Ann. Probab. Vol. 39, No. 6, 2011, (2385-2423)
Research Report No. 28, 2009, *Probab. Statist. Group Manchester* (33 pp)

Predicting the Ultimate Supremum of a Stable Lévy Process with No Negative Jumps

Violetta Bernyk, Robert C. Dalang & Goran Peskir

Given a stable Lévy process $X = (X_t)_{0 \le t \le T}$ of index $\alpha \in (1,2)$ with no negative jumps, and letting $S_t = \sup_{0 \le s \le t} X_s$ denote its running supremum for $t \in [0,T]$, we consider the optimal prediction problem

$$V = \inf_{0 \le \tau \le T} \mathsf{E}(S_T - X_\tau)^p$$

where the infimum is taken over all stopping times τ of X, and the error parameter $p \in (1,\alpha)$ is given and fixed. Reducing the optimal prediction problem to a fractional free-boundary problem of Riemann-Liouville type, and finding an explicit solution to the latter, we show that there exists $\alpha_* \in (1,2)$ (equal to 1.57 approximately) and a strictly increasing function $p_* : (\alpha_*, 2) \to (1,2)$ satisfying $p_*(\alpha_*+) = 1$, $p_*(2-) = 2$ and $p_*(\alpha) < \alpha$ for $\alpha \in (\alpha_*, 2)$ such that for every $\alpha \in (\alpha_*, 2)$ and $p \in (1, p_*(\alpha))$ the following stopping time is optimal

$$\tau_* = \inf\{\, t \in [0,T] : S_t - X_t \ge z_* (T-t)^{1/\alpha} \,\}$$

where $z_* \in (0,\infty)$ is the unique root to a transcendental equation (with parameters α and p). Moreover, if either $\alpha \in (1,\alpha_*)$ or $p \in (p_*(\alpha), \alpha)$ then it is not optimal to stop at $t \in [0,T)$ when $S_t - X_t$ is sufficiently large. The existence of the breakdown points α_* and $p_*(\alpha)$ stands in sharp contrast with the Brownian motion case (formally corresponding to $\alpha = 2$), and the phenomenon itself may be attributed to the interplay between the jump structure (admitting a transition from lighter to heavier tails) and the individual preferences (represented by the error parameter p).

1. Introduction

Stopping a stochastic process $X = (X_t)_{0 \le t \le T}$ as close as possible to its ultimate supremum $S_T = \sup_{0 \le s \le T} X_s$ is an objective of both practical and theoretical interest. Speaking in general terms, the optimal prediction problem can be formulated as follows

$$V = \inf_{0 \le \tau \le T} d(X_\tau, S_T) \tag{1.1}$$

where the infimum is taken over all stopping times τ of X, and d is a distance/error function (for example $d(X_\tau, S_T) = \mathsf{E}(S_T - X_\tau)^p$ where $p > 0$ is a parameter quantifying the error).

Mathematics Subject Classification 2000. Primary 60G40, 60J75, 45J05. Secondary 60G25, 47G20, 26A33.
Key words and phrases: Optimal prediction, optimal stopping, ultimate supremum, stable Lévy process with no negative jumps, spectrally positive, fractional free-boundary problem, Riemann-Liouville fractional derivative, Caputo fractional derivative, stochastic process reflected at its supremum, infinitesimal generator, weakly singular Volterra integral equation, polar kernel, smooth fit, curved boundary.

Variants of these problems have been studied in the past mostly in discrete time (see e.g. [14], [9], [4], [12]), and the case of continuous time has been studied in the recent papers [11] and [17] when X is a standard Brownian motion. This study was extended in [6] to the case of Brownian motion with drift. It was observed there that the existence of a non-zero drift leads to optimal stopping boundaries having a complex structure which in some cases appears to be counter-intuitive. For other optimal prediction problems studied to date we refer to [22], [24], [7], [23], [8] (see also [18, Chap. VIII]). In these problems it is assumed that the underlying process has continuous sample paths.

The purpose of the present paper is to initiate a study of the optimal prediction problems for processes with jumps in continuous time, and to examine the extent to which the jump structure influences the resulting optimal stopping boundaries. To stay close to the more familiar case of Brownian motion we study the case when X is a stable Lévy process of index $\alpha \in (1,2)$, and to focus on one particular aspect of the jump structure we consider the case when X jumps upwards only (i.e. when X has no negative jumps). It turns out that already these hypotheses lead to a complicated optimal prediction problem, which apart from initial similarities with the case of Brownian motion (through the scaling property and deterministic time-change arguments) requires novel arguments to be developed in order to find a solution. These complications are primarily attributed to the underlying jump structure which leads to the relatively unexplored avenue of integro-differential equations (fractional calculus) instead of more familiar differential equations. Yet another difficulty (that the law of S_T was not available in the literature prior to the present study) is now overcome by the accompanying paper [2], and the knowledge of this law plays a key role in our treatment of the optimal prediction problem below.

Our main findings (Theorem 11) can be summarised as follows. Given a stable Lévy process $X = (X_t)_{0 \le t \le T}$ of index $\alpha \in (1,2)$ with no negative jumps, and letting $S_t = \sup_{0 \le s \le t} X_s$ denote its running supremum for $t \in [0,T]$, we consider the optimal prediction problem

$$V = \inf_{0 \le \tau \le T} \mathsf{E}(S_T - X_\tau)^p \tag{1.2}$$

where the infimum is taken over all stopping times τ of X, and the error parameter $p \in (1,\alpha)$ is given and fixed (we will see in Section 2 below why the restriction to this interval is natural). Reducing the optimal prediction problem to a fractional free-boundary problem of Riemann-Liouville type, and finding an explicit solution to the latter, we show that there exists $\alpha_* \in (1,2)$ (equal to 1.57 approximately) and a strictly increasing function $p_* : (\alpha_*, 2) \to (1,2)$ satisfying $p_*(\alpha_*+) = 1$, $p_*(2-) = 2$ and $p_*(\alpha) < \alpha$ for $\alpha \in (\alpha_*, 2)$ such that for every $\alpha \in (\alpha_*, 2)$ and $p \in (1, p_*(\alpha))$ the following stopping time is optimal

$$\tau_* = \inf \{ \, t \in [0,T] : S_t - X_t \ge z_* (T-t)^{1/\alpha} \, \} \tag{1.3}$$

where $z_* \in (0,\infty)$ is the unique root to a transcendental equation (with parameters α and p). This extends the analogous results for a standard Brownian motion X derived in [11] and [17] when $p = 2$ and $p \in (1,2)$ respectively. Moreover, if either $\alpha \in (1, \alpha_*)$ or $p \in (p_*(\alpha), \alpha)$ then it is not optimal to stop at $t \in [0,T)$ when $S_t - X_t$ is sufficiently large. The existence of the breakdown points α_* and $p_*(\alpha)$ stands in sharp contrast with the Brownian motion case (formally corresponding to $\alpha = 2$), and the phenomenon itself may be attributed to the interplay between the jump structure (admitting a transition from lighter to heavier tails) and the individual preferences (represented by the error parameter p). In particular, recalling that the index α quantifies the heaviness of the upward tails of the process X , we see that the result may be broadly interpreted as follows: *the heavier the upward tails the larger the optimal stopping time*. While this conclusion is close to naive intuition, and the interpretation itself may also be extended to account for the individual preferences, the fact that the solution method can detect the breakdown points exactly appears to be of considerable practical and theoretical interest. Other interesting features of the problem include the remarkable probabilistic representation of the solution to the Itô/Riemann-Liouville/Caputo free-boundary problem that is novel in the case of Brownian motion as well.

2. The optimal prediction problem

1. Let $X = (X_t)_{t \ge 0}$ be a stable Lévy process of index $\alpha \in (1,2)$ whose characteristic function is given by

$$\mathsf{E} e^{i\lambda X_t} = \exp\left(t \int_0^\infty (e^{i\lambda x} - 1 - i\lambda x)\, \frac{c}{x^{1+\alpha}}\, dx \right) = e^{c\,\Gamma(-\alpha)(-i\lambda)^\alpha t} \tag{2.1}$$

for $\lambda \in I\!R$ and $t \ge 0$ with $c > 0$. Let $S = (S_t)_{t \ge 0}$ denote the supremum process of X , i.e.

$$S_t = \sup_{0 \le s \le t} X_s \tag{2.2}$$

for $t \ge 0$. Consider the optimal prediction problem

$$V = \inf_{0 \le \tau \le T} \mathsf{E}(S_T - X_\tau)^p \tag{2.3}$$

where the infimum is taken over all stopping times τ of X (i.e. stopping times with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s : 0 \le s \le t)$ generated by X for $t \ge 0$). It is assumed in (2.3) that the error parameter $p \in (1, \alpha)$ and the terminal time $T > 0$ are given and fixed (we will see below that there is no restriction in assuming that $T = 1$).

2. The following properties of X are readily deduced from (2.1) using standard means (see e.g. [3] and [15]): the law of $(X_{\sigma t})_{t \ge 0}$ is the same as the law of $(\sigma^{1/\alpha} X_t)_{t \ge 0}$ for each $\sigma > 0$ given and fixed (scaling property); X is a martingale with $\mathsf{E} X_t = 0$ for all $t \ge 0$; X jumps upwards (only) and creeps downwards (in the sense that $\mathsf{P}(X_{\rho_x} = x) = 1$ for $x < 0$ where $\rho_x = \inf\{\, t \ge 0 : X_t < x \,\}$ is the first entry time of X into $(-\infty, x)$); X has sample paths of unbounded variation; X oscillates from $-\infty$ to $+\infty$ (in the sense that $\liminf_{t \to \infty} X_t = -\infty$ and $\limsup_{t \to \infty} X_t = +\infty$ both a.s.); the starting point 0 of X is regular (for both $(-\infty, 0)$ and $(0, +\infty)$). Note also that the Lévy measure ν of X equals

$$\nu(dx) = \frac{c}{x^{1+\alpha}}\, dx \tag{2.4}$$

on the Borel σ-algebra of $(0, \infty)$. Setting e.g. $c = 1/(2\Gamma(-\alpha))$ we see from (2.1) that $X = X(\alpha)$ converges in law to a standard Brownian motion B as $\alpha \uparrow 2$. We moreover see from (2.4) that when α is closer to 2 then the (upward) jumps of X have lighter tails, and when α is closer to 1 then the (upward) jumps of X have heavier tails. Thus, in many ways, the process X resembles a standard Brownian motion B, however, the existence of (upward) jumps of X represents a notable exception. Note also that X_t is not equal in law to $-X_t$ for fixed $t>0$ unlike in the case of B.

3. The error parameter p in the problem (2.3) is assumed to belong to $(1,\alpha)$ for two reasons. Firstly, it is well known (see e.g. [21, p. 159]) that for a Lévy process $X=(X_t)_{t\geq 0}$ and a number $p>0$ given and fixed, the following three facts are equivalent: (i) $\mathsf{E} X_t^p < \infty$ for some/all $t>0$; (ii) $\mathsf{E}\sup_{0\leq s\leq t} X_s^p < \infty$ for some/all $t>0$; (iii) $\int_1^\infty x^p\,\nu(dx) < \infty$. In the case of our process X when ν is given by (2.4) above, it is easily seen that (iii) holds (and thus both expected values in (i) and (ii) are finite) if and only if $p<\alpha$. In particular, the latter condition then also implies that the value V in (2.3) is finite. Secondly, if $p=1$ then the optimal prediction problem (2.3) is trivial since $\mathsf{E} X_\tau = 0$ for every (bounded) stopping time τ of X due to the martingale property of X. Hence $p\in(1,\alpha)$ represents a natural assumption on the error parameter.

4. Note that there is no loss of generality if we assume that $T=1$ in the problem (2.3). Indeed, if we set $V=V(T)$ to indicate dependence on $T>0$ in (2.3), then by the scaling property of X we see that $V(T)=T^{p/\alpha}V(1)$ and there is a simple one-to-one correspondence between the stopping times τ in the problem $V(T)$ and the stopping times σ in the problem $V(1)$ (obtained by setting $\sigma=\tau/T$). For this reason we will often assume in the sequel that the horizon T in (2.3) equals 1.

5. *Projecting future onto present.* One of the key initial difficulties in the optimal prediction problem (2.3) is that the expression after the expectation sign contains the random variable S_T and as such depends on the (ultimate) future of the process X that is unknown at the present (stopping) time $\tau\in[0,T)$. In our first step therefore (similarly to [11] and [17]) we will project the future states of X onto the present/past states of X by conditioning with respect to $\mathcal{F}_\tau^X$ and exploiting stationary/independent increments of X. As already mentioned above, we may and do assume that $T=1$ in the sequel.

To this end note that we have

$$\begin{aligned}
\text{(2.5)}\qquad \mathsf{E}\big((S_1-X_t)^p\,|\,\mathcal{F}_t^X\big) &= \mathsf{E}\Big(\Big(\sup_{0\leq s\leq t}(X_s-X_t)\vee\sup_{t\leq s\leq 1}(X_s-X_t)\Big)^p\,|\,\mathcal{F}_t^X\Big)\\
&= \Big(\mathsf{E}(y\vee S_{1-t})^p\Big)\Big|_{y=S_t-X_t}
\end{aligned}$$

since $\sup_{t\leq s\leq 1}(X_s-X_t) \overset{\text{law}}{=} S_{1-t}$ is independent from $\mathcal{F}_t^X$ and S_t-X_t is $\mathcal{F}_t^X$-measurable. Moreover, we can write

$$\begin{aligned}
\text{(2.6)}\qquad \mathsf{E}(y\vee S_{1-t})^p &= \int_0^\infty \mathsf{P}\big((y\vee S_{1-t})^p>z\big)\,dz = y^p+\int_{y^p}^\infty \mathsf{P}\big(S_{1-t}^p>z\big)\,dz\\
&= y^p+\int_{y^p}^\infty \mathsf{P}\big((1-t)^{p/\alpha}S_1^p>z\big)\,dz\\
&= (1-t)^{p/\alpha}\Big[\Big(\frac{y}{(1-t)^{1/\alpha}}\Big)^p+\int_{\left(\frac{y}{(1-t)^{1/\alpha}}\right)^p}^\infty \mathsf{P}\big(S_1^p>w\big)\,dw\Big]\\
&=: F(t,y)
\end{aligned}$$

upon using that $S_{1-t} \overset{\text{law}}{=} (1-t)^{1/\alpha} S_1$ by the scaling property of X and substituting $w = z/(1-t)^{p/\alpha}$. Combining (2.5) and (2.6) we get

$$\mathsf{E}\big((S_1 - X_t)^p \,|\, \mathcal{F}_t^X\big) = F(t, S_t - X_t) \tag{2.7}$$

for all $t \geq 0$. Using the fact that each stopping time τ of X is the limit of a decreasing sequence of discrete stopping times τ_n of X as $n \to \infty$, it is easily verified using Hunt's lemma (see e.g. [26, p. 236]) that (2.7) extends as follows

$$\mathsf{E}\big((S_1 - X_\tau)^p \,|\, \mathcal{F}_\tau^X\big) = F(\tau, S_\tau - X_\tau) \tag{2.8}$$

for all stopping times τ of X with values in $[0,1]$. Setting

$$Y_t = S_t - X_t \tag{2.9}$$

for $t \geq 0$ it is well known (see e.g. [3]) that $Y = (Y_t)_{t \geq 0}$ is a time-homogeneous (strong) Markov process with respect to $(\mathcal{F}_t^X)_{t \geq 0}$ (obtained by reflecting X at its supremum S). Taking E on both sides in (2.8) and using the notation (2.9) we see that the optimal prediction problem (2.3) reduces to the optimal stopping problem

$$V = \inf_{0 \leq \tau \leq 1} \mathsf{E}\, F(\tau, Y_\tau) \tag{2.10}$$

where the infimum is taken over all stopping times τ of X . This optimal stopping problem is two-dimensional (see e.g. [18, Sect. 6]) since the underlying (strong) Markov process is the time-space process $((t, Y_t))_{0 \leq t \leq 1}$ and the horizon 1 is finite. We will now show (similarly to [11]) that this problem can further be reduced to a one-dimensional infinite-horizon optimal stopping problem for a (killed) Markov process $Z = (Z_s)_{s \geq 0}$. It should be noted that the time-change arguments used in [11] when X is a standard Brownian motion are not directly applicable in the present context (due to the absence of Lévy's characterisation theorem).

