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Stochastic Differential Equations for Sticky Brownian Motion

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We study (i) the SDE system

$$dX_t = I(X_t \neq 0)\, dB_t$$
$$I(X_t = 0)\, dt = \tfrac{1}{\mu}\, d\ell_t^0(X)$$

for Brownian motion X in $I\!R$ sticky at 0, and (ii) the SDE system

$$dX_t = \tfrac{1}{2}\, d\ell_t^0(X) + I(X_t > 0)\, dB_t$$
$$I(X_t = 0)\, dt = \tfrac{1}{2\mu}\, d\ell_t^0(X)$$

for reflecting Brownian motion X in $I\!R_+$ sticky at 0, where X starts at x in the state space, $\mu \in (0, \infty)$ is a given constant, $\ell^0(X)$ is a local time of X at 0, and B is a standard Brownian motion. We prove that both systems (i) have a jointly unique weak solution and (ii) have no strong solution. The latter fact verifies Skorokhod's conjecture on sticky Brownian motion and provides alternative arguments to those given in the literature.

1. Introduction

Sticky boundary behaviour of diffusion processes was discovered by Feller in his 1952 paper [9] (see also [10] and [11]). The problem considered by Feller was to describe domains of the infinitesimal generators associated with strong Markov processes X in $[0, \infty)$ that behave like a standard Brownian motion B while in $(0, \infty)$. Using a semigroup approach (initiated by Hille and Yosida a few years earlier) Feller showed that a possible boundary behaviour of X at 0 is described by the following condition

$$f'(0+) = \tfrac{1}{2\mu} f''(0+) \tag{1.1}$$

where $\mu \in (0, \infty)$ is a given and fixed constant. In addition to the classic boundary conditions of (i) Dirichlet ($f(0+) = 0$), (ii) Neumann ($f'(0+) = 0$) and (iii) Robin ($f'(0+) = (1/2\mu) f(0+)$) corresponding to (i) killing, (ii) instantaneous reflection and (iii) elastic boundary behaviour of X at 0 respectively, the key novelty of the condition (1.1) is the appearance of the second derivative $f''(0+)$ measuring the 'stickiness' of X at 0. (The formal limiting case $f'(0+) = 0$ for $\mu \uparrow \infty$ corresponds to instantaneous reflection of X at 0, and the formal limiting case $f''(0+) = 0$ for $\mu \downarrow 0$ corresponds to absorption of X at 0.) For this reason,

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and to distinguish it from the global process considered below, the process X will be called a *sticky reflecting Brownian motion* (as one of Feller's Brownian motions in the terminology of [14, Section 5.7] noting that the adjective 'reflecting' is often omitted in the literature).

The problem of how to construct the sample paths of X in a canonical manner was solved by Itô and McKean [13, Section 10]). They showed that X can be obtained from the reflecting Brownian motion $|B|$ by the time change $t \mapsto T_t := A_t^{-1}$ where $A_t = t+(1/\mu)\ell_t^0(B)$ for $t \ge 0$ and $\ell^0(B)$ is the local time of B at 0. When $X = |B_T|$ is away from 0, then both the additive functional A and the 'new clock' T run ordinarily (like the 'old clock') and hence the motion of X is the same as the motion of $|B|$. When X is at 0 however, then A gets an extra increase from the local time $\ell^0(B)$ causing the new clock T to slow down and forcing X to stay longer at 0 (in comparison with the reflecting Brownian motion $|B|$ under the old clock). Thus the two processes X and $|B|$ have the same sample paths and the two motions differ only by their speeds (the former being slower).

The Itô–McKean construction has become a basic building block for diffusion processes exhibiting sticky boundary behaviour (in one dimension) being also applicable in the case where the sticky point is not necessarily a boundary point. For example, considering the unconstrained Brownian motion B instead of $|B|$ above, and recalling that the time-changed process $X = B_T$ has the speed measure $m = 2\nu$ when $t \mapsto T_t$ is the (right) inverse of $t \mapsto A_t = \int_{I\!R} \ell_t^x(B)\, \nu(dx)$ for $t \ge 0$ where ν is a Borel measure (see e.g. [19, p. 417]), we see that the global analogue of the sticky reflecting condition (1.1) becomes

$$f'(0+) - f'(0-) = \tfrac{1}{\mu} f''(0\pm) \tag{1.2}$$

where we also recall that $(df/ds)(0+) - (df/ds)(0-) = m(\{0\}) I\!L_X f(0)$ with the scale function $s(x) = x$ for $x \in I\!R$ and the infinitesimal generator $I\!L_X = (1/2)\, d^2/dx^2$ (see e.g. [19, p. 309] where the factor 2 is not needed in the fifth line). For the reasons stated above the process X will be called a *sticky Brownian motion* (using the same letter will cause no ambiguity since it will be clear from the context which process is considered).

The problem whether the sticky reflecting Brownian motion arising from (1.1) above can be obtained from a stochastic differential equation (SDE) driven by B has been considered in the literature. Skorokhod conjectured that the stochastic differential equation has no strong solution and Chitashvili published a technical report [3] in 1989 claiming a proof (the paper was published after his death in 1997). In the same year Warren [22] derived a remarkable conditional probability relation that implies no strong existence as a by-product (see also [23] and [24] for more general results on the non-cosiness of filtrations). The two approaches have no common points and the latter makes use of explicit calculations which may not be fully available in the case of more general sticky processes. Renewed interest in the sticky boundary SDEs and related results are presented in a recent paper [15].

The present paper grew out from our (i) inability to follow Chitashvili's arguments in [3] and (ii) desire to make the arguments applicable to other sticky processes. This led us to undertake a fresh approach to the study of SDEs with sticky boundary behaviour with the aim of providing canonical/rigorous and (hopefully) readable arguments for the existence and uniqueness of solutions. The present paper contains basic results of this kind for the sticky Brownian motions arising from (1.1) and (1.2) above (the first one being reflecting).

Reversing the historical arrow we first study the sticky Brownian motion X arising from (1.2) above (Section 2). Building on the Itô–McKean construction we find that the SDE system for the process X is given as follows:

$$dX_t = I(X_t \neq 0)\, dB_t \tag{1.3}$$

$$I(X_t = 0)\, dt = \tfrac{1}{\mu}\, d\ell_t^0(X) \tag{1.4}$$

where $X_0 = x$ in $I\!R$ and $\ell^0(X)$ is the local time[1] of X at 0. We first show that the system (1.3)–(1.4) has a jointly unique weak solution (Theorem 1). We then prove that the system (1.3)–(1.4) has no strong solution (Theorem 3) thus verifying Skorokhod's conjecture in this case. We further study the sticky reflecting Brownian motion X arising from (1.1) above (Section 3). Building on the Itô–McKean construction in the same manner as earlier we find that the SDE system for the process X is given as follows:

$$dX_t = \tfrac{1}{2}\, d\ell_t^0(X) + I(X_t > 0)\, dB_t \tag{1.5}$$

$$I(X_t = 0)\, dt = \tfrac{1}{2\mu}\, d\ell_t^0(X) \tag{1.6}$$

where $X_0 = x$ in $[0, \infty)$ and $\ell^0(X)$ is the local time[1] of X at 0. Unlike the system (1.3)+(1.4) where the boundary condition (1.4) is unrelated to the equation (1.3), we show that the system (1.5)+(1.6) is equivalent to the single equation

$$dX_t = \mu I(X_t = 0)\, dt + I(X_t > 0)\, dB_t \tag{1.7}$$

obtained by incorporating (1.6) into (1.5). It will be also seen that any X solving the system (1.5)+(1.6) or equivalently the equation (1.7) must be non-negative as needed. We first show that the system (1.5)+(1.6) or equivalently the equation (1.7) has a jointly unique weak solution (Theorem 5). We then prove that the system (1.5)+(1.6) or equivalently the equation (1.7) has no strong solution (Theorem 6), thus verifying Skorokhod's conjecture in this case as well, and providing alternative arguments to those mentioned above.

