i Z_2 \|_\psi = \sqrt{2} \, . \tag{2.10}$$

Similarly, from (2.8) we get:

$$\begin{aligned} E|Z_1 + i Z_2|^{2p} &= E(Z_1^2 + Z_2^2)^p \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)^p \exp(-x^2 - y^2) \; dx \, dy \\ &= \frac{1}{\pi} \int_{0}^{\infty} \int_{0}^{2\pi} r^{2p+1} \exp(-r^2) \; dr = 2 \int_{0}^{\infty} r^{2p+1} \exp(-r^2) \; dr = p! \end{aligned} \tag{2.11}$$

for all $p \geq 1$. Finally, to distinguish from the real case in (1.1) where $E(\varepsilon_i)^2 = 1$ for $i \geq 1$, it is interesting to observe that $E(\sigma_k)^2 = 0$, although $E|\sigma_k|^2 = 1$ for $k \geq 1$.

3. Kahane-Khintchine inequalities for complex Steinhaus variables

In this section we present the main results of the paper. We begin with the following basic fact.

Lemma 3.1

Let $\{e^{i\varphi_k}\}_{k\geq 1}$ *be a complex Steinhaus sequence. Then the inequality is satisfied:*

$$E\Big|\sum_{k=1}^{n} z_k e^{i\varphi_k}\Big|^{2p} \leq p!\,\Big(\sum_{k=1}^{n}|z_k|^2\Big)^p \tag{3.1}$$

for all $z_1,\dots,z_n \in \mathbf{C}$ *, and all integers* n *,* $p \geq 1$ *. The constant* $p!$ *is the best possible.*

Proof. Since $z_k e^{\varphi_k} \sim |z_k| e^{\varphi_k}$ for $k \geq 1$, it is no restriction to assume that the given numbers $z_1,\dots,z_n$ belong to $\mathbf{R}_+$ for $n \geq 1$. In order to clarify the combinatorial argument in the general case below, we first verify (3.1) for $n = p = 2$. For this, note that we have:

$$Ee^{i(\varphi_j-\varphi_k+\varphi_l-\varphi_m)} = \begin{cases} 1\ , & \text{if } \ (\mathrm{j},\mathrm{l}) \in \{(\mathrm{k},\mathrm{m}),(\mathrm{m},\mathrm{k})\} \\ 0\ , & \text{otherwise} \end{cases} \tag{3.2}$$

for all $j,k,l,m \geq 1$. From this fact we obtain:

$$\begin{aligned} E\Big|\sum_{k=1}^{n} z_k e^{i\varphi_k}\Big|^{2p} &= E\Big|\sum_{k=1}^{2} z_k e^{i\varphi_k}\Big|^{4} = E\Big(\sum_{j=1}^{2}\sum_{k=1}^{2} z_j z_k e^{i(\varphi_j-\varphi_k)}\Big)\Big(\sum_{l=1}^{2}\sum_{m=1}^{2} z_l z_m e^{i(\varphi_l-\varphi_m)}\Big) \\ &= \sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2} z_j z_k z_l z_m E e^{i(\varphi_j-\varphi_k+\varphi_l-\varphi_m)} = z_1^4 + 4z_1^2 z_2^2 + z_2^4 \\ &= 2\Big(z_1^4/2 + 2z_1^2 z_2^2 + z_2^4/2\Big) \leq 2(z_1^2+z_2^2)^2 \end{aligned}$$

and the proof of (3.1) in this case is complete.

The general case follows from the analogue of (3.2) by the same combinatorial pattern:

$$E\Big|\sum_{k=1}^{n} z_k e^{i\varphi_k}\Big|^{2p} = \sum_{p_i\geq 0,\, p_1+\dots+p_n=p} C_{p_1,\dots,p_n}\cdot z_1^{2p_1}\cdot\ldots\cdot z_n^{2p_n} \tag{3.3}$$

$$C_{p_1,\dots,p_n} = \binom{p}{p_1}^2\binom{p-p_1}{p_2}^2\binom{p-p_1-\dots-p_{n-1}}{p_n}^2 = \Big(\frac{p!}{p_1!\dots p_n!}\Big)^2 \tag{3.4}$$

being valid for all $p \geq 1$, where $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = n$. Combining (3.3) and (3.4) we get:

$$\begin{aligned} E\Big|\sum_{k=1}^{n} z_k e^{i\varphi_k}\Big|^{2p} &\leq p! \sum_{p_i\geq 0,\, p_1+\dots+p_n=p} \frac{p!}{p_1!\dots p_n!}\cdot z_1^{2p_1}\cdot\ldots\cdot z_n^{2p_n} \\ &= p!\,\Big(\sum_{k=1}^{n}|z_k|^2\Big)^p \end{aligned}$$

for all $p \geq 1$. Thus the proof of (3.1) is complete.