5. *Deterministic time change.* Motivated by the form of the function F in (2.6) we now introduce the deterministic time change

$$t(s) = 1 - e^{-\alpha s} \tag{2.11}$$

where $t(s) \in [0,1)$ is the 'old' time and $s \in [0, \infty)$ is a 'new' time. Note that $\tau = t(\sigma)$ is a stopping time with respect to $(\mathcal{F}_t^X)_{t \geq 0}$ if and only if $\sigma = t^{(-1)}(\tau)$ is a stopping time with respect to $(\mathcal{F}_{t(s)}^X)_{s \geq 0}$. Letting F_{S_1} denote the distribution function of S_1 and setting

$$G(z) = \mathsf{E}(z \vee S_1)^p = z^p + \int_{z^p}^\infty \big(1 - F_{S_1}(w^{1/p})\big)\, dw \tag{2.12}$$

for $z \geq 0$, we see from (2.6) and (2.12) that

$$F(t, S_t - X_t) = e^{-ps} G(Z_s) \tag{2.13}$$

for all $t = t(s) \in [0,1)$ and all $s \in [0, \infty)$ satisfying (2.11), where $Z = (Z_s)_{s \geq 0}$ is a new stochastic process defined by

$$Z_s = e^s \big(S_{t(s)} - X_{t(s)}\big) \tag{2.14}$$

for $s \geq 0$. It turns out that Z is a time-homogeneous (strong) Markov process. Moreover, the following proposition reveals that one can enable Z to start at arbitrary points and still preserve the (strong) Markov property. This fact will play a prominent role in the main proof below.

Proposition 1. *The stochastic process* $Z = (Z_s)_{s\geq 0}$ *defined in* (2.14) *is a time-homogenous (strong) Markov process with respect to the filtration* $(\mathcal{F}^X_{t(s)})_{s\geq 0}$. *Moreover, if we set*

$$Z^z_s = e^s \big(z \vee S_{t(s)} - X_{t(s)} \big) \tag{2.15}$$

for $s \geq 0$ *and* $z \in I\!R_+$, *then* $\mathsf{P}_z := Law\big((Z^z_s)_{s\geq 0} \,|\, \mathsf{P}\big)$ *defines a family of probability measures on the canonical space of càdlàg functions* $(D_+, \mathcal{B}(D_+))$ *under which the coordinate process* $C = (C_s)_{s\geq 0}$ *is (strong) Markov with* $\mathsf{P}_z(C_0 = z) = 1$ *for* $z \in I\!R_+$.

Proof. We have

$$\begin{aligned} Z^z_{s+h} &= e^{s+h} \Big(z \vee S_{t(s+h)} - X_{t(s+h)} \Big) \\ &= e^{s+h} \bigg(\Big[\Big(z \vee S_{t(s)} - X_{t(s)} \Big) \vee \Big(\sup_{t(s)\leq r\leq t(s+h)} (X_r - X_{t(s)}) \Big) \Big] - \big(X_{t(s+h)} - X_{t(s)} \big) \bigg) \\ &= e^{h} \bigg(\Big[Z^z_s \vee e^s \Big(\sup_{t(s)\leq r\leq t(s+h)} (X_r - X_{t(s)}) \Big) \Big] - e^s \big(X_{t(s+h)} - X_{t(s)} \big) \bigg) \end{aligned} \tag{2.16}$$

for $s \geq 0$ and $h \geq 0$ given and fixed. By stationary independent increments and the scaling property of X we see that

$$\begin{aligned} \sup_{t(s)\leq r\leq t(s+h)} (X_r - X_{t(s)}) &= \sup_{1-e^{-\alpha s}\leq r\leq 1-e^{-\alpha(s+h)}} (X_r - X_{1-e^{-\alpha s}}) \overset{\text{law}}{=} \sup_{0\leq r\leq e^{-\alpha s}(1-e^{-\alpha h})} X_r \\ &\overset{\text{law}}{=} \sup_{0\leq re^{\alpha s}\leq 1-e^{-\alpha h}} X_{(re^{\alpha s})/e^{\alpha s}} \overset{\text{law}}{=} e^{-s} \sup_{0\leq r\leq 1-e^{-\alpha h}} X_r = e^{-s} S_{t(h)} \end{aligned} \tag{2.17}$$

and likewise

$$X_{t(s+h)} - X_{t(s)} = X_{1-e^{\alpha(s+h)}} - X_{1-e^{-\alpha s}} \overset{\text{law}}{=} X_{e^{-\alpha s}(1-e^{-\alpha h})} \overset{\text{law}}{=} e^{-s} X_{1-e^{-\alpha h}} = e^{-s} X_{t(h)} \tag{2.18}$$

both being independent from $\mathcal{F}^X_{t(s)}$. Combining (2.16)-(2.18) we get

$$\mathsf{E}\Big(f\big(Z^z_{s+h}\big) \,|\, \mathcal{F}^X_{t(s)} \Big) = \mathsf{E}\Big(f\Big(e^h (w \vee S_{t(h)} - X_{t(h)}) \Big) \Big) \Big|_{w=Z^z_s} \tag{2.19}$$

for any (bounded) measurable function $f : I\!R_+ \rightarrow I\!R$ from where all the claims follow by standard means [observe that the deterministic function on the right-hand side of (2.19) does not depend on s (implying that Z is a time-homogenous Markov process) as well as that it defines a continuous and bounded function of w whenever f is so (Feller property) implying that Z is a strong Markov process]. This completes the proof. □

Note from (2.14) that Z is a transient process (satisfying $Z_s \rightarrow \infty$ as $s \rightarrow \infty$) having downwards jumps only (since X jumps upwards). The state space of Z equals $I\!R_+$.

3. The optimal stopping problem

1. From (2.10) and (2.13) we see that the optimal prediction problem (2.3) reduces to the optimal stopping problem

$$V = \inf_{0 \le \sigma < \infty} \mathsf{E}\, e^{-p\sigma} G(Z_\sigma) \tag{3.1}$$

where the infimum is taken over all stopping times σ with respect to $(\mathcal{F}^X_{t(s)})_{s \ge 0}$. This optimal stopping problem is one-dimensional and the horizon is infinite. The exponential term $(e^{-ps})_{s \ge 0}$ in (3.1) corresponds to a new (strong) Markov process $\widetilde{Z}$ which may be identified with Z killed at rate p .

2. To tackle the problem (3.1) we need to enable Z to start at any point in the state space $I\!R_+$. This can be done using the result of Proposition 1 above, and it leads to the following variational extension of (3.1):

$$V(z) = \inf_{0 \le \sigma < \infty} \mathsf{E}_z\, e^{-p\sigma} G(Z_\sigma) \tag{3.2}$$

where the infimum is taken over all stopping times σ with respect to $(\mathcal{F}^X_{t(s)})_{s \ge 0}$, and the process Z starts at z under P_z . Moreover, by the result of Proposition 1 we know that P_z can be realised by (2.15) in terms of $Z^z = (Z^z_s)_{s \ge 0}$ under P , and this fact will be useful below when analysing properties of the mapping $z \mapsto V(z)$ on $I\!R_+$.

3. Before we turn to a more detailed analysis of the problem (3.2) let us state some basic properties of G and V that will be useful throughout. Recall that $f(z) \sim g(z)$ as $z \to z_0$ means that $\lim_{z \to z_0} f(z)/g(z) = 1$ for $z_0 \in [-\infty, \infty]$.

Proposition 2. *The gain function G from* (2.12) *above and the value function V from* (3.2) *above satisfy the following properties:*

(3.3) $z \mapsto G(z)$ *is (strictly) increasing and convex on* $I\!R_+$ *with* $G(0) = \mathsf{E} S_1^p > 0$;

(3.4) $z \mapsto V(z)$ *is increasing and continuous on* $I\!R_+$;

(3.5) $z^p \le V(z) \le G(z)$ *for all* $z \in I\!R_+$;

(3.6) $G(z) \sim z^p$ and $V(z) \sim z^p$ as $z \to \infty$.

Proof. (3.3): Recalling that F_{S_1} denotes the distribution function of S_1 , and letting f_{S_1} denote the density function of S_1 , we find from the final expression in (2.12) that $G'(z) = p z^{p-1} F_{S_1}(z) > 0$ and $G''(z) = p(p-1) z^{p-2} F_{S_1}(z) + p z^{p-1} f_{S_1}(z) > 0$ for all $z > 0$ implying that $z \mapsto G(z)$ is (strictly) increasing and convex respectively. Likewise we also see from the middle expression in (2.12) that $G(0) = \mathsf{E} S_1^p > 0$ as claimed.

(3.4): Letting σ be a given and fixed stopping time, we see from (2.15) that $z \mapsto Z^z_\sigma$ is increasing so that $z \mapsto G(Z^z_\sigma)$ is increasing, and the fact that $z \mapsto V(z)$ is increasing follows directly from the definition (3.2). To show that $z \mapsto V(z)$ is continuous, take $z_1 < z_2$ in $I\!R_+$ and note by the mean value theorem and (2.15) that

$$0 \le G(Z^{z_2}_\sigma) - G(Z^{z_1}_\sigma) = G'(\xi)(Z^{z_2}_\sigma - Z^{z_1}_\sigma) = G'(\xi)\, e^{\sigma} \big(z_2 \vee S_{t(\sigma)} - z_1 \vee S_{t(\sigma)} \big) \tag{3.7}$$

$$\leq p\,\xi^{p-1} F_{S_1}(\xi)\, e^{\sigma}(z_2 - z_1)$$

where $\xi \in (Z_\sigma^{z_1}, Z_\sigma^{z_2})$. Since $0 \leq \xi \leq e^{\sigma}(z_2 \vee S_1 - I_1)$, where we set $I_1 = \inf_{0 \leq t \leq 1} X_t$, it follows from (3.7) that

$$0 \leq \mathsf{E} e^{-p\sigma} G(Z_\sigma^{z_2}) - \mathsf{E} e^{-p\sigma} G(Z_\sigma^{z_1}) \leq p\, \mathsf{E}(z_2 \vee S_1 - I_1)^{p-1}(z_2 - z_1)\,. \tag{3.8}$$

Taking the infimum over all stopping times σ it follows that

$$0 \leq V(z_2) - V(z_1) \leq K(z_2 - z_1) \tag{3.9}$$

where $K = p\, \mathsf{E}(z_2 \vee S_1 - I_1)^{p-1} < \infty$. This implies that V is continuous on $I\!R_+$ (as well as Lipschitz continuous on compact sets in $I\!R_+$).

(3.5): The second inequality is obvious so let us derive the first inequality. For this, fix any $z \in I\!R_+$ and note that $G(z) \geq z^p$ and Jensen's inequality imply that

$$\begin{aligned} V(z) &\geq \inf_{0 \leq \sigma < \infty} \mathsf{E} e^{-p\sigma} (Z_\sigma^z)^p \geq \Big(\inf_{0 \leq \sigma < \infty} \mathsf{E} e^{-\sigma} Z_\sigma^z \Big)^p = \Big(\inf_{0 \leq \sigma < \infty} \mathsf{E} \big(z \vee S_{t(\sigma)} - X_{t(\sigma)} \big) \Big)^p \\ &= \Big(\inf_{0 \leq \tau \leq 1} \mathsf{E} \big(z \vee S_\tau - X_\tau \big) \Big)^p = z^p \end{aligned} \tag{3.10}$$

upon using that there is a one-to-one correspondence between σ and τ as stated following (2.11) above. Note also that for the final equality we use the fact that $\mathsf{E} X_\tau = 0$ since X is a martingale. This establishes the first inequality in (3.5) as claimed.

(3.6): Note that (2.12) above implies that $G(z)/z^p \rightarrow 1$ as $z \rightarrow \infty$, so that $V(z)/z^p \rightarrow 1$ as $z \rightarrow \infty$ follows by (3.5). This completes the proof. □

4. *Existence of an optimal stopping time.* General theory of optimal stopping for Markov processes (see e.g. [18]) can be used to establish the existence of an optimal stopping time in the problem (3.2). For this, let $C = \{\, z \in I\!R_+ : V(z) < G(z) \,\}$ denote the (open) continuation set, let $D = \{\, z \in I\!R_+ : V(z) = G(z) \,\}$ denote the (closed) stopping set, and note that

$$\mathsf{E} \Big(\sup_{s \geq 0} e^{-ps} G(Z_s^z) \Big) < \infty \tag{3.11}$$

since $e^{-ps} G(Z_s^z) = e^{-ps} \big((Z_s^z)^p + \int_{(Z_s^z)^p}^\infty \mathsf{P}(S_1^p > w)\, dw \big) \leq (z \vee S_1 - I_1)^p + \mathsf{E} S_1^p$ for all $s \geq 0$, and the latter random variable clearly is integrable for each $z \in I\!R_+$. Moreover, by (3.3) and (3.4) we know that the gain function $z \mapsto G(z)$ is lower semicontinuous on $I\!R_+$ and the value function $z \mapsto V(z)$ is upper semicontinuous on $I\!R_+$. Hence by Corollary 2.9 and Remark 2.10 in [18, pp. 46-48] we can conclude that the first entry time of Z into D given by

$$\sigma_D = \inf \{\, s \geq 0 : Z_s \in D \,\} \tag{3.12}$$

is an optimal stopping time in (3.2). This stopping time is not necessarily finite valued (when the set in (3.12) is empty) and the value $e^{-p\,\sigma_D} G(Z_{\sigma_D}^z)$ in (3.2) can be formally assigned as $(z \vee S_1 - X_1)^p$ when $\sigma_D = \infty$ since by (2.12) and (2.15) we have

$$e^{-ps} G(Z_s^z) \rightarrow (z \vee S_1 - X_1)^p \tag{3.13}$$

as $s \to \infty$. This is in agreement with the usual hypothesis from general theory introduced to cover the case of infinite-valued stopping times.

5. In addition to these general facts, it may be noted that the optimal stopping problem (3.2) plays an auxiliary role in tackling the optimal prediction problem (2.3), and it is clear from our considerations above that we only need to compute $V(z)$ for $z = 0$. Thus if we set $z_* = \inf D$ then either $z_* < \infty$ when $D \neq \emptyset$ (so that $z_* \in D$ since D is closed) or $z_* = \infty$ when $D = \emptyset$. In the first case (when $D \neq \emptyset$) the first entry time of Z to z_* given by

$$\sigma_{z_*} = \inf \{ \, s \geq 0 : Z_s = z_* \, \} \tag{3.14}$$

is optimal in (3.2) under P_z for $z = 0$. It should be recalled here that Z jumps downwards only and creeps upwards in $I\!R_+$ so that Z will hit any point in $(0, \infty)$ with probability one due to its transience to $+\infty$. Recalling further the time change (2.11) we see that (3.14) translates into the fact that the stopping time

$$\tau_* = \inf \{ \, t \in [0,1] : S_t - X_t \geq z_* (1-t)^{1/\alpha} \, \} \tag{3.15}$$

is optimal in (2.3) with $T = 1$. In the second case (when $D = \emptyset$) we see that the optimal stopping time σ_{z_*} in (3.2) equals $+\infty$ under P_z for $z = 0$. In this case we have

$$V(z) = \mathsf{E}(z \vee S_1 - X_1)^p \tag{3.16}$$

for all $z \in I\!R_+$ and the time change (2.11) implies that $\tau_* \equiv 1$ is optimal in (2.3) with

$$V = \mathsf{E}(S_1 - X_1)^p. \tag{3.17}$$

A central question therefore becomes to examine when $[0, z_*) \subseteq C$ with $z_* \in D$ (it will be shown in Section 5 below that z_* cannot be zero). We will tackle this question by forming a free-boundary problem on $[0, z_*)$ for V defined in (3.2). For this we first need to determine the infinitesimal characteristics of Z .