2. Sticky Brownian motion

In this section we consider the SDE system

$$dX_t = I(X_t \neq 0)\, dB_t \tag{2.1}$$

$$I(X_t = 0)\, dt = \tfrac{1}{\mu}\, d\ell_t^0(X) \tag{2.2}$$

for Brownian motion X in $I\!R$ sticky at 0, where $X_0 = x$ in $I\!R$, $\mu \in (0, \infty)$ is a given constant, $\ell^0(X)$ is the local time of X at 0, and B is a standard Brownian motion.

Recall that a pair of $I\!F$-adapted stochastic processes (X, B) defined on a filtered probability space $(\Omega, \mathcal{F}, I\!F, \mathsf{P})$ is said to be a *solution* to (2.1) if B is a standard Brownian motion that is a martingale with respect to $I\!F$ and the integral equation

$$X_t = x + \int_0^t I(X_s \neq 0)\, dB_s \tag{2.3}$$

[1] Unless stated otherwise by the local time of X we always mean the *semimartingale* right *local time* of X where we recall that this local time coincides with the symmetric (and left) local time when X is a continuous martingale as in (1.3) above (see e.g. [19, p. 227]). Formal definitions of the semimartingale *symmetric* and *right* local times are recalled following (2.4) and (3.2) below. A formal definition of the semimartingale *left* local time is recalled following (3.5) below.

is satisfied for all $t \geq 0$ where the integral with respect to B is understood in Itô's sense. Every solution (X, B) is also referred to as a *weak solution*. A solution (X, B) is called a *strong solution* if X is adapted to the natural filtration $I\!F^B = (\sigma(B_s \,|\, s \in [0,t]))_{t \geq 0}$ of B. When it is clear from the context which Brownian motion B is being addressed we will often simply say that X is a *weak* or *strong solution* to (2.1) respectively. Note that the existence of a strong solution implies the existence of a weak solution (since strong solutions are weak solutions by definition as well).

In addition to X solving (2.1) we also require that X satisfies (2.2) in the sense that

$$\int_0^t I(X_s = 0)\, ds = \tfrac{1}{\mu}\, \ell_t^0(X) \tag{2.4}$$

for all $t \geq 0$ where $\ell_t^0(X) = \mathsf{P}\text{-}\lim_{\varepsilon \downarrow 0} (1/2\varepsilon) \int_0^t I(-\varepsilon \leq X_s \leq \varepsilon)\, d\langle X, X \rangle_s$ is the local time of X at 0 with $\langle X, X \rangle_t = \int_0^t I(X_s \neq 0)\, ds$ for $t \geq 0$ upon recalling that the symmetric local time just defined and the right (or left) local time for the continuous martingale X coincide (cf. footnote 1 above). Note that B itself solves (2.3) in place of X with $x = 0$, however this solution fails to satisfy (2.4) (unless $\mu = \infty$ formally). The reason why the solution X to (2.1) is not unique is that the diffusion coefficient $x \mapsto I(x \neq 0)$ in (2.1) vanishes at 0 while the equation (2.1) itself does not determine the sojourn time of X at 0. The latter thus appears as an additional degree of freedom which is being specified by the condition (2.2). For more general SDEs with degenerate diffusion coefficients and the concepts of (i) fundamental solution, (ii) time delay (at zeros of the diffusion coefficient), and (iii) general solution, see [7]+[8] and the references therein.

Recall that (i) *uniqueness in law* and (ii) *joint uniqueness in law* hold for the system (2.1)+(2.2) if for any two solutions (X^1, B^1) and (X^2, B^2) that are not necessarily defined on the same filtered probability space we have (i) $X^1 \sim X^2$ and (ii) $(X^1, B^1) \sim (X^2, B^2)$ respectively. Solutions that are jointly unique in law are called *jointly unique weak solutions*. Clearly joint uniqueness in law implies uniqueness in law. Recall also that *pathwise uniqueness* holds for the system (2.1)+(2.2) if for any two solutions (X^1, B) and (X^2, B) defined on the same filtered probability space we have $X_t^1 = X_t^2$ outside a set of probability measure zero for all $t \geq 0$.

1. *Weak existence and uniqueness*. Note that the diffusion coefficient $x \mapsto I(x \neq 0)$ in the equation (2.1) is both discontinuous and degenerate (taking value zero) so that the standard results on the existence and uniqueness of solutions are not directly applicable (see e.g. [12, Chapter IV]). We begin by deriving a basic existence and uniqueness result in Theorem 1 below. Note that the weak existence and uniqueness can also be deduced from the general result in [8, Theorem 5.18] where the uniqueness in law was derived by appealing to uniqueness of the solution to a martingale problem. The proof of the uniqueness in law presented below is constructive and makes use of Lévy's characterisation theorem instead (see e.g. [19, p.150]). Building on this construction and making use of Knight's theorem [16] (see [19, pp. 183–184]) we further derive the joint uniqueness in law which to our knowledge has not been established before and which plays an important role in the proof of Theorem 3 below.

Theorem 1. *The system* (2.1)+(2.2) *has a jointly unique weak solution.*

Proof. 1. We show that the system (2.1)+(2.2) has a weak solution. For this, let B^1 be a standard Brownian motion defined on a probability space $(\Omega^1, \mathcal{F}^1, \mathsf{P}^1)$ and let $I\!F^1$ be the natural filtration of B^1 . For $x \in I\!R$ given and fixed set

$$W_t^1 = x + B_t^1 \tag{2.5}$$

and following [13, p. 186 & pp. 200–201] consider the additive functional

$$A_t = t + \tfrac{1}{\mu}\ell_t^0(W^1) \tag{2.6}$$

where $\ell_t^0(W^1) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0}(1/2\varepsilon)\int_0^t I(-\varepsilon \le W_s^1 \le \varepsilon)\,ds$ is the local time of W^1 at 0 for $t \ge 0$. Note that $t \mapsto A_t$ is continuous and strictly increasing with $A_t \uparrow \infty$ as $t \uparrow \infty$ so that its (proper) inverse $t \mapsto T_t$ defined by

$$T_t = A_t^{-1} \tag{2.7}$$

is well defined (finite) for all $t \ge 0$ and satisfies the same properties itself. Moreover, since $A = (A_t)_{t\ge 0}$ is adapted to $I\!F^1$ it follows that each T_t is a stopping time with respect to $I\!F^1$, so that $T = (T_t)_{t\ge 0}$ defines a time change with respect to $I\!F^1$. The fact that $t \mapsto T_t$ is continuous and strictly increasing with $T_t < \infty$ for $t \ge 0$ (or equivalently $A_t \uparrow \infty$ as $t \uparrow \infty$) implies that standard time change transformations are applicable to continuous semimartingales and their stochastic integrals without extra conditions on their sample paths (see e.g. [19, pp. 7–9 & pp. 179–181]) and they will be used in the sequel with no explicit mention.

Consider the time-changed process

$$X_t = W_{T_t}^1 = x + B_{T_t}^1 \tag{2.8}$$

for $t \ge 0$. Since B^1 is a continuous martingale with respect to $I\!F^1$ it follows that $B_T^1 = (B_{T_t}^1)_{t\ge 0}$ is a continuous martingale with respect to $I\!F_T^1 = (\mathcal{F}_{T_t}^1)_{t\ge 0}$. Note further that

$$B_{T_t}^1 = \int_0^{T_t} I(W_s^1 \ne 0)\,dB_s^1 = \int_0^t I(W_{T_s}^1 \ne 0)\,dB_{T_s}^1 = \int_0^t I(X_s \ne 0)\,dB_{T_s}^1 \tag{2.9}$$

for $t \ge 0$. Moreover, we have

$$\begin{aligned}\langle B_T^1, B_T^1\rangle_t = T_t &= \int_0^{T_t} I(W_s^1 \ne 0)\,ds = \int_0^{T_t} I(W_s^1 \ne 0)\,(ds + \tfrac{1}{\mu}d\ell_s^0(W^1)) \\ &= \int_0^{T_t} I(W_s^1 \ne 0)\,dA_s = \int_0^t I(W_{T_s}^1 \ne 0)\,dA_{T_s} = \int_0^t I(X_s \ne 0)\,ds\end{aligned} \tag{2.10}$$

for $t \ge 0$.