For the last statement, notice that by the central limit theorem (2.7) with (2.11) we obtain:

$$\lim_{n\to\infty} E\Big|\frac{1}{\sqrt{n}}\sum_{k=1}^{n} e^{i\varphi_k}\Big|^{2p} = E|Z_1+iZ_2|^{2p} = p!$$

for all $p \geq 1$. Thus the last statement follows by putting $z_1 = \ldots = z_n = 1/\sqrt{n}$ in (3.1), and passing to the limit when $n \to \infty$. These facts complete the proof. $\square$

Theorem 3.2
Let $\{e^{i\varphi_k}\}_{k\geq 1}$ *be a complex Steinhaus sequence. Then the inequality is satisfied:*

$$\Big\| \frac{1}{C\big(\sum_{k=1}^{n}|z_k|^2\big)^{1/2}} \sum_{k=1}^{n} z_k e^{i\varphi_k} \Big\|_{\Upsilon_\psi} \leq \frac{C^2}{C^2-1} - 1 \tag{3.5}$$

for all $z_1, \ldots, z_n \in \mathbf{C}$, *all* $n \geq 1$, *and all* $C > 1$. *The estimate is the best possible (in the sense described in the proof below).*

Proof. It is no restriction to assume that the given numbers $z_1, \ldots, z_n$ belong to $\mathbf{R}_+$ for $n \geq 1$, as well as that $\sum_{k=1}^{n}|z_k|^2 = 1$. Denote $S_n = \sum_{k=1}^{n} z_k e^{i\varphi_k}$, then by (3.1) and Taylor expansion we get:

$$E\exp\left(\frac{|S_n|^2}{C^2}\right) = \sum_{p=0}^{\infty}\frac{E|S_n|^{2p}}{p!\,C^{2p}} \leq \sum_{p=0}^{\infty}\frac{1}{C^{2p}} = \frac{C^2}{C^2-1} \tag{3.6}$$

for all $C > 1$. Hence (3.5) follows straightforward by definition of the Orlicz norm $\|\cdot\|_{\Upsilon_\psi}$.

For the last statement notice that putting $z_1 = \ldots = z_n = 1/\sqrt{n}$ in (3.6), we obtain by the central limit theorem (2.7) with (2.9):

$$\lim_{n\to\infty} E\exp\left(\frac{|S_n|^2}{C^2}\right) = \frac{C^2}{C^2-1} \tag{3.7}$$

for all $C > 1$. Hence we see that with this choice inequality (3.5) eventually becomes an equality. This fact completes the proof. $\square$

Theorem 3.3
Let $\{e^{i\varphi_k}\}_{k\geq 1}$ *be a complex Steinhaus sequence. Then the inequality is satisfied:*

$$\Big\| \sum_{k=1}^{n} z_k e^{i\varphi_k} \Big\|_{\psi} \leq \sqrt{2}\,\Big(\sum_{k=1}^{n}|z_k|^2\Big)^{1/2} \tag{3.8}$$

for all $z_1, \ldots, z_n \in \mathbf{C}$, *and all* $n \geq 1$. *The constant* $\sqrt{2}$ *is the best possible.*

Proof. It is no restriction to assume that the given numbers $z_1, \ldots, z_n$ belong to $\mathbf{R}_+$ for $n \geq 1$, as well as that $\sum_{k=1}^{n}|z_k|^2 = 1$. Identifying $C^2/C^2-1 = 2$ in (3.6), we obtain $C = \sqrt{2}$. Thus (3.8) is satisfied, and the proof of the first part is complete.

For the last statement take $z_1 = \ldots = z_n = 1/\sqrt{n}$ in (3.8), then by (3.7) we easily find:

$$\lim_{n\to\infty} \Big\| \frac{1}{\sqrt{n}}\sum_{k=1}^{n} e^{i\varphi_k} \Big\|_{\psi} = \sqrt{2}\ .$$

This fact completes the proof. $\square$

Corollary 3.4
Let $\{e^{i\varphi_k}\}_{k\geq 1}$ *be a complex Steinhaus sequence. Then the inequality is satisfied:*

$$\Big\| \sum_{k=1}^{n} z_k e^{i\varphi_k} \Big\|_{\psi} \leq \sqrt{2}\, n^{(\frac{1}{2}-\frac{1}{\alpha})^+} \Big(\sum_{k=1}^{n} |z_k|^{\alpha} \Big)^{1/\alpha} \tag{3.9}$$

for all $z_1, \dots, z_n \in \mathbf{C}$ *, all* $n \geq 1$ *, and all* $\alpha > 0$ *.*