4. The free-boundary problem

1. The following proposition determines the action of the infinitesimal generator of the process Z defined in (2.14) in terms of the action of the infinitesimal generator of the reflected process $Y = S - X$. Below we let $C_b^2(I\!R_+)$ denote the class of twice continuously differentiable functions $F : I\!R_+ \to I\!R$ such that F' and F'' are bounded on $I\!R_+$.

Proposition 3. *The infinitesimal generator* $I\!L_Z$ *of the process* Z *is given by*

$$I\!L_Z F(z) = z F'(z) + \alpha I\!L_Y F(z) \tag{4.1}$$

for any $F \in C_b^2(I\!R_+)$ *satisfying* (4.6) *below, where* $I\!L_Y$ *denotes the infinitesimal generator of the process* Y *.*

Proof. By the mean value theorem we have

$$I\!L_Z F(z) = \lim_{s \downarrow 0} \frac{1}{s} \, \mathsf{E}\big(F(Z_s^z) - F(z) \big) \tag{4.2}$$

$$
\begin{aligned}
&= \lim_{s\downarrow 0} \frac{1}{s}\, \mathsf{E}\Big(F\big(e^s(z\vee S_{t(s)}-X_{t(s)})\big) - F\big(z\vee S_{t(s)}-X_{t(s)}\big) \\
&\qquad\qquad + F\big(z\vee S_{t(s)}-X_{t(s)}\big)-F(z)\Big) \\
&= \lim_{s\downarrow 0} \frac{e^s-1}{s}\, \mathsf{E}\Big(F'(\xi_s)(z\vee S_{t(s)}-X_{t(s)})\Big) \\
&\quad + \lim_{s\downarrow 0} \frac{t(s)}{s}\bigg(\frac{1}{t(s)}\Big[\mathsf{E}F\big(z\vee S_{t(s)}-X_{t(s)}\big)-F(z)\Big]\bigg) \\
&= zF'(z) + \alpha \mathbb{L}_Y F(z)
\end{aligned}
$$

where for the second last limit we use that $(e^s-1)/s \to 1$ and $F'(\xi_s) \to F'(z)$ as $s\downarrow 0$ since $\xi_s \in (z\vee S_{t(s)}-X_{t(s)}, e^s(z\vee S_{t(s)}-X_{t(s)}))$, and for the last limit we use that $t(s)/s \to \alpha$ as $s\downarrow 0$ and the result of Proposition 4 below. This completes the proof. □

2. The following proposition determines the action of the infinitesimal generator of the reflected process $Y = S-X$. We refer to the Appendix for the analogous result in the case of a general (strictly) stable Lévy process X .

Proposition 4. *The infinitesimal generator* $\mathbb{L}_Y$ *of the reflected process* $Y = S-X$ *takes any of the following three forms for* $y>0$ *given and fixed:*

(4.3) *Itô's form*

$$
\mathbb{L}_Y F(y) = \int_0^y \Big(F(y-x)-F(y)+F'(y)\,x\Big)\,\frac{c}{x^{1+\alpha}}\,dx + \frac{c\,(F(0)-F(y))}{\alpha\, y^\alpha} + \frac{c\,F'(y)}{(\alpha-1)\,y^{\alpha-1}}
$$

(4.4) *Riemann-Liouville's form*

$$
\mathbb{L}_Y F(y) = \frac{c}{\alpha(\alpha-1)}\,\frac{d^2}{dy^2}\int_0^y \frac{F(x)}{(y-x)^{\alpha-1}}\,dx + \frac{cF(0)}{\alpha\, y^\alpha}
$$

(4.5) *Caputo's form*

$$
\mathbb{L}_Y F(y) = \frac{c}{\alpha(\alpha-1)}\int_0^y \frac{F''(x)}{(y-x)^{\alpha-1}}\,dx
$$

whenever $F\in C_b^2(\mathbb{R}_+)$ *satisfies*

$$
F'(0+) = 0 \quad \textit{(normal reflection).} \tag{4.6}
$$

Proof. It is enough to establish (4.3) since (4.4) and (4.5) can then be derived by (repeated) integration by parts using (4.6) (note that the equivalence of (4.3)-(4.5) under (4.6) remain valid for any $F\in C^1[0,\infty)\cap C^2(0,\infty)$ satisfying $|F''(x)| = O(x^{\alpha-2})$ as $x\downarrow 0$ since $\alpha-2>-1$). For this, fix $t>0$ and note that by Itô's formula we have

$$
F(Y_t) = F(Y_0) + \int_0^t F'(Y_{s-})\,dY_s + \sum_{0<s\le t}\Big(F(Y_s)-F(Y_{s-})-F'(Y_{s-})\,\Delta Y_s\Big) \tag{4.7}
$$

since $[Y,Y]^c \equiv 0$. Indeed, the latter equality follows by recalling that X is a quadratic pure jump semimartingale (i.e. $[X,X]^c = 0$) since it is a Lévy process with no Brownian component (see [19, p. 71]), the process S is a quadratic pure jump semimartingale since it is of bounded variation (see Theorem 26 in [19, p. 71]), and the sum/difference of two quadratic pure jump semimartingales is a quadratic pure jump semimartingale (this can be easily verified using Theorem 28 in [19, p. 75] for example).

Since X jumps upwards and creeps downwards, it follows that $dS_s = \Delta S_s$ in terms of a suggestive notation, and hence from (4.7) we get

$$F(Y_t) = F(Y_0) + M_t + \sum_{0<s\le t} \Big(F(Y_{s-}+\Delta Y_s) - F(Y_{s-}) + F'(Y_{s-})\,\Delta X_s \Big) \tag{4.8}$$

where $M_t = -\int_0^t F'(Y_{s-})\,dX_s$ is a local martingale for $t \ge 0$. By the BDG inequality (see e.g. [18, p. 63]) combined with the facts that F' is bounded on $I\!R_+$ and $\mathsf{E}[X,X]_t^q < \infty$ with $q = 1/2$ since $[X,X]$ is a stable process of index $\alpha/2 > q$ (with Lévy measure $c\,dx/(2x^{1+\alpha/2})$ as is easily verified directly from definition) it follows that $\mathsf{E}\sup_{0\le s\le t} |M_s| < \infty$ and hence M is a martingale. The right-hand side of this identity can be further rewritten as follows

$$\begin{aligned} F(Y_t) = F(Y_0) + M_t + \sum_{0<s\le t} \Big(&\big[F(Y_{s-}-\Delta X_s) - F(Y_{s-}) + F'(Y_{s-})\,\Delta X_s\big]\, I(\Delta X_s \le Y_{s-}) \\ &+ \big[F(0) - F(Y_{s-}) + F'(Y_{s-})\,\Delta X_s\big]\, I(\Delta X_s > Y_{s-}) \Big) \end{aligned} \tag{4.9}$$

upon using that $\Delta X_s \le Y_{s-}$ if and only if $X_s \le S_{s-}$ so that $\Delta S_s = 0$, and $\Delta X_s > Y_{s-}$ if and only if $X_s > S_{s-}$ so that $S_s = X_s$ i.e. $Y_s = 0$. Taking E_y on both sides of (4.9), where P_y denotes a probability measure under which $Y_0 = y$, and applying the compensation formula (see e.g. [20, p. 475]) we find that

$$\begin{aligned} \mathsf{E}_y F(Y_t) - F(y) = \mathsf{E}_y \bigg[\int_0^t ds \, \bigg(&\int_0^{Y_s} \big[F(Y_s - x) - F(Y_s) + F'(Y_s)\,x\big]\, \nu(dx) \\ &+ \int_{Y_s}^\infty \big[F(0) - F(Y_s) + F'(Y_s)\,x\big]\, \nu(dx) \bigg) \bigg] \end{aligned} \tag{4.10}$$

for all $y > 0$. The applicability of this formula (see e.g. [15, p. 97]) follows from the facts that $|F'(y)| \le Cy$ and $|F''(y)| \le C$ for all $y \ge 0$ with some $C > 0$ so that the mean value theorem yields the existence of $\xi_{s,x} \in (Y_s - x, Y_s)$ and $\eta_s \in (0, Y_s)$ such that

$$\begin{aligned} &\mathsf{E}_y \bigg[\int_0^t ds \, \bigg(\int_0^{Y_s} \big|F(Y_s - x) - F(Y_s) + F'(Y_s)\,x\big|\, \nu(dx) \\ &\qquad\qquad + \int_{Y_s}^\infty \big|F(0) - F(Y_s) + F'(Y_s)\,x\big|\, \nu(dx) \bigg) \bigg] \\ &\le \mathsf{E}_y \bigg[\int_0^t ds \, \bigg(\int_0^{Y_s} \tfrac{1}{2}\, |F''(\xi_{s,x})|\, x^2 \frac{c}{x^{1+\alpha}}\, dx \\ &\qquad\qquad + \int_{Y_s}^\infty \Big(|F'(\eta_s)|\, Y_s + |F'(Y_s)|\, x \Big) \frac{c}{x^{1+\alpha}}\, dx \bigg) \bigg] \\ &\le c\, \mathsf{E}_y \bigg[\int_0^t ds \, \Big(\tfrac{C}{2(2-\alpha)} + \tfrac{C}{\alpha} + \tfrac{C}{\alpha-1} \Big) Y_s^{2-\alpha} \bigg] \end{aligned} \tag{4.11}$$

$$\leq c\Big(\tfrac{C}{2(2-\alpha)}+\tfrac{C}{\alpha}+\tfrac{C}{\alpha-1}\Big)\tfrac{\alpha}{2}\, t^{2/\alpha}\, \mathsf{E}_y(S_1-I_1)^{2-\alpha}<\infty$$

since $2-\alpha\in(0,\alpha)$ and where we also use the scaling property of X . Dividing both sides of (4.10) by t , letting $t\downarrow 0$ and using the dominated convergence theorem, we get

$$\mathbb{L}_Y F(y)=\int_0^y \big[F(y-x)-F(y)+F'(y)\,x\big]\,\nu(dx) \\ +\big[F(0)-F(y)\big]\int_y^\infty \nu(dx)+F'(y)\int_y^\infty x\,\nu(dx) \tag{4.12}$$

which is easily verified to be equal to the right-hand side of (4.3) for all $y>0$ upon using (2.4). This completes the proof. $\square$

3. It will be shown in Section 5 below that the continuation set C in the optimal stopping problem (3.2) always contains the interval $[0,\varepsilon)$ for some $\varepsilon>0$ sufficiently small, so that the optimal stopping point z_* from (3.14) is always strictly larger than zero. Moreover, we now show that the value function V from (3.2) is smooth from the left at z_* whenever $D\neq\emptyset$.

Proposition 5 (Smooth fit). *If the optimal stopping point z_* from* (3.14) *is finite, then the value function V from* (3.2) *is differentiable from the left at z_* and we have*

$$V'_-(z_*)=G'(z_*)\,. \tag{4.13}$$

Proof. To simplify the notation let us write b in place of z_* . Then $[0,b)\subseteq C$ and $b\in D$ so that $V(b)=G(b)$. Hence $(V(b-\varepsilon)-V(b))/(-\varepsilon)\geq(G(b-\varepsilon)-G(b))/(-\varepsilon)$ for all $\varepsilon>0$ sufficiently small, and letting $\varepsilon\downarrow 0$ we obtain

$$\liminf_{\varepsilon\downarrow 0}\frac{V(b-\varepsilon)-V(b)}{-\varepsilon}\geq G'(b)\,. \tag{4.14}$$

To derive a reverse inequality, note that the stopping time

$$\sigma_\varepsilon=\inf\{\,s\geq 0: Z_s^{b-\varepsilon}\geq b\,\} \tag{4.15}$$

is optimal for $V(b-\varepsilon)$ under P (recall that Z creeps upwards). Hence by the mean value theorem we find that

$$V(b-\varepsilon)-V(b)\geq \mathsf{E}\big(e^{-p\sigma_\varepsilon}G(Z^{b-\varepsilon}_{\sigma_\varepsilon})\big)-\mathsf{E}\big(e^{-p\sigma_\varepsilon}G(Z^{b}_{\sigma_\varepsilon})\big)=\mathsf{E}\big(e^{-p\sigma_\varepsilon}G'(\xi_\varepsilon)(Z^{b-\varepsilon}_{\sigma_\varepsilon}-Z^{b}_{\sigma_\varepsilon})\big) \\ =\mathsf{E}\Big(e^{-p\sigma_\varepsilon}G'(\xi_\varepsilon)\Big(e^{\sigma_\varepsilon}\big((b-\varepsilon)\vee S_{t(\sigma_\varepsilon)}-b\vee S_{t(\sigma_\varepsilon)}\big)\Big)\Big) \\ \geq -\varepsilon\,\mathsf{E}\Big(e^{-p\sigma_\varepsilon}G'(\xi_\varepsilon)\,e^{\sigma_\varepsilon}I(S_{t(\sigma_\varepsilon)}<b)\Big) \tag{4.16}$$

where $\xi_\varepsilon\in(Z^{b-\varepsilon}_{\sigma_\varepsilon},Z^{b}_{\sigma_\varepsilon})$ for $\varepsilon\in(0,b)$.

We claim that $\sigma_\varepsilon\to 0$ P-a.s. as $\varepsilon\downarrow 0$. Indeed, setting

$$\rho_\varepsilon=\inf\{\,s\geq 0:(b-\varepsilon)\vee S_{t(s)}-X_{t(s)}\geq b\,\} \tag{4.17}$$

$$\tau_\varepsilon = \inf\{\, t \ge 0 : (b-\varepsilon) \vee S_t - X_t \ge b \,\} \tag{4.18}$$

we see that $\sigma_\varepsilon \le \rho_\varepsilon$ and $\rho_\varepsilon = t^{-1}(\tau_\varepsilon)$ for all $\varepsilon > 0$. Since $t^{-1}(0+) = 0$ it is therefore sufficient to show that $\tau_\varepsilon \to 0$ P-a.s. as $\varepsilon \downarrow 0$. For this, note that

$$\tau_\varepsilon \le \inf\{\, t \ge 0 : (b-\varepsilon) - X_t \ge b \,\} = \inf\{\, t \ge 0 : X_t \le -\varepsilon \,\} =: \gamma_\varepsilon \tag{4.19}$$

and $\gamma_\varepsilon \downarrow 0$ P-a.s. as $\varepsilon \downarrow 0$ since the starting point 0 of X is regular for $(-\infty, 0)$. Hence $\sigma_\varepsilon \to 0$ P-a.s. for $\varepsilon \downarrow 0$ as claimed.

Dividing both sides of (4.16) by $-\varepsilon$, letting $\varepsilon \downarrow 0$, and using the dominated convergence theorem (upon noting that $\xi_\varepsilon \le b + (Z^b_{\sigma_\varepsilon} - Z^{b-\varepsilon}_{\sigma_\varepsilon}) \le b + \varepsilon e^{\sigma_\varepsilon} \le (b+\varepsilon)\, e^{\sigma_\varepsilon}$ and recalling that $G'(z) = p\, z^{p-1} F_{S_1}(z) \le 2 z^{p-1}$ for all $z \ge 0$ so that $0 \le e^{-p\sigma_\varepsilon} G'(\xi_\varepsilon)\, e^{\sigma_\varepsilon} I(S_{t(\sigma_\varepsilon)} < b) \le 2 e^{(-p+1)\sigma_\varepsilon}$ $(b+\varepsilon)^{p-1} e^{(p-1)\sigma_\varepsilon} = 2(b+\varepsilon)^{p-1} \le 2(b+1)^{p-1}$ as $\varepsilon \downarrow 0$), we get

$$\limsup_{\varepsilon \downarrow 0} \frac{V(b-\varepsilon) - V(b)}{-\varepsilon} \le G'(b)\,. \tag{4.20}$$

Combining (4.14) and (4.20) we see that V is differentiable from the left at b and that (4.13) holds as claimed. This completes the proof. □

4. Returning to the case when $[0, z_*) \subseteq C$ with $z_* \in D$, recalling the general fact on the killed Dirichlet problem (which suggests that $z \mapsto V(z) = \mathsf{E}_z\, e^{-p\sigma_{z_*}} G(Z_{\sigma_{z_*}})$ should solve $\mathbb{L}_Z V = pV$ in $[0, z_*)$ due to the strong Markov property of Z) see e.g. [18, pp. 130-132], and making use of the facts from Propositions 3-5, we can formulate the following free-boundary problem for the value function V defined in (3.2) above:

$$zV'(z) + \alpha \mathbb{L}_Y V(z) - pV(z) = 0 \quad \text{for } z \in [0, z_*) \tag{4.21}$$

$$V(z_*) = G(z_*) \quad \text{(instantaneous stopping)} \tag{4.22}$$

$$V'(z_*) = G'(z_*) \quad \text{(smooth fit)} \tag{4.23}$$

$$V'(0) = 0 \quad \text{(normal reflection)} \tag{4.24}$$

where $z_* \in (0, \infty)$ is the (unknown) boundary point to be found along with V on $[0, z_*)$. Whilst the infinitesimal generator $\mathbb{L}_Y$ in (4.21) can take any of the three forms (4.3)-(4.5) from Proposition 4, it turns out that the Caputo form (4.5) is most convenient for the analysis of the problem (4.21)-(4.24) to be performed.