The identities (2.8)–(2.10) motivate to use a variant of Doob's martingale representation theorem in order to achieve (2.1). For this, take another Brownian motion B^0 defined on a probability space $(\Omega^0, \mathcal{F}^0, \mathsf{P}^0)$ and let $I\!F^0$ denote the natural filtration of B^0 . Set $\Omega = \Omega^1 \times \Omega^0$, $\mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^0$, $I\!F = I\!F_T^1 \times I\!F^0$ and $\mathsf{P} = \mathsf{P}^1 \times \mathsf{P}^0$. Then $(\Omega, \mathcal{F}, I\!F, \mathsf{P})$ is a filtered probability space. Extend all random variables Z^1 and Z^0 defined on Ω^1 and Ω^0 respectively to Ω by setting $Z^1(\omega) := Z^1(\omega_1)$ and $Z^0(\omega) := Z^0(\omega_0)$ for $\omega = (\omega_1, \omega_0) \in \Omega$. Then it is easily seen that B_T^1 and B^0 remain (continuous) martingales with respect to $I\!F$ and B^0 remains a standard Brownian motion on $(\Omega, \mathcal{F}, \mathsf{P})$ as well (note that B^1_T and B^0 are independent). It follows therefore that the process B defined by

$$B_t := B^1_{T_t} + \int_0^t I(X_s\!=\!0)\, dB^0_s \tag{2.11}$$

for $t \geq 0$ is a continuous martingale with respect to $I\!\!F$. From (2.9) and (2.10) we see that

$$\langle B, B\rangle_t = \langle B^1_T, B^1_T\rangle_t + \int_0^t I(X_s\!=\!0)\, ds = \int_0^t I(X_s\!\neq\!0)\, ds + \int_0^t I(X_s\!=\!0)\, ds = t \tag{2.12}$$

for all $t \geq 0$ and hence by Lévy's characterisation theorem it follows that B is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathsf{P})$.

From (2.11) we see that

$$\int_0^t I(X_s\!\neq\!0)\, dB_s = \int_0^t I(X_s\!\neq\!0)\, dB^1_{T_s} \tag{2.13}$$

for $t \geq 0$ which by (2.8) and (2.9) shows that X and B solve (2.1). Moreover, we have

$$\begin{aligned} \int_0^t I(X_s\!=\!0)\, ds &= \int_0^t I(W^1_{T_s}\!=\!0)\, dA_{T_s} = \int_0^{T_t} I(W^1_s\!=\!0)\, dA_s \\ &= \int_0^{T_t} I(W^1_s\!=\!0)\, \big(ds + \tfrac{1}{\mu} d\ell^0_s(W^1)\big) = \tfrac{1}{\mu}\ell^0_{T_t}(W^1) \\ &= \tfrac{1}{\mu}\ell^0_t(W^1_T) = \tfrac{1}{\mu}\ell^0_t(X) \end{aligned} \tag{2.14}$$

where we use that $\langle X, X\rangle_t = \langle W^1_T, W^1_T\rangle_t = \langle B^1_T, B^1_T\rangle_t = T_t$ for $t \geq 0$. From (2.14) we see that X satisfies (2.2) and this completes the proof of weak existence.

2. We show that uniqueness in law holds for the system (2.1)+(2.2). For this, we will undo the time change from the previous part of the proof starting with the notation afresh. Suppose that X and B solve (2.1) subject to (2.2). Consider the additive functional

$$T_t = \int_0^t I(X_s\!\neq\!0)\, ds \tag{2.15}$$

for $t \geq 0$ and note that $T_t \uparrow \infty$ as $t \uparrow \infty$. Indeed, to verify this claim note by the Itô–Tanaka formula using (2.1) and (2.2) that

$$|X_t| = |x| + \int_0^t \mathrm{sign}(X_s)\, dX_s + \ell^0_t(X) = |x| + M_t + \mu \int_0^t I(X_s\!=\!0)\, ds \tag{2.16}$$

where $\mathrm{sign}(x) := -I(x \leq 0) + I(x > 0)$ for $x \in I\!\!R$ and M is a continuous martingale with $\langle M, M\rangle_t = T_t \uparrow T_\infty$ as $t \uparrow \infty$. It follows therefore that $M_t \to M_\infty$ in $I\!\!R$ almost surely on $\{T_\infty\!<\!\infty\}$ as $t \to \infty$. Setting further $T^0_t = \int_0^t I(X_s\!=\!0)\, ds$ and noting that $T_t + T^0_t = t$ for $t \geq 0$ we see that $T^0_t \uparrow \infty$ on $\{T_\infty\!<\!\infty\}$ as $t \uparrow \infty$. But then $|X_t| = |x| + M_t + \mu T^0_t \to \infty$ almost surely on $\{T_\infty\!<\!\infty\}$ as $t \to \infty$ contradicting the fact that $T^0_t \uparrow \infty$ on $\{T_\infty\!<\!\infty\}$ unless its probability is zero. This shows that $T_\infty = \infty$ (with probability one) as claimed.

Since $T_t \uparrow \infty$ as $t \uparrow \infty$ it follows that its (right) inverse $t \mapsto A_t$ defined by

$$A_t = \inf \{ s \geq 0 \mid T_s > t \} \tag{2.17}$$

is finite for all $t \geq 0$. Note that $t \mapsto A_t$ is increasing and right-continuous on $I\!R_+$. Moreover, since $T = (T_t)_{t \geq 0}$ is adapted to $I\!F$ it follows that each A_t is a stopping time with respect to $I\!F$, so that $A = (A_t)_{t \geq 0}$ defines a time change with respect to $I\!F$.

Consider the time-changed process

$$B_t^1 = X_{A_t} - x \tag{2.18}$$

for $t \geq 0$. Since X is a continuous martingale with respect to $I\!F$ it follows that B^1 is a continuous martingale with respect to $I\!F_A$. Moreover, we have

$$\langle B^1, B^1 \rangle_t = \langle X_A, X_A \rangle_t = \langle X, X \rangle_{A_t} = T_{A_t} = t \tag{2.19}$$

where we use that $t \mapsto T_t$ is constant on each $[A_{s-}, A_s]$ and therefore the same is true for $t \mapsto X_t$ upon recalling that X is a continuous martingale with $\langle X, X \rangle_t = T_t$ for $t \geq 0$. It follows therefore by Lévy's characterisation theorem that B^1 is a standard Brownian motion. Moreover, recalling (2.2) we see that

$$\begin{aligned} t = T_{A_t} &= \int_0^{A_t} I(X_s \neq 0)\, ds = A_t - \int_0^{A_t} I(X_s = 0)\, ds \\ &= A_t - \tfrac{1}{\mu} \ell^0_{A_t}(X) = A_t - \tfrac{1}{\mu} \ell^0_t(X_A) = A_t - \tfrac{1}{\mu} \ell^0_t(x + B^1) \end{aligned} \tag{2.20}$$

from where it follows that

$$A_t = t + \tfrac{1}{\mu} \ell^0_t(x + B^1) \tag{2.21}$$

for $t \geq 0$. This shows that $t \mapsto A_t$ is strictly increasing (and continuous) and hence

$$T_t = A_t^{-1} \tag{2.22}$$

is the proper inverse for $t \geq 0$ (implying also that $t \mapsto T_t$ is strictly increasing and continuous). It follows in particular that $A_{T_t} = t$ so that

$$X_t = X_{A_{T_t}} = x + B^1_{T_t} \tag{2.23}$$

for $t \geq 0$. From (2.21)–(2.23) we see that X is a well-determined measurable functional of the standard Brownian motion B^1. This shows that the law of X solving (2.1)+(2.2) is uniquely determined and this completes the proof of weak uniqueness.