Proof. It follows from (3.8) by Jensen's inequality for $\alpha \geq 2$, and the fact that $x \mapsto x^{\alpha/2}$ is subadditive on $\mathbf{R}_+$ for $0 < \alpha < 2$. □

Theorem 3.5
Let $\{e^{i\varphi_k}\}_{k\geq 1}$ *be a complex Steinhaus sequence. Then the inequality is satisfied:*

$$\Big\| \frac{1}{\big(\sum_{k=1}^{n} |z_k|^2\big)^{1/2}} \sum_{k=1}^{n} z_k e^{i\varphi_k} \Big\|_{T_\psi} \leq \frac{1+\sqrt{5}}{2} \tag{3.10}$$

for all $z_1, \dots, z_n \in \mathbf{C}$ *, and all* $n \geq 1$ *. The constant* $(1+\sqrt{5})/2$ *is the best possible.*

Proof. It is no restriction to assume that the given numbers $z_1, \dots, z_n$ belong to $\mathbf{R}_+$ for $n \geq 1$, as well as that $\sum_{k=1}^{n} |z_k|^2 = 1$. Identifying $C^2/C^2 - 1 = C$ in (3.6), we obtain $C = (1+\sqrt{5})/2$. Thus (3.10) is satisfied, and the first part of the proof is complete.

For the last statement take $z_1 = \ldots = z_n = 1/\sqrt{n}$ in (3.10), then by (3.7) we easily find:

$$\lim_{n\to\infty} \Big\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{i\varphi_k} \Big\|_{T_\psi} = \frac{1+\sqrt{5}}{2} \ .$$

This fact completes the proof. □

The preceding results may be additionally strengthened by the next two facts which are also of interest in themselves (recall (2.1)-(2.3)).

Theorem 3.6
Let $\{e^{i\varphi_k}\}_{k\geq 1}$ *be a complex Steinhaus sequence. Then the function:*

$$(|z_1|, \dots, |z_n|) \longmapsto E\Big| \sum_{k=1}^{n} \sqrt{|z_k|}\, e^{i\varphi_k} \Big|^{2p}$$

is Schur-concave on $\mathbf{R}_+^n$ *, for all* $n \geq 1$ *, and all* $p \geq 1$ *.*

Proof. Let the function be denoted by Φ . It is no restriction to assume that Φ is defined on the set D of all $(|z_1|, \dots, |z_n|) \in \mathbf{R}_+^n$ such that $\sum_{k=1}^{n} |z_k| = 1$. Put:

$$\Psi(p_1, \dots, p_n) = \frac{p!}{p_1! \cdot \ldots \cdot p_n!}$$

for all $p_1, \dots, p_n \in \mathbf{Z}_+$ with $\sum_{i=1}^{n} p_i = p$. Then by (3.3) and (3.4) we find:

$$\Phi(|z_1|, \dots, |z_n|) = E_{|z|} \Psi(X)$$

where $X = (X_1, \dots, X_n)$ is a random vector from the multinomial distribution with parameters $|z| = (|z_1|, \dots, |z_n|)$ and $p \geq 1$, where $\sum_{k=1}^n |z_k| = 1$. Thus by Rinott's result (2.6), the proof will be completed as soon as we show that Ψ is Schur-concave. This is evidently true if and only if $\log \Psi$ is Schur-concave. Notice that:

$$\log \Psi(p_1, \dots, p_n) = \log p! - \sum_{k=1}^n \log \Gamma(p_k + 1)$$

for all $p_1, \dots, p_n \in \mathbf{Z}_+$ with $\sum_{i=1}^n p_i = p$. Thus the proof will follow as soon as we show that:

$$(p_1, \dots, p_n) \mapsto \sum_{k=1}^n \log \Gamma(p_k + 1)$$

is Schur-convex on $\mathbf{R}_+^n$. However, this follows by Schur's proposition 2.1, since $x \mapsto \log \Gamma(x)$ is known to be convex on $]0, \infty[$. The proof is complete. □

Corollary 3.7
Let $\{e^{i\varphi_k}\}_{k \geq 1}$ *be a complex Steinhaus sequence. Then the function:*

$$(|z_1|, \dots, |z_n|) \longmapsto E \exp \Big| \sum_{k=1}^n \sqrt{|z_k|}\, e^{i\varphi_k} \Big|^2$$

is Schur-concave on $\mathbf{R}_+^n$ *, for all* $n \geq 1$ *.*

Proof. It follows from Theorem 3.6 by Taylor expansion. □

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