For this reason let us rewrite the equation (4.21) in the Caputo form as

$$zF'(z) + \frac{c}{\alpha-1} \int_0^z \frac{F''(x)}{(z-x)^{\alpha-1}}\, dx - pF(z) = 0 \tag{4.25}$$

for $z \in (0, b]$ and $F : [0, b] \to \mathbb{R}$ with $b \in (0, \infty)$ given and fixed. The proof of Proposition 6 below shows that the natural solution space for this equation is one dimensional (once $F'(0)$ is set to 0). More precisely, let S_b denote the class of functions $F : [0, b] \to \mathbb{R}$ satisfying the following three conditions:

$$F \in C^1[0, b] \cap C^2(0, b] \tag{4.26}$$

$$(4.27) \qquad |F''(z)| = O(z^{\alpha-2}) \quad \text{as} \;\; z \downarrow 0$$

$$(4.28) \qquad F'(0) = 0\, .$$

Note that F'' is assumed to exist (and be continuous) on $(0,b]$ but may be unbounded (locally at zero). Note also that (4.26)-(4.28) imply that $|F'(z)| = O(z^{\alpha-1})$ as $z \downarrow 0$. For further reference let us also recall the following well-known identity (see e.g. (3.191) in [10, p. 333] and (6.2.2) in [1, p. 258]):

$$(4.29) \qquad \int_0^z x^{\mu-1}(z-x)^{\nu-1}\, dx = z^{\mu+\nu-1}\, \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

for $\mu > 0$ and $\nu > 0$.

Proposition 6. *The equation* (4.25) *has a unique solution* F *in* S_b *satisfying*

$$(4.30) \qquad F(0) = a_0$$

whenever $a_0 \in I\!\!R$ *is given and fixed. Moreover, the following explicit representation is valid:*

$$(4.31) \qquad F(z) = a_0 \sum_{n=0}^{\infty} \frac{1}{(-c\Gamma(-\alpha))^n} \Big(-\frac{p}{\alpha}\Big)_n \frac{z^{\alpha n}}{\Gamma(\alpha n+1)}$$

for $z \in [0,b]$ *where* $(q)_n = q(q+1)\cdots(q+n-1)$ *for* $n \geq 1$ *and* $(q)_0 = 1$ *with* $q = -p/\alpha$.

Proof. 1. *Uniqueness.* We will establish the uniqueness of solution by reducing the integro-differential equation (4.25) to a Volterra integral equation of the second kind. For this, let us introduce the following substitution in the equation (4.25):

$$(4.32) \qquad \varphi(z) = \int_0^z \frac{F''(x)}{(z-x)^{\alpha-1}}\, dx$$

for $z > 0$ upon extending F from $[0,b]$ to a bounded C^2 function on $(0,\infty)$ with bounded support in $I\!\!R_+$. Let $I\!\!L[f](\lambda) = \int_0^\infty e^{-\lambda x} f(x)\, dx$ denote the Laplace transform of a function $f : I\!\!R_+ \rightarrow I\!\!R$ with $\lambda > 0$, and let $I\!\!L^{-1}$ denote the inverse Laplace transform. By (4.27) and (4.29) we see that $I\!\!L[\varphi](\lambda)$ is well defined and finite for all $\lambda > 0$. Applying first $I\!\!L$ and then $I\!\!L^{-1}$ on both sides of (4.32) using the well-known properties (i) $I\!\!L[\int_0^x f_1(y) f_2(x-y)\, dy](\lambda) = I\!\!L[f_1](\lambda) I\!\!L[f_2](\lambda)$, (ii) $I\!\!L[f''](\lambda) = \lambda^2 I\!\!L[f](\lambda) - \lambda f(0) - f'(0)$ and (iii) $I\!\!L[x^\rho](\lambda) = \Gamma(\rho+1)/\lambda^{\rho+1}$ for $\rho > -1$, one finds using (4.28) that

$$(4.33) \qquad F(z) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^z (z-x)^{\alpha-1} \varphi(x)\, dx + F(0)$$

for $z \in (0,b]$. Inserting this expression back into (4.25) we obtain

$$(4.34) \qquad \int_0^z K(z,x)\, \varphi(x)\, dx + \varphi(z) = \psi$$

where K and ψ are given by

$$(4.35) \qquad K(z,x) = \frac{\alpha-1}{c\Gamma(\alpha)\Gamma(2-\alpha)}\, \frac{(\alpha-1-p)z + px}{(z-x)^{2-\alpha}}$$

$$\psi = \frac{p(\alpha-1)}{c}\, F(0) \tag{4.36}$$

for $z \in (0,b]$ and $x \in (0,z)$. We may now recognise (4.34) as a Volterra integral equation of the second kind with a weakly singular kernel K (the kernel is said to be weakly singular since the exponent $2-\alpha$ in the singular term $(z-x)^{2-\alpha}$ belongs to the interval $(0,1)$). Moreover, since ψ defines a bounded function on $[0,b]$, it is well known (see e.g. [13, Theorem 7, p. 35]) that the equation (4.34) can have at most one solution φ (in the class of locally integrable functions), and by means of the identity (4.33) this fact translates directly into the uniqueness of solution for the equation (4.25) as claimed. This completes the first part of the proof.

2. *Existence.* Seeking a solution to (4.25) of the form

$$F(z) = \sum_{n=0}^{\infty} a_n\, z^{\beta n+\gamma} \tag{4.37}$$

and inserting it into (4.25) upon differentiating and integrating formally term by term and making use of the identity (4.29), a lengthy but straightforward calculation shows that $\beta = \alpha$, $\gamma = 0$ and the series coefficients satisfy

$$a_{n+1} = \frac{1}{c\Gamma(-\alpha)} \Big(\frac{p}{\alpha} - n\Big) \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n+1)+1)}\, a_n \tag{4.38}$$

for $n = 0, 1, \ldots$ This yields the candidate series representation (4.31). Moreover, setting $b_n = (1/(-c\Gamma(-\alpha))^n)\,(-p/\alpha)_n\,(z^{\alpha n}/\Gamma(\alpha n+1))$ for $n \geq 1$ and using the well-known fact that $\Gamma(\alpha n+1)/\Gamma(\alpha(n+1)+1) \sim (\alpha n)^{-\alpha}$ as $n \to \infty$ (see (6.1.47) in [1, p. 257]), it is easily verified that $b_{n+1}/b_n \to 0$ as $n \to \infty$. Hence by the ratio test we can conclude that the series in (4.31) converges absolutely for every $z \in [0,b]$. A direct verification also shows that the function F defined by the series in (4.31) belongs to S_b. These facts justify the formal steps leading to (4.38) above, and the proof is complete. □

5. Before we continue our analysis of the free-boundary problem (4.21)-(4.24), let us make precise the following consequence of Itô's formula and the optional sampling theorem. Note that G satisfies both (4.39) and (4.40) below since $|G''(z)| = O(z^{p+\alpha-3})$ as $z \downarrow 0$ and $|G''(z)| = O(z^{p-2})$ as $z \uparrow \infty$. This is easily seen upon recalling the expression for G'' from the proof of (3.3) above and using the asymptotic relations (5.14), (5.15) and (5.17) below. Recall also that F from Proposition 6 satisfies (4.39) below.

Proposition 7. *Let* $F : I\!\!R_+ \to I\!\!R$ *be a function from* $C^1[0,\infty) \cap C^2(0,\infty)$ *satisfying*

$$|F''(z)| = O(z^{\alpha-2}) \;\; \textit{as}\;\; z \downarrow 0 \;\;\text{and}\;\; F'(0) = 0\,. \tag{4.39}$$

Let σ *be a stopping time of* Z *such that either*

$$|F''(z)| = O(z^{\beta}) \;\; \textit{as}\;\; z \uparrow \infty \;\; \textit{for some}\;\; \beta < \alpha-2 \tag{4.40}$$

and $\sigma \leq k$ *for some* $k \geq 1$, *or* $\sigma \leq \sigma_m$ *for some* $m \geq 1$ *where* $\sigma_m = \inf\{\, s \geq 0 : Z_s = m\,\}$. *Then the following identity holds:*

$$\mathsf{E}_z e^{-p\sigma} F(Z_\sigma) = F(z) + \mathsf{E}_z \int_0^{\sigma} e^{-ps}\, (I\!\!L_Z F - pF)(Z_s)\, ds \tag{4.41}$$

for all $z \in I\!R_+$.

Proof. Under P_z with $z \in I\!R_+$ by Itô's formula we get

$$
(4.42) \qquad e^{-ps} F(Z_s) = F(z) - p\int_0^s e^{-pr} F(Z_r)\, dr + \int_0^s e^{-pr} F'(Z_r)\, Z_r\, dr + M_s + J_s
$$

where $M_s = -\int_0^s e^{(-p+1)r} F'(Z_{r-})\, dX_{t(r)}$ is a local martingale and $J_s = \sum_{0<r\le s} e^{-pr}\big(F(Z_r) - F(Z_{r-}) + e^r F'(Z_{r-})\Delta X_{t(r)}\big)$ for $s \ge 0$ (upon noting that $dZ_r = Z_r\, dr + e^r\, dY_{t(r)}$ from (2.15) above). Note also that Z is a quadratic pure jump semimartingale (i.e. $[Z,Z]^c = 0$) for the reasons outlined following (4.7) above. Note further that similarly to (4.9) we find that

$$
\begin{aligned}
(4.43) \qquad J_s = \sum_{0<r\le s} e^{-pr}\bigg(&\big[F(e^r Y_{t(r)-} - e^r \Delta X_{t(r)}) \\
&- F(e^r Y_{t(r)-}) + e^r F'(e^r Y_{t(r)-})\, \Delta X_{t(r)}\big]\, I(\Delta X_{t(r)} \le Y_{t(r)-}) \\
&+ \big[F(0) - F(e^r Y_{t(r)-}) + e^r F'(e^r Y_{t(r)-})\, \Delta X_{t(r)}\big]\, I(\Delta X_{t(r)} > Y_{t(r)-})\bigg)
\end{aligned}
$$

upon using that $\Delta X_{t(r)} \le Y_{t(r)-}$ if and only if $X_{t(r)} \le S_{t(r)-}$ so that $\Delta S_{t(r)} = 0$, and $\Delta X_{t(r)} > Y_{t(r)-}$ if and only if $X_{t(r)} > S_{t(r)-}$ so that $S_{t(r)} = X_{t(r)}$ i.e. $Y_{t(r)} = 0$. Setting $v = t(r)$ this further reads

$$
\begin{aligned}
(4.44) \qquad J_s = \sum_{0<v\le t(s)} e^{-p\, t^{-1}(v)}\bigg(&\big[F(e^{t^{-1}(v)} Y_{v-} - e^{t^{-1}(v)} \Delta X_v) \\
&- F(e^{t^{-1}(v)} Y_{v-}) + e^{t^{-1}(v)} F'(e^{t^{-1}(v)} Y_{v-})\, \Delta X_v\big]\, I(\Delta X_v \le Y_{v-}) \\
&+ \big[F(0) - F(e^{t^{-1}(v)} Y_{v-}) + e^{t^{-1}(v)} F'(e^{t^{-1}(v)} Y_{v-})\, \Delta X_v\big]\, I(\Delta X_v > Y_{v-})\bigg).
\end{aligned}
$$

The compensator K of J is given by

$$
\begin{aligned}
(4.45) \qquad K_s = \int_0^{t(s)} e^{-p t^{-1}(v)}\, dv\, \bigg(&\int_0^{Y_v} \big[F(e^{t^{-1}(v)} Y_v - e^{t^{-1}(v)} x) \\
&- F(e^{t^{-1}(v)} Y_v) + e^{t^{-1}(v)} F'(e^{t^{-1}(v)} Y_v)\, x\big]\, \nu(dx) \\
&+ \int_{Y_v}^\infty \big[F(0) - F(e^{t^{-1}(v)} Y_v) + e^{t^{-1}(v)} F'(e^{t^{-1}(v)} Y_v)\, x\big]\, \nu(dx)\bigg).
\end{aligned}
$$

Setting $r = t^{-1}(v)$ and $y = e^r x$ we see that $dv = \alpha e^{-\alpha r} dr$ and $dx = e^{-r} dy$ so that $\nu(dx) = c\, dx/x^{1+\alpha} = (e^{(1+\alpha)r} c\, dx)/y^{1+\alpha} = (e^{\alpha r} c\, dy)/y^{1+\alpha} = e^{\alpha r}\nu(dy)$. This shows that

$$
\begin{aligned}
(4.46) \qquad K_s = \alpha \int_0^s e^{-pr}\, dr\, \bigg(&\int_0^{Z_r} \big[F(Z_r - y) - F(Z_r) + F'(Z_r)\, y\big]\, \nu(dy) \\
&+ \int_{Z_r}^\infty \big[F(0) - F(Z_r) + F'(Z_r)\, y\big]\, \nu(dy)\bigg) = \alpha \int_0^s e^{-pr}\, I\!L_Y F(Z_r)\, dr
\end{aligned}
$$

upon recalling the argument following (4.12) above to obtain the final equality (where $I\!L_Y F$ denotes the action of $I\!L_Y$ on F given by the right-hand side of (4.3)-(4.5)).

If $m \geq 1$ is given and fixed then (4.39) implies the existence of $C > 0$ such that $|F'(z)| \leq Cz^{\alpha-1}$ and $|F''(z)| \leq Cz^{\alpha-2}$ for all $z \in (0, m]$. This combined with the mean value theorem yields the existence of $\xi_{r,y} \in (Z_r - y, Z_r)$ and $\eta_r \in (0, Z_r)$ such that

$$
\begin{aligned}
(4.47)\qquad & \mathsf{E}_z\Big[\int_0^{s\wedge\sigma_m} e^{-pr}\,dr\,\Big(\int_0^{Z_r} \big|F(Z_r - y) - F(Z_r) + F'(Z_r)\,y\big|\,\nu(dy) \\
&\qquad + \int_{Z_r}^{\infty} \big|F(0) - F(Z_r) + F'(Z_r)\,y\big|\,\nu(dy)\Big)\Big] \\
&\leq \mathsf{E}_z\Big[\int_0^{s\wedge\sigma_m} e^{-pr}\,dr\,\Big(\int_0^{Z_r} \tfrac{1}{2}\,|F''(\xi_{r,y})|\,y^2\,\frac{c}{y^{1+\alpha}}\,dy \\
&\qquad + \int_{Z_r}^{\infty} \Big(|F'(\eta_r)|\,Z_r + |F'(Z_r)|\,y\Big)\,\frac{c}{y^{1+\alpha}}\,dy\Big)\Big] \\
&\leq c\,\mathsf{E}_z\Big[\int_0^{s\wedge\sigma_m} e^{-pr}\,dr\,\Big(\tfrac{C}{2}\int_0^{Z_r} (Z_r - y)^{\alpha-2}\,y^{1-\alpha}\,dy \\
&\qquad + C\,Z_r^{\alpha}\int_{Z_r}^{\infty} y^{-1-\alpha}\,dy + C\,Z_r^{\alpha-1}\int_{Z_r}^{\infty} y^{-\alpha}\,dy\Big)\Big] \\
&= c\Big(\tfrac{C}{2}\Gamma(2-\alpha)\Gamma(\alpha-1) + \tfrac{C}{\alpha} + \tfrac{C}{\alpha-1}\Big)\,\mathsf{E}_z\Big[\int_0^{s\wedge\sigma_m} e^{-pr}\,dr\Big] < \infty
\end{aligned}
$$

upon using (4.29) in the final equality. It follows that $N_{s\wedge\sigma_m} := J_{s\wedge\sigma_m} - K_{s\wedge\sigma_m}$ is a martingale under P_z for $s \geq 0$ (see e.g. [15, p. 97]). This shows that $N := J - K$ is a local martingale (with $(\sigma_m)_{m\geq1}$ as a localisation sequence of stopping times).