3. We show that joint uniqueness in law holds for the system (2.1)+(2.2). For this, we will continue our considerations starting with (2.15) above upon assuming that X and B solve (2.1) subject to (2.2). Note from (2.1) that

$$B_t = X_t - x + X_t^0 \tag{2.24}$$

where the process X^0 is defined by

$$X_t^0 = \int_0^t I(X_s\!=\!0)\, dB_s \tag{2.25}$$

for $t \ge 0$. Note further that X^0 is a continuous martingale with respect to $I\!\!F$ such that $\langle X^0, X^0\rangle_t = \int_0^t I(X_s\!=\!0)\, ds = T_t^0 \uparrow \infty$ almost surely as $t \uparrow \infty$. Indeed, to see this recall that $T_t = \langle M, M\rangle_t \uparrow \infty$ almost surely as $t \uparrow \infty$ so that $\liminf_{t\to\infty} M_t = -\infty$ and $\limsup_{t\to\infty} M_t = +\infty$ almost surely from where the claim follows by (2.16) since otherwise its left-hand side could not remain non-negative. Defining the (right) inverse of $t \mapsto T_t^0$ by $A_t^0 = \inf\{\, s \ge 0 \mid T_s^0 > t\,\}$ it then follows by the Dambis–Dubins–Schwarz theorem [4]–[5] (see [19, p. 181]) that

$$B_t^0 = X_{A_t^0}^0 \tag{2.26}$$

is a standard Brownian motion such that

$$X_t^0 = B_{T_t^0}^0 \tag{2.27}$$

for $t \ge 0$. Moreover, it is clear from (2.15)+(2.17) that (2.18) can also be seen as an application of the Dambis–Dubins–Schwarz theorem for the continuous martingale X, and since X and X^0 are orthogonal in the sense that $\langle X, X^0\rangle_t = \int_0^t I(X_s\!\neq\!0)\, I(X_s\!=\!0)\, ds = 0$ for all $t \ge 0$, it follows by Knight's theorem [16] (see [19, pp. 183–184]) that the standard Brownian motions B^1 and B^0 are independent. Because T_t^0 is a measurable functional of X we see from (2.21)–(2.23) and (2.24)+(2.27) that (X, B) is a well-determined measurable functional of the two-dimensional Brownian motion (B^1, B^0). This shows that the law of (X, B) solving (2.1)+(2.2) is uniquely determined and the proof is complete. $\square$

2. *No strong existence.* We continue by proving in Theorem 3 below that the system (2.1)+(2.2) has no strong solution thus verifying Skorokhod's conjecture in this case. This will be accomplished in several steps and we will begin by providing a brief review of the results and facts that will be used in the proof below.

2.1. All processes under consideration are continuous and can be seen as random elements taking values in the space $C = \{\, x : [0,\infty) \to I\!\!R \mid x \text{ is continuous }\}$ equipped with the topology of uniform convergence on compacts that is induced by the metric

$$d_\infty(x,y) = \sum_{N=1}^{\infty} \frac{1}{2^N} \Big(1 \wedge \sup_{0\le t\le N} |x(t)-y(t)| \Big) \tag{2.28}$$

for $x, y \in C$. For x_n and x in C recall that x_n converges to x in this topology if and only if x_n converges uniformly to x on $[0,N]$ as $n \to \infty$ for every $N > 0$. For random elements X^n and X taking values in C we will write $X^n \xrightarrow{\text{d}} X$ if X^n converges in distribution to X relative to this topology as $n \to \infty$. We will write $X^n \xrightarrow{\text{p}} X$ if X^n converges to X in probability (meaning that $d_\infty(X^n, X)$ converges to zero in probability), and we will write $X^n \xrightarrow{\text{a.s.}} X$ if X^n converges to X almost surely (meaning that $d_\infty(X^n, X)$ converges to zero almost surely) as $n \to \infty$. The three convergence relations extend to random elements taking values in C^n equipped with the product topology induced by the metric d_∞ in the standard manner when the dimension n is strictly larger than one.

2.2. Weak convergence of stochastic integrals has been studied since the 1980s by a number of authors and general results of this kind have been established in the space D equipped with the Skorokhod topology of which the space C is a special and simpler case. We will make use of the following known relations (see Theorem 7.10 in [17] combined with (7.14) and the second remark following the proof of this theorem). Let $S^n = M^n + A^n$ for $n \geq 1$ be a continuous semimartingale where M^n is a continuous local martingale and A^n is a continuous adapted process with a finite total variation $V(A^n)$ satisfying

$$\sup_{n \geq 1} \mathsf{E} \int_0^t dV_s(A^n) < \infty \tag{2.29}$$

for every $t \geq 0$. Let H^n be a continuous adapted process (implying that the stochastic integral $H^n \cdot S^n = (\int_0^t H^n_s \, dS^n_s)_{t \geq 0}$ is well defined) and let H and S be continuous processes. Then the following two relations are valid:

$$\text{If } (H^n, S^n) \xrightarrow{\mathsf{d}} (H, S) \text{ then } (H^n, S^n, H^n \cdot S^n) \xrightarrow{\mathsf{d}} (H, S, H \cdot S) \text{ as } n \to \infty ; \tag{2.30}$$

$$\text{If } (H^n, S^n) \xrightarrow{\mathsf{P}} (H, S) \text{ then } (H^n, S^n, H^n \cdot S^n) \xrightarrow{\mathsf{P}} (H, S, H \cdot S) \text{ as } n \to \infty . \tag{2.31}$$

Implicit in these conclusions is the important fact that S is a (continuous) semimartingale with respect to the natural filtration of (H, S) so that the stochastic integral $H \cdot S$ is well defined (note that all processes in (2.31) are defined on the same filtered probability space while the filtered probability spaces in (2.30) may vary with $n \geq 1$). The relations (2.30)+(2.31) including the semimartingale fact about the limit are also valid for finite-dimensional vector/ matrix valued processes in which case (2.29) needs to be verified for each coordinate process. In essence (2.30)+(2.31) is a consequence of the well-known fact that Itô's integral of continuous (or left-continuous with right limits) processes H and S can be equivalently defined as the limit (in probability) of the Riemann–Stieltjes sums

$$\int_0^t H_s \, dS_s = \lim_{\|\pi\| \to 0} \sum_{i=1}^n H_{t_{i-1}} (S_{t_i} - S_{t_{i-1}}) \tag{2.32}$$

where π denotes the partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ and $\|\pi\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$ stands for its diameter. After transferring weak convergence from the left-hand side in (2.30) to almost sure convergence on another probability space using Skorokhod's representation theorem (see e.g. [1, pp. 70–71]), we see that the implication in (2.30) reduces to the implication in (2.31), and the latter implication follows from the fact that the Riemann–Stieltjes sums converge uniformly over the (good) sequences of continuous functions where the laws of H^n and S^n are supported for $n \geq 1$. For further details see [17] and the references therein.

2.3. A key fact of independent interest that will be used in the proof below may be stated as follows (note that the converse implication holds as well but we will make no use of it).

Lemma 2. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, let $(C, \|\cdot\|)$ be a separable Banach space with the Borel σ-algebra $\mathcal{B}(C)$, let X be a random element from Ω into C , and let f_n and f be measurable functions from C into C for $n \geq 1$. Then we have*

$$(f_n(X), X) \xrightarrow{\mathsf{d}} (f(X), X) \implies f_n(X) \xrightarrow{\mathsf{P}} f(X) \tag{2.33}$$

as $n \to \infty$.

Proof. We first show that the left-hand convergence in (2.33) implies that

$$(f_n(X), f(X)) \xrightarrow{\mathsf{d}} (f(X), f(X)) \tag{2.34}$$

as $n \to \infty$. For this, recall that $\mathsf{P}_X(B) = \mathsf{P}(X \in B)$ for $B \in \mathcal{B}(C)$ defines a probability measure on $(C, \mathcal{B}(C))$, and since C is a separable Banach space it follows that there exists a sequence of continuous functions $g_n : C \to C$ for $n \geq 1$ such that $g_n \to f$ in P_X-measure (see [25]). This means that $g_n(X) \to f(X)$ in P-probability as $n \to \infty$.