Let σ be a stopping time of Z such that $\sigma \leq \sigma_m$ for some $m \geq 1$. Choose a localisation sequence of stopping times $(\rho_n)_{n\geq1}$ for the local martingale M. Subtracting and adding K_s on the right-hand side of (4.42), replacing s by $\sigma \wedge \rho_n$, taking E_z on both sides and applying the optional sampling theorem, we obtain

$$
(4.48)\qquad \mathsf{E}_z e^{-p(\sigma\wedge\rho_n)} F(Z_{\sigma\wedge\rho_n}) = F(z) + \mathsf{E}_z \int_0^{\sigma\wedge\rho_n} e^{-pr}\,(\mathbb{L}_Z F - pF)(Z_r)\,dr
$$

for all $z \in \mathbb{R}_+$ and all $n \geq 1$ (upon recalling (4.46) and the action of $\mathbb{L}_Z$ in (4.1) above). Moreover, it is easily seen from (4.5) using (4.39) and (4.29) that $z \mapsto \mathbb{L}_Y F(z)$ is bounded on $[0, m]$ (and so are F and F' by continuity). Letting $n \to \infty$ in (4.48) and using the dominated convergence theorem we see that (4.41) holds as claimed in this case.

Let us now assume that (4.40) holds with $\sigma \leq k$ for some $k \geq 1$. Choose again a localisation sequence of stopping times $(\rho_n)_{n\geq1}$, however, this time for both the local martingale M and and the local martingale N. Subtracting and adding K_s on the right-hand side of (4.42), replacing s by $\sigma \wedge \rho_n$, taking E_z on both sides and applying the optional sampling theorem, we again obtain (4.48) for all $z \in \mathbb{R}_+$ and all $n \geq 1$. Moreover, it is easily seen from (4.5) using (4.39)+(4.40) and (4.29) that $|\mathbb{L}_Y F(z)| \leq C_3(1+z^{\beta+2-\alpha})$ for all $z \in \mathbb{R}_+$ with some $C_3 > 0$. Likewise it is easily verified that (4.39) and (4.40) imply that $|F(z)| \leq C_4(1+z^{\beta+2})$ and $|F'(z)| \leq C_5(1+z^{\beta+1})$ for all $z \in \mathbb{R}_+$ with some $C_4 > 0$ and $C_5 > 0$. Hence we see that there exists $C_6 > 0$ such that

$$
(4.49)\qquad |F(Z_s^z)| + \big|(\mathbb{L}_Z F - pF)(Z_s^z)\big| \leq C_6\,(1+(Z_s^z)^{\beta+2}) \leq C_6\big(1 + e^{k(\beta+2)}(z + S_1 - I_1)^{\beta+2}\big)
$$

for all $s \in [0,k]$ where the right-hand side defines an integrable random variable since $\beta+2 \in (0,\alpha)$. (Note that without loss of generality we can assume that β is close enough to $\alpha-2$ so that $\beta+2>0$.) Letting $n \to \infty$ in (4.48) and using the dominated convergence theorem (twice) we see that (4.41) holds as claimed. This completes the proof. $\square$

6. We now establish a remarkable probabilistic representation of the global solution (4.31) to the equation (4.25). For this, let us set

$$V_1(z) = \mathsf{E}(z \vee S_1 - X_1)^p \tag{4.50}$$

for all $z \in I\!\!R_+$. From (3.13) we see formally that $V_1(z) = \mathsf{E}_z e^{-p\sigma_\infty} G(Z_{\sigma_\infty})$ for all $z \in I\!\!R_+$ where $\sigma_\infty = \inf\{\, s \geq 0 : Z_s = \infty \,\}$, and this suggests that $z \mapsto V_1(z)$ should solve the equation (4.25) on $I\!\!R_+$. This can be derived rigourously as follows.

Proposition 8. *Let* F_1 *denote the global solution* (4.31) *to the equation* (4.25) *on* $I\!\!R_+$ *with* $F_1(0)=1$ *. Then the following identity holds:*

$$V_1(z) = a_1 F_1(z) \tag{4.51}$$

for all $z \in I\!\!R_+$ *where the constant* a_1 *is given explicitly by*

$$a_1 = \alpha \big(c\Gamma(-\alpha) \big)^{p/\alpha} \frac{\Gamma(p)}{\Gamma(p/\alpha)}\, . \tag{4.52}$$

Proof. 1. We first show that the identity (4.51) holds with some constant $a_1>0$. For this, fix an arbitrary $z_1>0$, set $F(z) = aF_1(z)$ for $z \in I\!\!R_+$ where $a = V_1(z_1)/F_1(z_1)$, and consider $\sigma_{z_1} = \inf\{\, s \geq 0 : Z_s = z_1 \,\}$. Then by (4.41) and (4.25) we find that

$$F(z) = \mathsf{E}_z e^{-p\sigma_{z_1}} F(Z_{\sigma_{z_1}}) = F(z_1)\, \mathsf{E}_z e^{-p\sigma_{z_1}} = V_1(z_1)\, \mathsf{E}_z e^{-p\sigma_{z_1}} \tag{4.53}$$

for all $z \in [0,z_1]$. In addition, consider $\sigma_n = \inf\{\, s \geq 0 : Z_s = n \,\}$ and set

$$V^n(z) = \mathsf{E}_z e^{-p\sigma_n} G(Z_{\sigma_n}) \tag{4.54}$$

for $n > z_1$ and $z \in [0,z_1]$. Note that (3.13) implies that $V^n(z) \to V_1(z)$ as $n \to \infty$ for all $z \in [0,z_1]$. Fixing $n > z_1$ and applying the strong Markov property of Z at σ_{z_1} we find that

$$\begin{aligned} V^n(Z_{\sigma_{z_1}}) &= \mathsf{E}_{Z_{\sigma_{z_1}}} e^{-p\sigma_n} G(Z_{\sigma_n}) = \mathsf{E}_z \big(e^{-p\sigma_n \circ \theta_{\sigma_{z_1}} - p\sigma_{z_1} + p\sigma_{z_1}} G(Z_{\sigma_n}) \circ \theta_{\sigma_{z_1}} \,|\, \mathcal{F}_{t(\sigma_{z_1})} \big) \\ &= e^{p\sigma_{z_1}}\, \mathsf{E}_z \big(e^{-p\sigma_n} G(Z_{\sigma_n}) \,|\, \mathcal{F}_{t(\sigma_{z_1})} \big) \end{aligned} \tag{4.55}$$

for all $z \in [0,z_1]$. Multiplying both sides by $e^{-p\sigma_{z_1}}$ and then taking E_z we get

$$V^n(z_1)\, \mathsf{E}_z e^{-p\sigma_{z_1}} = V^n(z) \tag{4.56}$$

for all $z \in [0,z_1]$ and $n > z_1$. Letting $n \to \infty$ we obtain

$$V_1(z_1)\, \mathsf{E}_z e^{-p\sigma_{z_1}} = V_1(z) \tag{4.57}$$

for all $z \in [0, z_1]$. Comparing (4.57) with (4.53) we see that $V_1(z) = F(z)$ for all $z \in [0, z_1]$. Since $z_1 > 0$ was arbitrary this establishes (4.51) with some constant $a_1 > 0$.

2. To derive (4.52) we may apply the Laplace transform $I\!L$ on both sides of (4.25) where $F(z) = V_1(z) = a_1 F_1(z)$ for $z \in I\!R_+$ so that $a_1 = V_1(0)$. Using the well-known properties (i)-(iii) recalled following (4.32) above and (iv) $I\!L[zF'(z)](\lambda) = -\lambda I\!L[F]'(\lambda) - I\!L[F](\lambda)$ for $\lambda > 0$, it can be verified using (4.28) that this leads to

$$I\!L[F]'(\lambda) + \Big(\frac{1+p}{\lambda} - \frac{c\Gamma(2-\alpha)}{\alpha-1}\,\lambda^{\alpha-1}\Big) I\!L[F](\lambda) = -F(0)\,\frac{c\Gamma(2-\alpha)}{\alpha-1}\,\lambda^{\alpha-2} \tag{4.58}$$

for $\lambda > 0$. Solving this equation under $I\!L[F](\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ (this condition is satisfied since $F(z) = V_1(z) \sim z^p$ as $z \rightarrow \infty$ by (4.50) above) we find that

$$I\!L[F](\lambda) = \frac{F(0)}{(c\Gamma(-\alpha))^{p/\alpha}}\,\frac{e^{c\Gamma(-\alpha)\,\lambda^\alpha}}{\lambda^{1+p}}\,\Gamma\big(1+p/\alpha, c\Gamma(-\alpha)\lambda^\alpha\big) \tag{4.59}$$

for $\lambda > 0$, where $\Gamma(a, x) = \int_x^\infty y^{a-1} e^{-y}\, dy$ denotes the incomplete gamma function for $a > 0$ and $x \geq 0$. Since $z \mapsto F(z)$ is increasing (by (4.50) above) we can use the Tauberian monotone density theorem (see e.g. [15, Theorem 5.14, p. 127]) which states that (i) $I\!L[F](\lambda) \sim \ell \lambda^{-\rho}$ as $\lambda \downarrow 0$ if and only if (ii) $F(z) \sim (\ell/\Gamma(\rho)) z^{\rho-1}$ as $z \uparrow \infty$ where $\rho > 0$ and $\ell > 0$. From (4.59) we see that (i) is satisfied with $\rho = 1+p$ and $\ell = \big(F(0)/(c\Gamma(-\alpha))^{p/\alpha}\big)\Gamma(1+p/\alpha)$ so that (ii) yields (4.52) since $F(z) = V_1(z) \sim z^p$ as $z \rightarrow \infty$. This completes the proof. $\square$

5. Predicting the ultimate supremum

1. We will begin by connecting our findings on the free-boundary problem from the previous section to the value function from (3.2).

Proposition 9. *If the optimal stopping point z_* from* (3.14) *is finite, then the value function V from* (3.2) *coincides on $[0, z_*]$ with F from* (4.31) *where a_0 is set to $V(0)$. In terms of the function V_1 from* (4.50) *this reads as follows:*

$$V(z) = a\, V_1(z) \tag{5.1}$$

for all $z \in [0, z_*]$ where $a = V(0)/a_1 \in (0, 1)$ and a_1 is given by (4.52) *above. If the optimal stopping point z_* is not finite (i.e. the optimal stopping set D is empty), then*

$$V(z) = V_1(z) \tag{5.2}$$

for all $z \in I\!R_+$.

Proof. If $z_* < \infty$ then

$$V(z) = \mathsf{E}_z e^{-p\sigma_{z_*}} G(Z_{\sigma_{z_*}}) = V(z_*)\, \mathsf{E}_z e^{-p\sigma_{z_*}} \tag{5.3}$$

for all $z \in [0, z_*]$. Moreover, if we set $F(z) = a_0 F_1(z)$ for all $z \in I\!R_+$ with $a_0 = V(z_*)/F_1(z_*)$ then by (4.41) and (4.25) we have

$$F(z) = \mathsf{E}_z e^{-p\sigma_{z_*}} F(Z_{\sigma_{z_*}}) = F(z_*)\, \mathsf{E}_z e^{-p\sigma_{z_*}} = V(z_*)\, \mathsf{E}_z e^{-p\sigma_{z_*}} \tag{5.4}$$

for all $z \in [0, z_*]$. Comparing (5.3) and (5.4) we see that $V(z) = F(z)$ for all $z \in [0, z_*]$. Hence $a_0 = V(0)$ and this establishes (5.1) upon recalling (4.51). If $z_* = \infty$ then (5.2) follows from (3.13) above. This completes the proof. $\square$

From (5.1) and (5.2) we see that the value function V is a constant multiple of the function V_1 from (4.50) up to the first contact point with G (when starting from 0 and moving towards ∞ in the state space). The unknown constant needs to be chosen so that the contact with G occurs smoothly. Since $V \le V_1$ this leads the following criterion for D to be non-empty:

$$z_* < \infty \text{ if and only if } \exists z_1 \in I\!R_+ \text{ such that } V_1(z_1) \ge G(z_1) \tag{5.5}$$

or equivalently, the following criterion for D to be empty:

$$z_* = \infty \text{ if and only if } V_1(z) < G(z) \text{ for all } z \in I\!R_+ . \tag{5.6}$$

We will continue our analysis by examining when (5.5) holds.

2. Consider the function $H : [0, \infty) \to I\!R$ defined by

$$H(z) = (I\!L_Z G - pG)(z) \tag{5.7}$$

for $z \ge 0$ where $H(0) := H(0+)$ exists by (5.11) below. Recall that (4.41) reads

$$\mathsf{E}_z e^{-p\sigma} G(Z_\sigma) = G(z) + \mathsf{E}_z \int_0^\sigma e^{-ps} H(Z_s)\, ds \tag{5.8}$$

for $z \in I\!R_+$ where σ is any stopping time of Z like in Proposition 7. Set

$$N = \{\, z \in [0, \infty) : H(z) < 0 \,\} \quad \text{and} \quad P = \{\, z \in [0, \infty) : H(z) \ge 0 \,\} . \tag{5.9}$$

Then the following two inclusions are valid:

$$N \subseteq C \quad \text{and} \quad D \subseteq P . \tag{5.10}$$

Indeed, to show the first inclusion (the second one then being obvious) take any $z \in N$ and choose $\varepsilon > 0$ small enough such that $(z{-}\varepsilon, z{+}\varepsilon) \cap I\!R_+ \subset N$ (note that N is open in $I\!R_+$). Inserting the stopping time $\sigma_\varepsilon = \inf \{ s \ge 0 : Z_s \notin (z{-}\varepsilon, z{+}\varepsilon) \}$ into (5.8) we see that $\mathsf{E}_z e^{-p\sigma_\varepsilon} G(Z_{\sigma_\varepsilon}) < G(z)$ since $H(Z_s) < 0$ for $s \in [0, \sigma_\varepsilon)$. Hence z belongs to C as claimed.

3. Motivated by the important role that the function H plays in the optimal stopping problem (3.2) we now determine its asymptotic behaviour at zero and infinity. Note that (5.11) below and (5.10) above imply (since H is continuous) that the continuation set C always contains the interval $[0, \varepsilon)$ for some $\varepsilon > 0$ sufficiently small so that the optimal stopping point z_* from (3.14) is always strictly larger than zero.

Proposition 10. *The following relations are valid:*

$$\lim_{z \downarrow 0} H(z) = -p\, G(0) = -p\, \mathsf{E} S_1^p < 0 \tag{5.11}$$

$$\lim_{z\uparrow\infty} z^{\alpha-p} H(z) = \frac{cp}{\Gamma(p-\alpha+1)} \Big(\Gamma(p-\alpha) - \Gamma(p)\,\Gamma(1-\alpha)\Big)\, . \tag{5.12}$$

Proof. Since $G'(z) = pz^{p-1}F_{S_1}(z)$ and $G''(z) = p(p-1)z^{p-2}F_{S_1}(z) + pz^{p-1}f_{S_1}(z)$ we see by (4.1) and (4.5) that

$$\begin{aligned} H(z) &= zG'(z) + \frac{c}{\alpha-1}\int_0^z \frac{G''(x)}{(z-x)^{\alpha-1}}\, dx - pG(z) \\ &= pz^p F_{S_1}(z) + \frac{c}{\alpha-1}\int_0^z \frac{p(p-1)x^{p-2}F_{S_1}(x) + px^{p-1}f_{S_1}(x)}{(z-x)^{\alpha-1}}\, dx - pG(z) \end{aligned} \tag{5.13}$$

for $z > 0$. Recall that the following asymptotic relations are valid (see [2, Corollary 3]):

$$f_{S_1}(z) \sim \frac{z^{\alpha-2}}{(c\Gamma(-\alpha))^{1-1/\alpha}\,\Gamma(\alpha-1)\,\Gamma(1/\alpha)} \quad \text{as } z\downarrow 0 \tag{5.14}$$

$$F_{S_1}(z) \sim \frac{z^{\alpha-1}}{(c\Gamma(-\alpha))^{1-1/\alpha}\,\Gamma(\alpha)\,\Gamma(1/\alpha)} \quad \text{as } z\downarrow 0\, . \tag{5.15}$$

Using (5.14) and (5.15) together with (4.29) it is readily verified that the integral in (5.13) tends to 0 as $z \downarrow 0$. This easily yields the first equality in (5.11) and the second equality follows from (3.3).