Take any bounded and uniformly continuous function $G : S \times S \to I\!R$. Then for every $\varepsilon > 0$ given and fixed, there exists $\delta > 0$ such that $|G(x_1, x_2) - G(y_1, y_2)| \leq \varepsilon$ whenever $\|y_1 - x_1\| + \|y_2 - x_2\| \leq \delta$ with x_i and y_i in C for $i = 1, 2$. It follows therefore that

$$\begin{aligned} &\big|\,\mathsf{E}\big[G(f_n(X), f(X))\big] - \mathsf{E}\big[G(f(X), f(X))\big]\,\big| \\ &\leq \big|\,\mathsf{E}\big[G(f_n(X), f(X)) - G(f_n(X), g_m(X))\big]\,\big| \\ &\quad + \big|\,\mathsf{E}\big[G(f_n(X), g_m(X)) - G(f(X), f(X))\big]\,\big| \\ &\leq \mathsf{E}\Big[\big|G(f_n(X), f(X)) - G(f_n(X), g_m(X))\big|\, I\big(\|g_m(X) - f(X)\| \leq \delta\big)\Big] \\ &\quad + \mathsf{E}\Big[\big|G(f_n(X), f(X)) - G(f_n(X), g_m(X))\big|\, I\big(\|g_m(X) - f(X)\| > \delta\big)\Big] \\ &\quad + \big|\,\mathsf{E}\big[G(f_n(X), g_m(X)) - G(f(X), f(X))\big]\,\big| \\ &\leq \varepsilon + 2K\,\mathsf{P}\big(\|g_m(X) - f(X)\| > \delta\big) + \big|\,\mathsf{E}\big[G(f_n(X), g_m(X)) - G(f(X), f(X))\big]\,\big| \end{aligned} \tag{2.35}$$

for all $n \geq 1$ and all $m \geq 1$ where $K > 0$ is an upper bound for $|G|$. Letting $n \to \infty$ and using the left-hand convergence in (2.33) combined with the continuous mapping theorem (see e.g. [1, pp. 20–22]) we get

$$\begin{aligned} &\limsup_{n\to\infty} \big|\,\mathsf{E}\big[G(f_n(X), f(X))\big] - \mathsf{E}\big[G(f(X), f(X))\big]\,\big| \\ &\leq \varepsilon + 2K\,\mathsf{P}\big(\|g_m(X) - f(X)\| > \delta\big) + \big|\,\mathsf{E}\big[G(f(X), g_m(X)) - G(f(X), f(X))\big]\,\big| \\ &\leq \varepsilon + 2K\,\mathsf{P}\big(\|g_m(X) - f(X)\| > \delta\big) \\ &\quad + \mathsf{E}\Big[\big|G(f(X), g_m(X)) - G(f(X), f(X))\big|\, I\big(\|g_m(X) - f(X)\| \leq \delta\big)\Big] \\ &\quad + \mathsf{E}\Big[\big|G(f(X), g_m(X)) - G(f(X), f(X))\big|\, I\big(\|g_m(X) - f(X)\| > \delta\big)\Big] \\ &\leq 2\varepsilon + 4K\,\mathsf{P}\big(\|g_m(X) - f(X)\| > \delta\big) \end{aligned} \tag{2.36}$$

for all $m \geq 1$. Letting first $m \to \infty$ and then $\varepsilon \downarrow 0$ we see that the left-hand side in (2.36) equals zero and hence (2.34) holds as claimed.

We next note that (2.34) combined with the continuous mapping theorem yields

$$\|f_n(X) - f(X)\| \xrightarrow{\mathsf{d}} \|f(X) - f(X)\| = 0 \tag{2.37}$$

as $n \to \infty$ and this implies the right-hand convergence in (2.33) as claimed. $\square$

2.4. The main result on strong existence of the stochastic differential system for sticky Brownian motion may now be stated as follows.

Theorem 3. *The system* (2.1)+(2.2) *has no strong solution.*

Proof. The central idea of the proof is to approximate the weak solution X constructed in the proof of Theorem 1 by strong solutions X^n to a regularised equation indexed by $n \in I\!N$. This will reduce the question of strong existence to the question of convergence in probability of the approximating sequence. Showing that the latter fails we will be able to conclude that strong existence fails as well. We begin by defining a regularised equation and constructing the approximating sequence of strong solutions by time change.

1. Fix any sequence $\delta_n \downarrow 0$ as $n \to \infty$ and consider the equation

$$dX_t^n = \sigma_n(X_t^n)\, dW_t^n \tag{2.38}$$

where $\sigma_n(x) = I(|x| > \delta_n) + \sqrt{2\mu\delta_n}\, I(|x| \le \delta_n)$ for $x \in I\!R$ and $n \ge 1$. There is no loss of generality to assume that $X_0^n = 0$ for $n \ge 1$. Since σ_n is (i) uniformly bounded by a strictly positive constant from below and (ii) of bounded variation on compacts, we know by Nakao's theorem [18] (see [20, pp. 266–267]) that the equation (2.38) has a unique strong solution X^n for any given standard Brownian motion W in place of W^n for $n \ge 1$. We will make use of this fact in the final part of the proof below. In order to connect to the solution X of (2.1)+(2.2) constructed in the proof of Theorem 1 above, we will first construct a (strong) solution (X^n, W^n) to (2.38) by time change for every $n \ge 1$.

For this, define the measure

$$\nu_n(dx) = \frac{dx}{\sigma_n^2(x)} \tag{2.39}$$

and consider the additive functional

$$\begin{aligned} A_t^n &= \int_{I\!R} \ell_t^x(B^1)\, d\nu_n(dx) = \int_{I\!R} \frac{\ell_t^x(B^1)}{\sigma_n^2(x)}\, dx = \int_0^t \frac{ds}{\sigma_n^2(B_s^1)} \\ &= \int_0^t I(|B_s^1| > \delta_n)\, ds + \frac{1}{2\mu\delta_n} \int_0^t I(|B_s^1| \le \delta_n)\, ds \end{aligned} \tag{2.40}$$

for $t \ge 0$ and $n \ge 1$ where B^1 is a standard Brownian motion as in the beginning of the proof of Theorem 1 above. Recalling (2.6) and (2.7) we see from (2.40) that

$$A_t^n \to t + \tfrac{1}{\mu}\, \ell_t^0(B^1) = A_t \tag{2.41}$$

$$T_t^n := (A_t^n)^{-1} \to (A_t)^{-1} = T_t \tag{2.42}$$

almost surely as $n \to \infty$ for all $t \ge 0$. Since the functions in (2.41) and (2.42) are increasing and continuous, by the well-known implication (see e.g. [21, Exc. 13, p. 167]) we know that

$$A_t^n \rightrightarrows A_t \quad \& \quad T_t^n \rightrightarrows T_t \tag{2.43}$$

for $t \in [0, N]$ as $n \to \infty$ with any $N > 0$ where the double arrows denote uniform convergence (almost surely). Recalling (2.8) it follows therefore by the (uniform on compacts) continuity of $t \mapsto B_t^1$ that

$$X_t^n := B_{T_t^n}^1 \rightrightarrows B_{T_t}^1 =: X_t \tag{2.44}$$

for $t \in [0,N]$ as $n \to \infty$ with any $N > 0$. Moreover, since $t \mapsto A_t^n$ is continuous and strictly increasing with $A_t^n \uparrow \infty$ as $t \uparrow \infty$, the same (time change) arguments as those used prior (2.8) show that X^n is a continuous martingale and hence the process

$$W_t^n := \int_0^t \frac{dX_s^n}{\sigma_n(X_s^n)} \tag{2.45}$$

is a continuous martingale satisfying

$$\begin{aligned} \langle W^n, W^n \rangle_t &= \int_0^t \frac{d\langle X^n, X^n \rangle_s}{\sigma_n^2(X_s^n)} = \int_0^t \frac{d\langle B^1_{T^n}, B^1_{T^n} \rangle_s}{\sigma_n^2(B^1_{T_s^n})} = \int_0^t \frac{d\langle B^1, B^1 \rangle_{T_s^n}}{\sigma_n^2(B^1_{T_s^n})} \\ &= \int_0^{T_t^n} \frac{d\langle B^1, B^1 \rangle_s}{\sigma_n^2(B_s^1)} = \int_0^{T_t^n} \frac{ds}{\sigma_n^2(B_s^1)} = A^n_{T_t^n} = t \end{aligned} \tag{2.46}$$

for $t \ge 0$ and $n \ge 1$. By Lévy's characterisation theorem we therefore see that W^n is a standard Brownian motion for $n \ge 1$. Moreover from (2.45) we see that (X^n, W^n) is a (strong) solution to (2.38) for $n \ge 1$. Note also that

$$X_t^n = B^1_{T_t^n} = \int_0^{T_t^n} I(B_s^1 \neq 0)\, dB_s^1 = \int_0^t I(B^1_{T_s^n} \neq 0)\, dB^1_{T_s^n} = \int_0^t I(X_s^n \neq 0)\, dX_s^n \tag{2.47}$$

for $t \ge 0$ and $n \ge 1$, and the same arguments show that

$$X_t = B^1_{T_t} = \int_0^{T_t} I(B_s^1 \neq 0)\, dB_s^1 = \int_0^t I(B^1_{T_s} \neq 0)\, dB^1_{T_s} = \int_0^t I(X_s \neq 0)\, dX_s \tag{2.48}$$

for $t \ge 0$ as already established in (2.9) above. Recall also that X solves (2.1) and (2.2) where B is defined in (2.11) (upon recalling (2.5)–(2.8) above).

2. From (2.44) we see that $\{\, X^n \mid n \ge 1 \,\}$ is tight in C, and since $\{\, W^n \mid n \ge 1 \,\}$ is tight in C, it follows that $\{\, (X^n, W^n) \mid n \ge 1 \,\}$ is tight in $C \times C$. Hence by Prohorov's theorem (see e.g. [1, Section 5]) we know that

$$(X^n, W^n) \xrightarrow{\mathsf{d}} (X^\infty, W^\infty) \tag{2.49}$$

as $n \to \infty$ (possibly over a subsequence which we again denote by n for simplicity) where (X^∞, W^∞) is a random element with the given limit law (note that $X^\infty \sim X$ and W^∞ is a standard Brownian motion). We will show below that this law coincides with the jointly unique solution law for (2.1)+(2.2) derived in Theorem 1. To this end we will first apply Skorokhod's representation theorem (see e.g. [1, pp. 70–71]) and conclude that there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathsf{P}})$ and random elements $(\tilde{X}^n, \tilde{W}^n) : \tilde{\Omega} \to C \times C$ such that

$$(\tilde{X}^n, \tilde{W}^n) \sim (X^n, W^n) \;\text{ for all }\; n \in I\!N \cup \{\infty\}\,; \tag{2.50}$$

$$(\tilde{X}^n, \tilde{W}^n) \xrightarrow{\text{a.s.}} (\tilde{X}^\infty, \tilde{W}^\infty) \;\text{ as }\; n \to \infty\,. \tag{2.51}$$

A closer look in the proof of this theorem shows that the structure of this probability space is rather complicated. Note however that since both X^n and W^n are continuous martingales (with respect to the filtration $I\!F^1_{T^n}$ defined as following (2.8) above) which therefore trivially satisfy (2.29) for all $n \in I\!N$, by the vector version of (2.30) applied to the continuous semimartingale $S^n = (X^n, W^n)$ (with $H^n \equiv 0$ for $n \in I\!N$ given and fixed as well as when tending to ∞) we see that $(\tilde{X}^n, \tilde{W}^n)$ is a continuous semimartingale with respect to its natural filtration for all $n \in I\!N \cup \{\infty\}$. It follows in particular that the stochastic integral $U(\tilde{X}^n) \cdot \tilde{W}^n$ is well defined for all $n \in I\!N \cup \{\infty\}$ whenever U is a continuous function. This fact will be used in the rest of the proof with no explicit mention.

3. We show that $(\tilde{X}^\infty, \tilde{W}^\infty)$ solves the system (2.1)+(2.2). For this, choose a continuous function $U_m : I\!R \rightarrow I\!R$ such that $U_m(x) = 0$ for $|x| \le 1/m$ and $U_m(x) = 1$ for $|x| \ge 2/m$ where $m \ge 1$ is given and fixed. Then (2.51) implies that

$$(2.52) \qquad (U_m(\tilde{X}^n), \tilde{W}^n) \xrightarrow{\mathsf{p}} (U_m(\tilde{X}^\infty), \tilde{W}^\infty)$$

as $n \rightarrow \infty$. It follows therefore by (2.31) that

$$(2.53) \qquad U_m(\tilde{X}^n) \cdot \tilde{W}^n \xrightarrow{\mathsf{p}} U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty$$

and hence again by (2.51) we find that

$$(2.54) \qquad (\tilde{X}^n, U_m(\tilde{X}^n) \cdot \tilde{W}^n) \xrightarrow{\mathsf{p}} (\tilde{X}^\infty, U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty)$$

as $n \rightarrow \infty$. From (2.50) we see that $(\tilde{X}^n, U_m(\tilde{X}^n) \cdot \tilde{W}^n) \sim (X^n, U_m(X^n) \cdot W^n)$ for all $n \ge 1$ and hence by (2.54) it follows that

$$(2.55) \qquad (X^n, U_m(X^n) \cdot W^n) \xrightarrow{\mathsf{d}} (\tilde{X}^\infty, U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty)$$

as $n \rightarrow \infty$. By the continuous mapping theorem this implies that

$$(2.56) \qquad X^n - U_m(X^n) \cdot W^n \xrightarrow{\mathsf{d}} \tilde{X}^\infty - U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty$$

as $n \rightarrow \infty$. Using (2.45) and taking $\delta_n \le 1/m$ for $n \ge n_m$ large enough upon recalling definitions of σ_n and U_m we find by (2.43) and (2.44) that

$$\begin{aligned}(2.57) \qquad X^n_t - \int_0^t U_m(X^n_s)\, dW^n_s &= X^n_t - \int_0^t U_m(X^n_s)\, \frac{dX^n_s}{\sigma_n(X^n_s)} \\ &= X^n_t - \int_0^t U_m(X^n_s)\, dX^n_s = B^1_{T^n_t} - \int_0^t U_m(B^1_{T^n_s})\, dB^1_{T^n_s} \\ &= B^1_{T^n_t} - \int_0^{T^n_t} U_m(B^1_s)\, dB^1_s \rightrightarrows B^1_{T_t} - \int_0^{T_t} U_m(B^1_s)\, dB^1_s \\ &= X_t - \int_0^t U_m(B^1_{T_s})\, dB^1_{T_s} = X_t - \int_0^t U_m(X_s)\, dX_s\end{aligned}$$

for $t \in [0, N]$ as $n \rightarrow \infty$ with any $N > 0$. Combining this with (2.56) we see that

$$(2.58) \qquad \tilde{X}^\infty - U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty \sim X - U_m(X) \cdot X\, .$$

By Markov's inequality and Doob's maximal inequality we find that

$$\mathsf{P}\Big(\sup_{0\le t\le N}\Big|X_t-\int_0^t I(X_s\!\neq\!0)\,dX_s-\Big(X_t-\int_0^t U_m(X_s)\,dX_s\Big)\Big|>\varepsilon\Big) \tag{2.59}$$
$$\le\frac{4}{\varepsilon^2}\,\mathsf{E}\Big[\int_0^N\big(I(X_s\!\neq\!0)-U_m(X_s)\big)^2\,d\langle X,X\rangle_s\Big]$$
$$=\frac{4}{\varepsilon^2}\,\mathsf{E}\Big[\int_0^N\big(I(X_s\!\neq\!0)-U_m(X_s)\big)^2\,I(X_s\!\neq\!0)\,ds\Big]\longrightarrow 0$$

as $m\to\infty$ by the dominated convergence theorem for any $N>0$. This shows that

$$X-U_m(X)\cdot X\stackrel{\mathsf{P}}{\longrightarrow}X-I(X\!\neq\!0)\cdot X=0 \tag{2.60}$$

as $m\to\infty$ where we also use (2.48) above. Combining this with (2.58) we see that

$$\tilde X^\infty-U_m(\tilde X^\infty)\cdot\tilde W^\infty\stackrel{\mathsf{P}}{\longrightarrow}0 \tag{2.61}$$