Moreover, using (2.12) above we can further rewrite (5.13) as follows:

$$\begin{aligned} H(z) = &-pz^p\big(1-F_{S_1}(z)\big) + \frac{cp(p-1)}{\alpha-1}\int_0^z \frac{x^{p-2}}{(z-x)^{\alpha-1}}\, dx - p\int_{z^p}^\infty \big(1-F_{S_1}(x^{1/p})\big)\, dx \\ &- \frac{cp(p-1)}{\alpha-1}\int_0^z \frac{x^{p-2}(1-F_{S_1}(x))}{(z-x)^{\alpha-1}}\, dx + \frac{cp}{\alpha-1}\int_0^z \frac{x^{p-1}f_{S_1}(x)}{(z-x)^{\alpha-1}}\, dx \end{aligned} \tag{5.16}$$

for $z > 0$. Recall that the following asymptotic relations are valid (cf. [2], [5], [16]):

$$f_{S_1}(z) \sim \frac{c}{z^{1+\alpha}} \quad \text{as } z\uparrow\infty \tag{5.17}$$

$$1-F_{S_1}(z) \sim \frac{c}{\alpha z^\alpha} \quad \text{as } z\uparrow\infty\, . \tag{5.18}$$

Using (5.17) and (5.18) together with (4.29) it is somewhat lengthy but still straightforward to verify that the final two integrals in (5.16) are $o(z^{p-\alpha})$ as $z\to\infty$, whilst the first three terms in (5.16) multiplied by $z^{\alpha-p}$ converge to the constant on the right-hand side of (5.12) as $z\to\infty$. This completes the proof. $\square$

4. Motivated by the identity (5.12) let us consider the function ℓ defined by

$$\ell(\alpha,p) = \frac{cp}{\Gamma(p-\alpha+1)}\Big(\Gamma(p-\alpha) - \Gamma(p)\,\Gamma(1-\alpha)\Big) \tag{5.19}$$

for $\alpha \in (1,2)$ and $p \in (1,\alpha)$. A direct examination of the right-hand side in (5.19) shows that there exist $\alpha_* \in (1,2)$ (equal to 1.57 approximately) and a strictly increasing function $p_* : (\alpha_*,2) \to (1,2)$ satisfying $p_*(\alpha_*+) = 1$, $p_*(2-) = 2$ and $p_*(\alpha) < \alpha$ for $\alpha \in (\alpha_*,2)$ such that (i) $\ell(\alpha,p) > 0$ if $\alpha \in (\alpha_*,2)$ and $p \in (1,p_*(\alpha))$; (ii) $\ell(\alpha,p) < 0$ if either $\alpha \in (1,\alpha_*)$ and $p \in (1,\alpha)$ or $\alpha \in [\alpha_*,2)$ and $p \in (p_*(\alpha),\alpha)$; and (iii) $\ell(\alpha,p_*(\alpha)) = 0$ for $\alpha \in (\alpha_*,2)$. Note that the properties (i)-(iii) do not depend on the value of the constant c in (2.4). Recall also from (5.12) above that

$$\ell(\alpha,p) = \lim_{z\uparrow\infty} z^{\alpha-p} H(z) \tag{5.20}$$

for all $\alpha \in (1,2)$ and $p \in (1,\alpha)$. In view of (5.10) this suggests that the sign of ℓ plays an important role in the problem (3.2).

Building on the facts presented in the previous sections, and extending these arguments further in the proof below, we can now present the main result of the paper. It should be recalled in the statement below that the function V_1 can be expressed probabilistically by (4.50) and analytically by (4.51)+(4.52) (where F_1 is given by (4.31) with $a_0 = 1$), and the probabilistic and analytic representations of the function G are given in (2.12) above (upon recalling that F_{S_1} admits an explicit series representation as shown in [2, Theorem 1]).

Theorem 11. (I): *If $\alpha \in (\alpha_*,2)$ and $p \in (1,p_*(\alpha))$ then there exists $z_* \in (0,\infty)$ such that the stopping time (3.14) is optimal in the problem (3.2) under P_z for $z \in [0,z_*]$. The optimal stopping point z_* can be characterised as the minimal $z \in (0,\infty)$ for which*

$$\beta_* V_1(z)\big|_{z=z_*} = G(z)\big|_{z=z_*} \tag{5.21}$$

where $\beta_* \in (0,1)$ is the minimal $\beta \in (0,1)$ for which the equation (5.21) has at least one root $z \in (0,\infty)$. The optimal z_* and β_* satisfy the smooth fit condition

$$\beta_* V_1'(z)\big|_{z=z_*} = G'(z)\big|_{z=z_*} \, . \tag{5.22}$$

The value function from (3.2) is given by $V(z) = \beta_* V_1(z) = \beta_* \, \mathsf{E}(z \vee S_1 - X_1)^p$ for $z \in [0,z_*]$.

(II): *The stopping time (1.3) is optimal in the problem (2.3) and the value from (2.3) is given by $V = T^{p/\alpha}\beta_* V_1(0) = T^{p/\alpha}\beta_* \, \mathsf{E}(S_1 - X_1)^p = T^{p/\alpha}\beta_* \, \alpha(c\Gamma(-\alpha))^{p/\alpha}\Gamma(p)/\Gamma(p/\alpha)$.*

Proof. Since Part II follows from Part I as discussed in Sections 2 and 3 above, it is enough to prove Part I. For this, we will first show that the assumptions $\alpha \in (\alpha_*,2)$ and $p \in (1,p_*(\alpha))$ imply the existence of $z_1 > 0$ (large enough) such that

$$V_1(z) > G(z) \tag{5.23}$$

for all $z \geq z_1$. We will then show how the knowledge of (5.23) combined with the properties and facts about V_1 and G derived in the previous sections yield the existence of β_* and z_* satisfying the remaining statements of Part I.

1. To prove (5.23) recall that the identity (4.41) is applicable to G in place of F with $\sigma \equiv n$ for $n \geq 1$. Letting $n \to \infty$ in this identity, using (3.13) combined with the fact that each $e^{-pn} G(Z_n^z)$ is dominated by $(z \vee S_1 - I_1)^p + \mathsf{E} S_1^p$ which clearly has finite expectation, as well as the fact that the function H is bounded (by the result of Proposition 10), it follows by the dominated convergence theorem that

$$\mathsf{E}(z \vee S_1 - X_1)^p = G(z) + \mathsf{E}\int_0^\infty e^{-ps} H(Z_s^z)\, ds \tag{5.24}$$

for all $z \geq 0$. Recognising the left-hand side of (5.24) as $V_1(z)$ we see that (5.23) will be established if we show the existence of $z_1 > 0$ (large enough) such that

$$I(z) := \mathsf{E} \int_0^\infty e^{-ps} H(Z_s^z)\, ds > 0 \tag{5.25}$$

for all $z \geq z_1$.

To show (5.25) recall from (i) following (5.19) above that $\ell := \ell(\alpha, p)$ in (5.20) is strictly positive when $\alpha \in (\alpha_*, 2)$ and $p \in (1, p_*(\alpha))$ are given and fixed. Hence for any given and fixed $\varepsilon > 0$ (small) there exists $z_\varepsilon > 0$ (large) such that

$$z^{\alpha-p} H(z) \geq \ell - \varepsilon \tag{5.26}$$

for all $z \geq z_\varepsilon$. Consider

$$J(z) := \mathsf{E} \int_0^\infty e^{-ps} H(Z_s^z)\, I(Z_s^z < z_\varepsilon)\, ds \tag{5.27}$$

$$K(z) := \mathsf{E} \int_0^\infty e^{-ps} H(Z_s^z)\, I(Z_s^z \geq z_\varepsilon)\, ds \tag{5.28}$$

and note that $I(z) = J(z) + K(z)$ for all $z \geq 0$.

Let $M > 0$ be large enough so that $|H(z)| \leq M$ for all $z \geq 0$. Then we have

$$|J(z)| \leq M \int_0^\infty e^{-ps}\, \mathsf{P}(Z_s^z < z_\varepsilon)\, ds \tag{5.29}$$

for all $z \geq 0$. Moreover, by (5.18) we see that

$$\begin{aligned} \mathsf{P}(Z_s^z < z_\varepsilon) &= \mathsf{P}\big(e^s (z \vee S_{t(s)} - X_{t(s)}) < z_\varepsilon\big) \leq \mathsf{P}\big(z \vee S_{t(s)} - S_{t(s)} < z_\varepsilon\big) \\ &\leq \mathsf{P}\big(z - S_{t(s)} < z_\varepsilon\big) \leq \mathsf{P}\big(S_1 > z - z_\varepsilon\big) \leq N \frac{c}{\alpha} (z - z_\varepsilon)^{-\alpha} \end{aligned} \tag{5.30}$$

for all $z > z_\varepsilon$ with some $N > 0$ large enough. Combining (5.29) and (5.30) we find that

$$|J(z)| \leq \frac{MNc}{p\alpha} (z - z_\varepsilon)^{-\alpha} \tag{5.31}$$

for all $z > z_\varepsilon$.

On the other hand, by (5.26) we see that

$$\begin{aligned} K(z) &= \mathsf{E} \int_0^\infty e^{-ps} H(Z_s^z)\, I(Z_s^z \geq z_\varepsilon)\, ds \geq (\ell - \varepsilon) \int_0^\infty e^{-ps}\, \mathsf{E}\big[(Z_s^z)^{p-\alpha} I(Z_s^z \geq z_\varepsilon)\big]\, ds \\ &= (\ell - \varepsilon) \int_0^\infty e^{-\alpha s}\, \mathsf{E}\big[(z \vee S_{t(s)} - X_{t(s)})^{p-\alpha} I(Z_s^z \geq z_\varepsilon)\big]\, ds \\ &= (\ell - \varepsilon)\, z^{p-\alpha} \int_0^\infty e^{-\alpha s}\, \mathsf{E}\big[\big(1 \vee \tfrac{S_{t(s)}}{z} - \tfrac{X_{t(s)}}{z}\big)^{p-\alpha} I(Z_s^z \geq z_\varepsilon)\big]\, ds \\ &\geq (\ell - \varepsilon)\, z^{p-\alpha} \int_0^\infty e^{-\alpha s}\, \mathsf{E}\big[(1 \vee S_1 - I_1)^{p-\alpha} I(Z_s^z \geq z_\varepsilon)\big]\, ds \end{aligned} \tag{5.32}$$

$$\geq \frac{(\ell-\varepsilon)}{\alpha}\,\big(\,\mathsf{E}(1\vee S_1-I_1)^{p-\alpha}-\delta\,\big)\,z^{p-\alpha}$$

for all $z \geq 1 \vee z_\delta$, where in the second last inequality we use that

$$1 \vee \tfrac{S_{t(s)}}{z} - \tfrac{X_{t(s)}}{z} \leq 1 \vee \tfrac{S_1}{z} - \tfrac{I_1}{z} \leq 1 \vee S_1 - I_1 \tag{5.33}$$

for all $s \geq 0$ and $z \geq 1$, and in the last inequality we use that

$$\lim_{z\to\infty} \int_0^\infty e^{-\alpha s}\,\mathsf{E}\big[(1\vee S_1-I_1)^{p-\alpha} I(Z_s^z \geq z_\varepsilon)\big]\,ds = \frac{1}{\alpha}\,\mathsf{E}(1\vee S_1-I_1)^{p-\alpha} < \infty \tag{5.34}$$

by the dominated convergence theorem since $Z_s^z \to \infty$ as $z \to \infty$ (from (5.34) we see that for given $\delta \in (0, \mathsf{E}(1\vee S_1-I_1)^{p-\alpha})$ there exists $z_\delta > 0$ such that the final inequality in (5.32) holds for all $z \geq z_\delta$). Since the right-hand side in (5.31) tends faster to zero than the right-hand side in (5.32) as $z \uparrow \infty$, we see that (5.23) holds with some $z_1 > 0$ large enough as claimed.

2. We now establish the existence of β_* and z_* satisfying the remaining statements of Part I. For this, recall that (5.23) holds for $z = z_1$ so that for some $\beta_1 \in (0,1)$ sufficiently close to 1 we have $\beta_1 V_1(z_1) > G(z_1)$. Since $\beta_1 V_1(z) \sim \beta_1 z^p < z^p \sim G(z)$ as $z \to \infty$ we also see that there exists $z_2 > z_1$ such that $\beta_1 V_1(z) < G(z)$ for all $z \geq z_2$. This shows that for some $\beta_0 \in (0,1)$ sufficiently close to 0 we have $\beta_0 V_1(z) < G(z)$ for all $z \geq 0$ (recall that $V_1(0) = \mathsf{E}(S_1-X_1)^p > 0$ and that V_1 is increasing). It follows therefore by continuity that there exists the smallest $\beta_* \in (\beta_0, \beta_1) \subset (0,1)$ such that the set $A = \{\, z \in I\!R_+ \mid \beta_* V_1(z) = G(z)\,\}$ is non-empty so that $\beta V_1(z) < G(z)$ for all $z \in I\!R_+$ if $\beta \in (0, \beta_*)$. Setting $w_* = \inf A$ we see that w_* belongs to A by continuity so that (5.21) holds for $z = w_*$. Moreover, since $V_1(z_1) > G(z_1)$ we know by (5.5) that $z_* = \inf D < \infty$ so that by (5.1) we have $V(z) = a_* V_1(z)$ for all $z \in [0, z_*]$ with some $a_* \in (0,1)$. By the construction of β_* and w_* it follows therefore that $\beta_* \leq a_*$ and $w_* \geq z_*$. If either $\beta_* < a_*$ or equivalently $w_* > z_*$ then since $\beta_* V_1(z) = \mathsf{E}e^{-p\,\sigma_{w_*}} \beta_* V_1(Z^z_{\sigma_{w_*}})$ for all $z \in [0, w_*]$ by the result of Proposition 7, and this further equals $\mathsf{E}e^{-p\,\sigma_{w_*}} G(Z^z_{\sigma_{w_*}})$ for all $z \in [0, w_*]$ by definition of σ_{w_*} , we see that $\beta_* V_1(0) \geq V(0)$ while at the same time $\beta_* V_1(0) < a_* V_1(0) = V(0)$ which is a contradiction. Thus $\beta_* = a_*$ and $w_* = z_*$ so that $V(z) = \beta_* V_1(z)$ for all $z \in [0, z_*]$ as claimed. The smooth fit condition (5.22) then follows by the result of Proposition 5. This completes the proof of Part I whence Part II follows as discussed above. □

5. In the final part of this section we briefly consider the case when the hypotheses of Theorem 11 are not satisfied.

Proposition 12. *If either $\alpha \in (1, \alpha_*)$ or $p \in (p_*(\alpha), \alpha)$ then there exists $z_1 > 0$ large enough such that $V_1(z) < G(z)$ for all $z \geq z_1$.*

Proof. This can be proved in exactly the same way as (5.23) above upon noting that $\ell := \ell(\alpha, p)$ in (5.20) is strictly negative when either $\alpha \in (1, \alpha_*)$ or $p \in (p_*(\alpha), \alpha)$ and replacing (5.26) with $z^{\alpha-p} H(z) \leq \ell+\varepsilon$ for all $z \geq z_\varepsilon$. This leads to (5.31) without changes and (5.32) holds with the inequalities reversed since $\ell+\varepsilon < 0$ in this case. Different rates of convergence in the resulting inequalities then complete the proof just as above. □

It follows from the result of Proposition 12 that the continuation set C contains the interval $[z_1, \infty)$ for some $z_1 > 0$ large enough when either $\alpha \in (1, \alpha_*)$ or $p \in (p_*(\alpha), \alpha)$. It shows that the stopping time (1.3) can no longer be optimal in this case (in the sense that it is not optimal to stop at $t \in [0, T)$ when $S_t - X_t$ is sufficiently large). This stands in sharp contrast with the Brownian motion case (formally corresponding to $\alpha = 2$) where it is optimal to stop in such a case. Recall also that the continuation set C always contains the interval $[0, \varepsilon)$ for some $\varepsilon > 0$ sufficiently small so that the stopping set D must be contained in $[\varepsilon, z_1 - \delta]$ for some $\delta > 0$. We do not know whether $V_1(z) < G(z)$ holds for all $z \in I\!\!R_+$ in this case, or equivalently, whether the stopping set D is empty (recall (5.6) above). This is an interesting open question. We refer to [6, Figure 1] for a related phenomenon in the presence of strictly positive drifts and the absence of jumps.