as $m\to\infty$. The same arguments as in (2.59) also imply that

$$\tilde X^\infty-U_m(\tilde X^\infty)\cdot\tilde W^\infty\stackrel{\mathsf{P}}{\longrightarrow}\tilde X^\infty-I(\tilde X^\infty\!\neq\!0)\cdot\tilde W^\infty \tag{2.62}$$

as $m\to\infty$. Comparing (2.61) and (2.62) we see that

$$\tilde X_t^\infty=\int_0^t I(\tilde X_s^\infty\!\neq\!0)\,d\tilde W_s^\infty \tag{2.63}$$

for all $t\ge 0$. Moreover, due to $\tilde X^n\sim X^n$ for all $n\ge 1$ we see that $\tilde X^n\stackrel{\mathrm{d}}{\to}\tilde X^\infty$ and $X^n\stackrel{\mathrm{d}}{\to}X$ as $n\to\infty$ imply that $\tilde X^\infty\sim X$. Since X satisfies (2.2) hence it follows that

$$\int_0^t I(\tilde X_s^\infty\!=\!0)\,ds=\tfrac{1}{\mu}\,\ell_t^0(\tilde X^\infty) \tag{2.64}$$

for all $t\ge 0$. From (2.63)+(2.64) we see that $(\tilde X^\infty,\tilde W^\infty)$ solves the system (2.1)+(2.2) as claimed. It follows in particular that the limit law in (2.49) coincides with the jointly unique solution law for (2.1)+(2.2) derived in Theorem 1 as stated prior to (2.50)+(2.51) above.

4. Suppose now that the system (2.1)+(2.2) has a strong solution (X,B) defined on some probability space $(\Omega,\mathcal{F},\mathsf{P})$. Then there exists a measurable functional $f:C\to C$ such that

$$X=f(B)\,. \tag{2.65}$$

Return to the beginning of the proof and apply Nakao's theorem to find a unique strong solution Y^n to the equation (2.38) driven by B so that

$$dY_t^n=\sigma_n(Y_t^n)\,dB_t \tag{2.66}$$

with $Y_0^n=0$ for $n\ge 1$. Since (Y^n,B) is a strong solution to (2.66) we know that there exists a measurable functional $f_n:C\to C$ such that

$$Y^n=f_n(B) \tag{2.67}$$

for $n \geq 1$. By the joint uniqueness in law for the equation (2.66) (which follows directly from the uniqueness in law since σ_n in (2.66) is strictly positive) we know that

$$(Y^n, B) \sim (X^n, W^n) \tag{2.68}$$

for $n \geq 1$. It follows therefore by (2.50)+(2.51) that

$$(Y^n, B) \xrightarrow{\mathsf{d}} (\tilde{X}^\infty, \tilde{W}^\infty) \tag{2.69}$$

as $n \to \infty$. Since $(\tilde{X}^\infty, \tilde{W}^\infty)$ solves (2.1)+(2.2) we know by the joint uniqueness in law established in Theorem 1 for this system that

$$(\tilde{X}^\infty, \tilde{W}^\infty) \sim (X, B)\,. \tag{2.70}$$

Combining (2.65)+(2.67) and (2.69)+(2.70) we see that

$$(f_n(B), B) \xrightarrow{\mathsf{d}} (f(B), B) \tag{2.71}$$

as $n \to \infty$. By Lemma 2 we can therefore conclude that

$$Y^n = f_n(B) \xrightarrow{\mathsf{P}} f(B) = X \tag{2.72}$$

as $n \to \infty$. In Proposition 4 below we will see however that Y^n does not converge in probability as $n \to \infty$. This fact contradicts (2.72) and we can therefore conclude that the system (2.1)+(2.2) has no strong solution as claimed. $\square$

2.5. In order to conclude the proof we need the following negative result on the regularisation of the stochastic differential equation for sticky Brownian motion.

Proposition 4. *Let Y^n be the unique strong solution to*

$$dY_t^n = \sigma_n(Y_t^n)\, dB_t \tag{2.73}$$

satisfying $Y_0^n = 0$ where $\sigma_n(x) = I(|x| > \delta_n) + \sqrt{2\mu\delta_n}\, I(|x| \leq \delta_n)$ for $x \in I\!R$ and $n \geq 1$ with $\delta_n \downarrow 0$ as $n \to \infty$. Then Y^n does not converge in probability (relative to d_∞) as $n \to \infty$.

Proof. 1. Passing to a subsequence if needed (which we again denote by n for simplicity) we can assume that $\delta_{n+1} \leq \delta_n/4$ for all $n \geq 1$. Set $Z^n = Y^n - Y^{n+1}$ for $n \geq 1$ given and fixed and note that

$$Z_t^n = \int_0^t \big(\sigma_n(Y_s^n) - \sigma_{n+1}(Y_s^{n+1})\big)\, dB_s \tag{2.74}$$

for $t \geq 0$. From (2.74) we see that Z^n is a martingale with

$$\begin{aligned} \langle Z^n, Z^n \rangle_t &= \int_0^t \big(\sigma_n(Y_s^n) - \sigma_{n+1}(Y_s^{n+1})\big)^2\, ds \\ &= \big(\sqrt{2\mu\delta_n} - \sqrt{2\mu\delta_{n+1}}\big)^2 \int_0^t I\big(|Y_s^n| \leq \delta_n\,,\, |Y_s^{n+1}| \leq \delta_{n+1}\big)\, ds \end{aligned} \tag{2.75}$$

$$+ \big(1-\sqrt{2\mu\delta_n}\big)^2 \int_0^t I\big(|Y_s^n| \le \delta_n\,,\, |Y_s^{n+1}| > \delta_{n+1}\big)\, ds$$
$$+ \big(1-\sqrt{2\mu\delta_{n+1}}\big)^2 \int_0^t I\big(|Y_s^n| > \delta_n\,,\, |Y_s^{n+1}| \le \delta_{n+1}\big)\, ds$$

for $t \ge 0$ from where we also see that the process $(Z^n)^2 - \langle Z^n, Z^n \rangle$ is a martingale.

2. We will now show that Z^n does not converge in probability (relative to d_∞) as $n \to \infty$. For this, note that $Z_0^n = 0$ and introduce the stopping time

$$\tau_\varepsilon^n = \inf\{\, t \ge 0 \mid |Z_t^n| = \varepsilon \,\} \tag{2.76}$$

for $\varepsilon > 0$ given and fixed (where we formally set $\inf \emptyset = \infty$). Next observe that

$$\mathsf{P}\Big(\sup_{0 \le s \le t} |Z_s^n| \ge \varepsilon \Big) = \mathsf{P}\big(\tau_\varepsilon^n \le t\big) \ge \frac{1}{t}\, \mathsf{E}\Big[I(\tau_\varepsilon^n \le t) \int_0^t I(|Y_s^n| \le \delta_n)\, ds \Big] \tag{2.77}$$
$$\ge \frac{1}{t}\, \mathsf{E} \int_0^t I(|Y_s^n| \le \delta_n)\, ds - \frac{1}{t}\, \mathsf{E} \int_0^{t \wedge \tau_\varepsilon^n} I(|Y_s^n| \le \delta_n)\, ds$$

for any $t > 0$ given and fixed.