6. Appendix

In this section we determine the action of the infinitesimal generator of the reflected process $Y = S - X$ when X is a general (strictly) stable Lévy process (see [25]). Set

$$\nu_\alpha(dx) = \frac{c_+}{x^{1+\alpha}} I(x>0)\, dx + \frac{c_-}{(-x)^{1+\alpha}} I(x<0)\, dx \tag{6.1}$$

where c_+ and c_- are non-negative constants (not both zero) and $\alpha \in (0, 2)$. For $\alpha = 1$ the two constants need to be identical (see e.g. [21, pp. 86-87] so that

$$\nu_1(dx) = \frac{c}{x^2} I(x \neq 0)\, dx \tag{6.2}$$

with $c > 0$. Recall that $C_b^2(I\!\!R_+)$ denotes the class of twice continuously differentiable functions $F : I\!\!R_+ \to I\!\!R$ such that F' and F'' are bounded on $I\!\!R_+$.

Proposition A.1. *Let $X = (X_t)_{t \geq 0}$ be a stable Lévy process of index $\alpha \in (1, 2)$ whose characteristic function is given by*

$$\mathsf{E} e^{i\lambda X_t} = \exp\left(t \int_{-\infty}^{\infty} (e^{i\lambda x} - 1 - i\lambda x)\, \nu_\alpha(dx) \right) = e^{(c_+(-i\lambda)^\alpha + c_-(i\lambda)^\alpha)\, \Gamma(-\alpha)\, t} \tag{6.3}$$

for $\lambda \in I\!\!R$ and $t \geq 0$. Then the infinitesimal generator $I\!\!L_Y$ of the reflected process $Y = S – X$ takes any of the following three forms for $y > 0$ given and fixed:

(6.4) *Itô's form*

$$I\!\!L_Y F(y) = \int_0^y \Big(F(y-x) - F(y) + F'(y)\, x \Big) \frac{c_+}{x^{1+\alpha}}\, dx + \frac{c_+ (F(0) - F(y))}{\alpha\, y^\alpha} + \frac{c_+ F'(y)}{(\alpha-1)\, y^{\alpha-1}} + \int_0^\infty \Big(F(y+x) - F(y) - F'(y)\, x \Big) \frac{c_-}{x^{1+\alpha}}\, dx$$

(6.5) *Riemann-Liouville's form*

$$I\!\!L_Y F(y) = \frac{c_+}{\alpha(\alpha-1)} \frac{d^2}{dy^2} \int_0^y \frac{F(x)}{(y-x)^{\alpha-1}}\, dx + \frac{c_+ F(0)}{\alpha\, y^\alpha} + \frac{c_-}{\alpha(\alpha-1)} \frac{d^2}{dy^2} \int_y^\infty \frac{F(x)}{(x-y)^{\alpha-1}}\, dx$$

(6.6) *Caputo's form*

$$I\!L_Y F(y) = \frac{c_+}{\alpha(\alpha-1)} \int_0^y \frac{F''(x)}{(y-x)^{\alpha-1}}\, dx + \frac{c_-}{\alpha(\alpha-1)} \int_y^\infty \frac{F''(x)}{(x-y)^{\alpha-1}}\, dx$$

whenever $F \in C_b^2(I\!R_+)$ *satisfies*

$$F'(0+) = 0 \quad \textit{(normal reflection)} \tag{6.7}$$

with $|F''(y)| = O(y^\gamma)$ *as* $y \to \infty$ *for some* $\gamma < \alpha-2$ *(as well as* $|F(y)| = O(y^\delta)$ *as* $y \to \infty$ *for some* $\delta < \alpha-2$ *in* (6.5) *above).*

Proof. As in the proof of Proposition 4 it is enough to derive (6.4). For this, fix $t > 0$ and note that by Itô's formula we have

$$F(Y_t) = F(Y_0) + \int_0^t F'(Y_{s-})(dS_s - dX_s) + \sum_{0<s\le t} \Big(F(Y_s) - F(Y_{s-}) - F'(Y_{s-})\,(\Delta S_s - \Delta X_s)\Big) \tag{6.8}$$

since $[Y,Y]^c \equiv 0$ for the same reasons as in (4.7). Letting $S_s = S_s^c + S_s^d$ be the decomposition of $s \mapsto S_s$ into continuous and discontinuous parts, and noting that $dS_s^d = \Delta S_s$, we see that (6.8) simplifies to

$$F(Y_t) = F(Y_0) + M_t + \int_0^t F'(Y_{s-})\, dS_s^c + \sum_{0<s\le t} \Big(F(Y_{s-} + \Delta Y_s) - F(Y_{s-}) + F'(Y_{s-})\,\Delta X_s\Big) \tag{6.9}$$

where $M_t = -\int_0^t F'(Y_{s-})\, dX_s$. Since F' is bounded the same argument as following (4.8) above shows that M is a martingale. If s belongs to the support of dS_s^c in $[0,t]$ then either $S_{s-\varepsilon}^c < S_s^c$ and therefore $S_{s-\varepsilon} < S_s$ for $\varepsilon > 0$ implying $Y_{s-} = 0$, or $S_s^c < S_{s+\varepsilon}^c$ and therefore $S_s < S_{s+\varepsilon}$ for $\varepsilon > 0$ implying $Y_s = 0$. Since there could be at most countably many s in $[0,t]$ for which $Y_s \neq Y_{s-}$, it follows using (6.7) that the integral with respect to dS_s^c in (6.9) is zero. Moreover, the right-hand side of (6.9) can further be rewritten as follows:

$$\begin{aligned} F(Y_t) = F(Y_0) + M_t + \sum_{0<s\le t} \Big(&\big[F(Y_{s-} - \Delta X_s) - F(Y_{s-}) + F'(Y_{s-})\,\Delta X_s\big]\, I(\Delta X_s \le Y_{s-}) \\ &+ \big[F(0) - F(Y_{s-}) + F'(Y_{s-})\,\Delta X_s\big]\, I(\Delta X_s > Y_{s-})\Big) \end{aligned} \tag{6.10}$$

using the same arguments as in (4.9) above. Taking E_y on both sides of (6.10), where P_y denotes a probability measure under which $Y_0 = y$, and applying the compensation formula (see e.g. [20, p. 475]) we find that

$$\begin{aligned} \mathsf{E}_y F(Y_t) - F(y) = \mathsf{E}_y\bigg[\int_0^t ds\, \bigg(&\int_{-\infty}^{Y_s} \big[F(Y_s - x) - F(Y_s) + F'(Y_s)\,x\big]\, \nu_\alpha(dx) \\ &+ \int_{Y_s}^\infty \big[F(0) - F(Y_s) + F'(Y_s)\,x\big]\, \nu_\alpha(dx)\bigg)\bigg] \end{aligned} \tag{6.11}$$

for all $y > 0$. The applicability of this formula (see e.g. [15, p. 97]) follows from the facts that $|F'(y)| \le C$ and $|F''(y)| \le C$ for all $y \ge 0$ with some $C > 0$ so that the mean value theorem yields the existence of $\xi_{s,x} \in (Y_s, Y_s + x)$ and $\eta_{s,x} \in (Y_s, Y_s + x)$ such that

$$\mathsf{E}_y\bigg[\int_0^t ds\, \bigg(\int_{-\infty}^0 \big|F(Y_s - x) - F(Y_s) + F'(Y_s)\,x\big|\, \nu_\alpha(dx)\bigg)\bigg] \tag{6.12}$$

$$= \mathsf{E}_y\Big[\int_0^t ds\,\Big(\int_0^\infty \big|F(Y_s{+}x)-F(Y_s)-F'(Y_s)\,x\big|\,\frac{c}{x^{1+\alpha}}\,dx\Big)\Big]$$

$$\le \mathsf{E}_y\Big[\int_0^t ds\,\Big(\int_0^1 \tfrac{1}{2}|F''(\xi_{s,x})|\,x^2\,\frac{c}{x^{1+\alpha}}\,dx + \int_1^\infty \Big(|F'(\eta_{s,x})|\,x + |F'(Y_s)|\,x\Big)\,\frac{c}{x^{1+\alpha}}\,dx\Big)\Big]$$

$$\le c\,\mathsf{E}_y\Big[\int_0^t ds\,\Big(\tfrac{C}{2(2-\alpha)} + \tfrac{2C}{\alpha-1}\Big)\Big] = \tfrac{c(7-3\alpha)C}{2(2-\alpha)(\alpha-1)}\,t < \infty$$

where the remaining two integrals (from 0 to Y_s and from Y_s to ∞) can be controlled (bound from above) in exactly the same way as in (4.11) above. Dividing both sides of (6.11) by t, letting $t\downarrow 0$ and using the dominated convergence theorem, we get

$$\mathbb{L}_Y F(y) = \int_{-\infty}^y \big[F(y{-}x)-F(y)+F'(y)\,x\big]\,\nu_\alpha(dx) + \big[F(0)-F(y)\big]\int_y^\infty \nu_\alpha(dx) + F'(y)\int_y^\infty x\,\nu_\alpha(dx) \tag{6.13}$$

which is easily verified to be equal to the right-hand side of (6.4) for all $y>0$ upon using (6.1). This completes the proof. ☐

Proposition A.2. *Let $X=(X_t)_{t\ge 0}$ be a stable Lévy process of index $\alpha\in(0,1)$ whose characteristic function is given by*

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big(t\int_{-\infty}^\infty (e^{i\lambda x}-1)\,\nu_\alpha(dx)\Big) = e^{(c_+(-i\lambda)^\alpha + c_-(i\lambda)^\alpha)\,\Gamma(-\alpha)\,t} \tag{6.14}$$

for $\lambda\in\mathbb{R}$ and $t\ge 0$. Then the infinitesimal generator $\mathbb{L}_Y$ of the reflected process $Y=S{-}X$ takes any of the following three forms for $y>0$ given and fixed:

(6.15) *Itô's form*

$$\mathbb{L}_Y F(y) = \int_0^y \big(F(y{-}x)-F(y)\big)\,\frac{c_+}{x^{1+\alpha}}\,dx + \frac{c_+\big(F(0)-F(y)\big)}{\alpha\,y^\alpha} + \int_0^\infty \big(F(y{+}x)-F(y)\big)\,\frac{c_-}{x^{1+\alpha}}\,dx$$

(6.16) *Riemann-Liouville's form*

$$\mathbb{L}_Y F(y) = -\frac{c_+}{\alpha}\,\frac{d}{dy}\int_0^y \frac{F(x)}{(y{-}x)^\alpha}\,dx + \frac{c_+F(0)}{\alpha\,y^\alpha} + \frac{c_-}{\alpha}\,\frac{d}{dy}\int_y^\infty \frac{F(x)}{(x{-}y)^\alpha}\,dx$$

(6.17) *Caputo's form*

$$\mathbb{L}_Y F(y) = -\frac{c_+}{\alpha}\int_0^y \frac{F'(x)}{(y{-}x)^\alpha}\,dx + \frac{c_-}{\alpha}\int_y^\infty \frac{F'(x)}{(x{-}y)^\alpha}\,dx$$

whenever $F\in C_b^2(\mathbb{R}_+)$ satisfies $|F'(y)|=O(y^\gamma)$ as $y\to\infty$ for some $\gamma<\alpha{-}1$ (as well as $|F(y)|=O(y^\delta)$ as $y\to\infty$ for some $\delta<\alpha{-}1$ in (6.16) above).

Proof. As in the proof of Proposition 4 it is enough to derive (6.15). For this, fix $t>0$ and note that since X is a pure jump semimartingale with bounded variation, we have $dX_s = \Delta X_s$ and $dS_s = \Delta S_s$ for $0<s\le t$, so that Itô's formula yields

$$F(Y_t) = F(Y_0) + \sum_{0<s\le t} \big(F(Y_s)-F(Y_{s-})\big)\,. \tag{6.18}$$

Proceeding as in (6.10), taking E_y on both sides of the resulting identity and applying the compensation formula (see e.g. [20, p. 475]), we find that

$$\begin{aligned}\mathsf{E}_y F(Y_t) - F(y) = \mathsf{E}_y\Big[\int_0^t ds\,\Big(&\int_{-\infty}^{Y_s} \big[F(Y_s-x)-F(Y_s)\big]\,\nu_\alpha(dx)\\ &+\int_{Y_s}^{\infty}\big[F(0)-F(Y_s)\big]\,\nu_\alpha(dx)\Big)\Big]\end{aligned} \tag{6.19}$$

for all $y>0$. The applicability of this formula (see e.g. [15, p. 97]) follows from the facts that $|F(y)|\le C$ and $|F'(y)|\le C$ for all $y\ge 0$ with some $C>0$ so that the mean value theorem yields the existence of $\xi^1_{s,x}\in(Y_s, Y_s{+}x)$, $\xi^2_{s,x}\in(Y_s{-}x, Y_s)$ and $\eta_s\in(0,Y_s)$ such that

$$\begin{aligned}&\mathsf{E}_y\Big[\int_0^t ds\,\Big(\int_{-\infty}^{Y_s}\big|F(Y_s-x)-F(Y_s)\big|\,\nu_\alpha(dx)+\int_{Y_s}^{\infty}\big|F(0)-F(Y_s)\big|\,\nu_\alpha(dx)\Big)\Big]\\
&\le \mathsf{E}_y\Big[\int_0^t ds\,\Big(\int_0^\infty \big|F(Y_s+x)-F(Y_s)\big|\,\frac{c}{x^{1+\alpha}}\,dx\\
&\qquad+\int_0^{Y_s}\big|F(Y_s-x)-F(Y_s)\big|\,\frac{c}{x^{1+\alpha}}\,dx+\int_{Y_s}^\infty\big|F(0)-F(Y_s)\big|\,\frac{c}{x^{1+\alpha}}\,dx\Big)\Big]\\
&\le \mathsf{E}_y\Big[\int_0^t ds\,\Big(\int_0^1 |F'(\xi^1_{s,x})|\,x\,\frac{c}{x^{1+\alpha}}\,dx+\int_1^\infty\big|F(Y_s+x)-F(Y_s)\big|\,\frac{c}{x^{1+\alpha}}\,dx\\
&\qquad+\int_0^1 |F'(\xi^2_{s,x})|\,x\,\frac{c}{x^{1+\alpha}}\,dx+\int_1^\infty\big|F(Y_s-x)-F(Y_s)\big|\,\frac{c}{x^{1+\alpha}}\,dx\\
&\qquad+\int_{Y_s}^1 |F'(\eta_s)|\,Y_s\,\frac{c}{x^{1+\alpha}}\,dx\,I(Y_s\le 1)+\int_1^\infty\big|F(0)-F(Y_s)\big|\,\frac{c}{x^{1+\alpha}}\,dx\Big)\Big]\\
&\le c\,\mathsf{E}_y\Big[\int_0^t ds\,\Big(\tfrac{2C}{1-\alpha}+\tfrac{6C}{\alpha}+\tfrac{C}{\alpha}\big(Y_s^{1-\alpha}-Y_s\big)\,I(Y_s\le 1)\Big)\Big]\le c\Big(\tfrac{2C}{1-\alpha}+\tfrac{7C}{\alpha}\Big)\,t<\infty\end{aligned} \tag{6.20}$$

upon using that $1-\alpha\in(0,1)$ in the final inequality. Dividing both sides of (6.19) by t, letting $t\downarrow 0$ and using the dominated convergence theorem, we get

$$I\!\!L_Y F(y) = \int_{-\infty}^{y}\big[F(y-x)-F(y)\big]\,\nu_\alpha(dx)+\big[F(0)-F(y)\big]\int_y^\infty \nu_\alpha(dx) \tag{6.21}$$

which is easily verified to be equal to the right-hand side of (6.15) for all $y>0$ upon using (6.1). This completes the proof. □

Proposition A.3. *Let $X=(X_t)_{t\ge 0}$ be a stable Lévy process of index 1 whose characteristic function is given by*

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big(t\int_{-\infty}^{\infty}\big(e^{i\lambda x}-1-i\lambda x\,I(|x|\le 1)\big)\,\nu_1(dx)\Big) = e^{-c|\lambda|\pi t} \tag{6.22}$$

for $\lambda \in I\!\!R$ *and* $t \geq 0$ *. Then the infinitesimal generator* $I\!\!L_Y$ *of the reflected process* $Y = S-X$ *takes any of the following three forms for* $y > 0$ *given and fixed:*

(6.23) *Itô's form*

$$I\!\!L_Y F(y) = \int_0^y \Big(F(y-x) - F(y) + F'(y)\,x \Big)\, \frac{c}{x^2}\; dx + \frac{c\,(F(0)-F(y))}{y}$$
$$+ \int_0^y \Big(F(y+x) - F(y) - F'(y)\,x \Big)\, \frac{c}{x^2}\; dx + \int_y^\infty \Big(F(y+x) - F(y) \Big)\, \frac{c}{x^2}\; dx$$

(6.24) *Riemann-Liouville's form*

$$I\!\!L_Y F(y) = c\, \frac{d^2}{dy^2} \int_0^\infty F(x)\, \log\bigg(\frac{1}{|y-x|} \bigg)\, dx + \frac{cF(0)}{y}$$

(6.25) *Caputo's form*

$$I\!\!L_Y F(y) = c \int_0^\infty F''(x)\, \log\bigg(\frac{1}{|y-x|} \bigg)\, dx$$

whenever $F \in C_b^2(I\!\!R_+)$ *satisfies*

$$F'(0+) = 0 \quad \textit{(normal reflection)} \tag{6.26}$$

with $|F''(y)| = O(y^\gamma)$ *as* $y \to \infty$ *for some* $\gamma < -1$ *(as well as* $|F(y)| = O(y^\delta)$ *as* $y \to \infty$ *for some* $\delta < -1$ *in* (6.24) *above).*

Proof. As in the proof of Proposition 4 it is enough to derive (6.23). Using the same arguments as in (6.8) and (6.9) we find that

$$F(Y_t) = F(Y_0) - \int_0^t F'(Y_{s-})\, dX_s + \sum_{0<s\leq t} \Big(F(Y_{s-} + \Delta Y_s) - F(Y_{s-}) + F'(Y_{s-})\, \Delta X_s \Big) \tag{6.27}$$

where $(\int_0^t F'(Y_{s-})\, dX_s)_{t\geq 0}$ is a local martingale. We can no longer claim that this process is a martingale, however, we note from (6.22) that $X_t = M_t + A_t$ with

$$\mathsf{E} e^{i\lambda M_t} = \exp\bigg(t \int_{|x|\leq 1} \big(e^{i\lambda x} - 1 - i\lambda x \big)\, \nu_1(dx) \bigg) \tag{6.28}$$

$$\mathsf{E} e^{i\lambda A_t} = \exp\bigg(t \int_{|x|>1} \big(e^{i\lambda x} - 1 \big)\, \nu_1(dx) \bigg) \tag{6.29}$$

from where we see that the (Lévy) process $M = (M_t)_{t\geq 0}$ is a martingale (whose Lévy measure has bounded support) and the bounded variation (Lévy) process $A = (A_t)_{t\geq 0}$ is given by

$$A_t = \sum_{0<s\leq t} \Delta X_s\, I(|\Delta X_s| > 1) \tag{6.30}$$

for $t \geq 0$. From (6.27)-(6.30) we see that

$$F(Y_t) = F(Y_0) - \int_0^t F'(Y_{s-})\, dM_s \tag{6.31}$$

$$
\begin{aligned}
&+\sum_{0<s\le t}\Big(F(Y_{s-}+\Delta Y_s)-F(Y_{s-})+F'(Y_{s-})\,\Delta X_s\,I(|\Delta X_s|\le 1)\Big)\\
&=F(Y_0)-\int_0^t F'(Y_{s-})\,dM_s\\
&+\sum_{0<s\le t}\Big(\big[F(Y_{s-}-\Delta X_s)-F(Y_{s-})+F'(Y_{s-})\,\Delta X_s\,I(|\Delta X_s|\le 1)\big]\,I(\Delta X_s\le Y_{s-})\\
&\qquad+\big[F(0)-F(Y_{s-})+F'(Y_{s-})\,\Delta X_s\,I(|\Delta X_s|\le 1)\big]\,I(\Delta X_s>Y_{s-})\Big)
\end{aligned}
$$

using the same arguments as in (4.9) above. Since F' is bounded and the Lévy measure of M has bounded support (implying $\mathsf{E}\sup_{0\le s\le t}|M_s|^q<\infty$ and hence $\mathsf{E}[M,M]^{q/2}<\infty$ for all $q>0$ by the BDG inequality) it also follows by the BDG inequality (with $q=1$) that $(\int_0^t F'(Y_{s-})\,dX_s)_{t\ge 0}$ is a martingale. Taking E_y on both sides of (6.31), where P_y denotes a probability measure under which $Y_0=y$, and applying the compensation formula (see e.g. [20, p. 475]) we find that

$$
\begin{aligned}
\text{(6.32)}\qquad \mathsf{E}_yF(Y_t)-F(y)=\mathsf{E}_y\bigg[\int_0^t ds\,\bigg(&\int_{-\infty}^{Y_s}\big[F(Y_s-x)-F(Y_s)+F'(Y_s)\,x\,I(|x|\le 1)\big]\,\nu_1(dx)\\
&+\int_{Y_s}^{\infty}\big[F(0)-F(Y_s)+F'(Y_s)\,x\,I(|x|\le 1)\big]\,\nu_1(dx)\bigg)\bigg]
\end{aligned}
$$

for all $y>0$. The applicability of this formula (see e.g. [15, p. 97]) follows from the facts that $|F(y)|\le C(1+y^{\gamma+2})$, $|F'(y)|\le C(y\wedge y^{\gamma+1})$ and $|F''(y)|\le C(1\wedge y^{\gamma})$ for all $y\ge 0$ with some $C>0$ so that the mean value theorem yields the existence of $\xi^1_{s,x}\in(Y_s,Y_s+x)$, $\xi^2_{s,x}\in(Y_s-x,Y_s))$ and $\eta_s\in(0,Y_s)$ such that

$$
\begin{aligned}
\text{(6.33)}\qquad &\mathsf{E}_y\bigg[\int_0^t ds\,\bigg(\int_{-\infty}^{Y_s}\big|F(Y_s-x)-F(Y_s)+F'(Y_s)\,x\,I(|x|\le 1)\big|\,\nu_1(dx)\\
&\qquad+\int_{Y_s}^{\infty}\big|F(0)-F(Y_s)+F'(Y_s)\,x\,I(|x|\le 1)\big|\,\nu_1(dx)\bigg)\bigg]\\
&\le\mathsf{E}_y\bigg[\int_0^t ds\,\bigg(\int_1^{\infty}\big|F(Y_s+x)-F(Y_s)\big|\,\nu_1(dx)\\
&\qquad+\int_0^1\big|F(Y_s+x)-F(Y_s)-F'(Y_s)\,x\big|\,\nu_1(dx)\\
&\qquad+\int_0^{Y_s}\big|F(Y_s-x)-F(Y_s)+F'(Y_s)\,x\big|\,\nu_1(dx)\,I(Y_s<1)\\
&\qquad+\int_0^1\big|F(Y_s-x)-F(Y_s)+F'(Y_s)\,x\big|\,\nu_1(dx)\,I(Y_s\ge 1)\\
&\qquad+\int_1^{Y_s}\big|F(Y_s-x)-F(Y_s)\big|\,\nu_1(dx)\,I(Y_s\ge 1)\\
&\qquad+\int_{Y_s}^1\big|F(0)-F(Y_s)+F'(Y_s)\,x\big|\,\nu_1(dx)\,I(Y_s<1)\\
&\qquad+\int_1^{\infty}\big|F(0)-F(Y_s)\big|\,\nu_1(dx)\bigg)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\le \mathsf{E}_y\Big[\int_0^t ds\,\Big(2C\int_1^\infty \big(1+(Y_s+x)^{\gamma+2}\big)\,\frac{c}{x^2}\,dx+\int_0^1 \tfrac{1}{2}|F''(\xi^1_{s,x})|\,x^2\,\frac{c}{x^2}\,dx \\
&\qquad\qquad +2\int_0^1 \tfrac{1}{2}|F''(\xi^2_{s,x})|\,x^2\,\frac{c}{x^2}\,dx+2C\,(1+Y_s^{\gamma+2})\int_1^{Y_s}\frac{1}{x^2}\,dx\;I(Y_s\ge 1) \\
&\qquad\qquad +\int_{Y_s}^1 \Big(|F'(\eta_s)|Y_s+|F'(Y_s)|x\Big)\,\frac{c}{x^2}\,dx\;I(Y_s<1) \\
&\qquad\qquad +\int_1^\infty \Big(|F(0)|+C\big(1+Y_s^{\gamma+2}\big)\Big)\,\frac{c}{x^2}\,dx\Big)\Big] \\
&\le c\,\mathsf{E}_y\Big[\int_0^t ds\,\Big(4C\big(1+Y_s^{\gamma+2}\big)\int_1^\infty\frac{1}{x^2}\,dx+2C\int_1^\infty x^\gamma\,dx+\tfrac{3}{2}C \\
&\qquad\qquad\quad +2C(1-Y_s)\,I(Y_s<1)+|F(0)|+C\big(1+Y_s^{\gamma+2}\big)\Big)\Big] \\
&\le c\,\mathsf{E}_y\Big[\int_0^t ds\,\Big(\tfrac{17}{2}C+5C\,Y_s^{\gamma+2}-\tfrac{2C}{\gamma+1}+|F(0)|\Big)\Big] \\
&\le c\,\Big[\big(\tfrac{17}{2}C-\tfrac{2C}{\gamma+1}+|F(0)|\big)\,t+\tfrac{5C}{\gamma+3}\,t^{\gamma+3}\,\mathsf{E}_y(S_1-I_1)^{\gamma+2}\Big]<\infty
\end{aligned}
$$

since $\gamma+2 \in (0,1)$ and where we also use the scaling property of X. (Note that without loss of generality we can assume that γ is close enough to -1 so that $\gamma+2>0$.) Dividing both sides of (6.32) by t, letting $t \downarrow 0$ and using the dominated convergence theorem, we get

$$
\begin{aligned}
\text{(6.34)}\qquad \mathbb{L}_Y F(y) &= \int_{-\infty}^{y}\big[F(y-x)-F(y)+F'(y)\,x\,I(|x|\le 1)\big]\,\nu_1(dx) \\
&\quad+\big[F(0)-F(y)\big]\int_y^\infty \nu_1(dx)+F'(y)\int_y^\infty x\,I(|x|\le 1)\,\nu_1(dx)
\end{aligned}
$$

for all $y>0$. Splitting the integral over $(-\infty,y]$ into integrals over $(-\infty,-y]$ and $[-y,y]$, noting that the third term of the resulting integral over $(-\infty,-y]$ cancels with the final term in (6.34), it is easily seen using (6.2) that the expression on the right-hand side of (6.34) coincides with the expression on the right-hand side of (6.23). This completes the proof. □

References

[1] Abramowitz, M. *and* Stegun, I. A. (1992). *Handbook of Mathematical Functions.* Dover Publ.

[2] Bernyk, V. Dalang, R. C. *and* Peskir, G. (2008). The law of the supremum of a stable Lévy process with no negative jumps. *Ann. Probab.* 36 (1777–1789).

[3] Bertoin, J. (1996). *Lévy Processes.* Cambridge Univ. Press.

[4] Boyce, W. M. (1970). Stopping rules for selling bonds. *Bell J. Econom. Manage. Sci.* 1 (27–53).

[5] Doney, R. A. (2008). A note on the supremum of a stable process. *Stochastics* 80 (151–155).

[6] Du Toit, J. *and* Peskir, G. (2007). The trap of complacency in predicting the maximum. *Ann. Probab.* 35 (340–365).

[7] Du Toit, J. *and* Peskir, G. (2008). Predicting the time of the ultimate maximum for Brownian motion with drift. *Proc. Math. Control Theory Finance* (Lisbon 2007), Springer (95–112).

[8] Du Toit, J. Peskir, G. *and* Shiryaev, A. N. (2008). Predicting the last zero of Brownian motion with drift. *Stochastics* 80 (229–245).

[9] Gilbert, J. P. *and* Mosteller, F. (1966). Recognizing the maximum of a sequence. *J. Amer. Statist. Assoc.* 61 (35–73).

[10] Gradshteyn, I. S. *and* Ryzhik, I. M. (1994). *Table of Integrals, Series, and Products.* Academic Press.

[11] Graversen, S. E. Peskir, G. *and* Shiryaev, A. N. (2001). Stopping Brownian motion without anticipation as close as possible to its ulimate maximum. *Theory Probab. Appl.* 45 (125–136).

[12] Griffeath, D. *and* Snell, J. L. (1974). Optimal stopping in the stock market. *Ann. Probab.* 2 (1–13).

[13] Hochstadt, H. (1973). *Integral Equations.* John Wiley.

[14] Karlin, S. (1962). Stochastic models and optimal policy for selling an asset. *Studies in Applied Probability and Management Science*, Stanford Univ. Press (148–158).

[15] Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer-Verlag.

[16] Patie, P. (2009). A few remarks on the supremum of stable processes. *Statist. Probab. Lett.* 79 (1125–1128).

[17] Pedersen, J. L. (2003). Optimal prediction of the ultimate maximum of Brownian motion. *Stoch. Stoch. Rep.* 75 (205–219).

[18] Peskir, G. *and* Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zürich, Birkhäuser.

[19] Protter, P. E. (2004). *Stochastic Integration and Differential Equations.* Springer-Verlag.

[20] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer-Verlag.

[21] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Univ. Press.

[22] Shiryaev, A. N. (2002). Quickest detection problems in the technical analysis of the financial data. *Proc. Math. Finance Bachelier Congress* (Paris, 2000), Springer (487–521).

[23] Shiryaev, A. N. (2009). On conditional-extremal problems of the quickest detection of nonpredictable times of the observable Brownian motion. *Theory Probab. Appl.* 53 (663–678).

[24] Urusov, M. A. (2005). On a property of the moment at which Brownian motion attains its maximum and some optimal stopping problems. *Theory Probab. Appl.* 49 (169–176).

[25] Watanabe, S. (1962). On stable processes with boundary conditions. *J. Math. Soc. Japan* 14 (170–198).

[26] Williams, D. (1991). *Probability with Martingales*. Cambridge University Press.

Violetta Bernyk
UBS AG
Europastrasse 1
8152 Opfikon
Switzerland
E-mail: violetta.bernyk@ubs.com

Robert C. Dalang
Institut de Mathématiques
Ecole Polytechnique Fédérale
Station 8
1015 Lausanne
Switzerland
E-mail: robert.dalang@epfl.ch

Goran Peskir
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
E-mail: goran@maths.man.ac.uk