3. To bound the first expectation on the right-hand side of (2.77) recall that $Y^n \sim X^n$ and $X^n = B^1_{T^n}$ so that we have

$$\mathsf{E} \int_0^t I(|Y_s^n| \le \delta_n)\, ds = \mathsf{E} \int_0^t I(|X_s^n| \le \delta_n)\, ds = \mathsf{E} \int_0^t I(|B^1_{T_s^n}| \le \delta_n)\, dA^n_{T_s^n} \tag{2.78}$$
$$= \mathsf{E} \int_0^{T_t^n} I(|B_s^1| \le \delta_n)\, dA_s^n = \mathsf{E}\Big(\frac{1}{2\mu\delta_n} \int_0^{T_t^n} I(|B_s^1| \le \delta_n)\, ds \Big)\,.$$

Letting $n \to \infty$ and using that $T_t^n \to T_t$ almost surely it follows by Fatou's lemma that

$$\liminf_{n\to\infty} \mathsf{E} \int_0^t I(|Y_s^n| \le \delta_n)\, ds \ge \mathsf{E}\Big(\liminf_{n\to\infty} \frac{1}{2\mu\delta_n} \int_0^{T_t^n} I(|B_s^1| \le \delta_n)\, ds \Big) \tag{2.79}$$
$$= \frac{1}{\mu}\, \mathsf{E}\, \ell^0_{T_t}(B^1) = \frac{1}{\mu}\, \mathsf{E}\, \ell^0_t(X) > 0\,.$$

4. To bound the second expectation on the right-hand side of (2.77) note that (2.75) and the inclusion $\{\, |Y_s^n| \le \delta_n\,, |Y_s^{n+1}| \le \delta_{n+1} \,\} \subseteq \{\, |Z_s^n| \le 2\delta_n \,\}$ yield

$$\mathsf{E} \int_0^{t \wedge \tau_\varepsilon^n} I(|Y_s^n| \le \delta_n)\, ds \tag{2.80}$$
$$= \mathsf{E} \int_0^{t \wedge \tau_\varepsilon^n} I(|Y_s^n| \le \delta_n\,,\, |Y_s^{n+1}| \le \delta_{n+1})\, ds + \mathsf{E} \int_0^{t \wedge \tau_\varepsilon^n} I(|Y_s^n| \le \delta_n\,,\, |Y_s^{n+1}| > \delta_{n+1})\, ds$$
$$\le \frac{2}{\mu\delta_n}\, \mathsf{E} \int_0^{t \wedge \tau_\varepsilon^n} I(|Y_s^n| \le \delta_n\,,\, |Y_s^{n+1}| \le \delta_{n+1})\, d\langle Z^n, Z^n \rangle_s + 4\, \mathsf{E} \langle Z^n, Z^n \rangle_{t \wedge \tau_\varepsilon^n}$$
$$\le \frac{2}{\mu\delta_n}\, \mathsf{E} \int_0^{t \wedge \tau_\varepsilon^n} I(|Z_s^n| \le 2\delta_n)\, d\langle Z^n, Z^n \rangle_s + 4\, \mathsf{E} \langle Z^n, Z^n \rangle_{t \wedge \tau_\varepsilon^n}$$

$$= \frac{8}{\mu}\,\frac{1}{4\delta_n}\int_{-2\delta_n}^{2\delta_n} \mathsf{E}\,\ell^z_{t\wedge\tau^n_\varepsilon}(Z^n)\,dz + 4\,\mathsf{E}\big(Z^n_{t\wedge\tau^n_\varepsilon}\big)^2$$

where in the first inequality we use that $(\sqrt{2\mu\delta_n}-\sqrt{2\mu\delta_{n+1}})^2 \geq \mu\delta_n/2$ (due to $\delta_{n+1} \leq \delta_n/4$) and $(1-\sqrt{2\mu\delta_n})^2 \geq 1/4$ whenever $\delta_n \in (0, 1/8\mu)$. From the Itô–Tanaka formula we find that

$$\ell^z_{t\wedge\tau^n_\varepsilon}(Z^n) = |Z^n_{t\wedge\tau^n_\varepsilon}-z| - |z| - \int_0^{t\wedge\tau^n_\varepsilon} \mathrm{sign}(Z^n_s-z)\,dZ^n_s \tag{2.81}$$
$$\leq |Z^n_{t\wedge\tau^n_\varepsilon}| - \int_0^{t\wedge\tau^n_\varepsilon} \mathrm{sign}(Z^n_s-z)\,dZ^n_s$$

so that taking expectations on both sides we get

$$\mathsf{E}\,\ell^z_{t\wedge\tau^n_\varepsilon}(Z^n) \leq \mathsf{E}|Z^n_{t\wedge\tau^n_\varepsilon}| \tag{2.82}$$

for all $z \in I\!\!R$. Inserting this into (2.80) we can conclude that

$$\mathsf{E}\int_0^{t\wedge\tau^n_\varepsilon} I(|Y^n_s| \leq \delta_n)\,ds \leq \tfrac{8}{\mu}\,\varepsilon + 4\varepsilon^2 \tag{2.83}$$

whenever $\delta_n \in (0, 1/8\mu)$.

5. Letting $n \to \infty$ in (2.77) and using (2.79)+(2.83) we find that

$$\liminf_{n\to\infty}\, \mathsf{P}\Big(\sup_{0\leq s\leq t} |Z^n_s| \geq \varepsilon\Big) \geq \frac{1}{\mu}\,\mathsf{E}\ell^0_t(X) - \frac{1}{t}\Big(\tfrac{8}{\mu}\,\varepsilon + 4\varepsilon^2\Big) > 0 \tag{2.84}$$

for any sufficiently small $\varepsilon > 0$ given and fixed. From (2.84) we see that Z^n does not converge in probability (relative to d_∞) as $n \to \infty$ and hence Y^n does not converge either. This completes the proof of Proposition 4 and Theorem 3 as claimed. $\square$

3. Sticky reflecting Brownian motion

In this section we consider the SDE system

$$dX_t = \tfrac{1}{2}\,d\ell^0_t(X) + I(X_t\!>\!0)\,dB_t \tag{3.1}$$

$$I(X_t\!=\!0)\,dt = \tfrac{1}{2\mu}\,d\ell^0_t(X) \tag{3.2}$$

for reflecting Brownian motion X in $I\!\!R_+$ sticky at 0 , where $X_0 = x$ in $I\!\!R_+$, $\mu \in (0,\infty)$ is a given constant, $\ell^0(X)$ is the local time of X at 0 , and B is a standard Brownian motion.

The notion of solution to (3.1)+(3.2) (weak and strong) and its uniqueness (in law and pathwise) are analogous to those given in the beginning of Section 2 for the system (2.1)+(2.2). The same is true for the comments on non-uniqueness of the solution to (3.1) and its specification by (3.2) (observe that the diffusion coefficient $x \mapsto I(x > 0)$ in (3.1) vanishes at 0). Recall that $\ell^0_t(X) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0}\,(1/\varepsilon)\int_0^t I(0 \leq X_s \leq \varepsilon)\,d\langle X,X\rangle_s$ in (3.2) is the right local time of X at 0 with $\langle X,X\rangle_t = \int_0^t I(X_s\!>\!0)\,ds$ for $t \geq 0$ (cf. footnote 1 above).

We will see below that the system (3.1)+(3.2) is equivalent to the single equation

$$dX_t = \mu I(X_t\!=\!0)\,dt + I(X_t\!>\!0)\,dB_t \tag{3.3}$$

with $X_0 = x$ in $\mathbb{R}_+$ obtained by incorporating (3.2) into (3.1). This equivalence is understood in the sense that X solves (3.1)+(3.2) if and only if X solves (3.3). Recall that a pair of $\mathbb{F}$-adapted stochastic processes (X, B) defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ is said to be a *solution* to (3.3) if B is a standard Brownian motion that is a martingale with respect to $\mathbb{F}$ and the integral equation

$$X_t = x + \mu \int_0^t I(X_s\!=\!0)\, ds + \int_0^t I(X_s\!>\!0)\, dB_s \tag{3.4}$$

is satisfied for all $t \geq 0$ where the integral with respect to B is understood in Itô's sense. Every solution (X, B) is also referred to as a *weak solution*. A solution (X, B) is called a *strong solution* if X is adapted to the natural filtration $\mathbb{F}^B$ of B. Recall that (i) *uniqueness in law* and (ii) *joint uniqueness in law* hold for the equation (3.3) if for any two solutions (X^1, B^1) and (X^2, B^2) that are not necessarily defined on the same filtered probability space we have (i) $X^1 \sim X^2$ and (ii) $(X^1, B^1) \sim (X^2, B^2)$ respectively. Solutions that are jointly unique in law are called *jointly unique weak solutions*. It is well known that uniqueness in law is equivalent to joint uniqueness in law for a large class of SDEs including the equation (3.3). Recall also that *pathwise uniqueness* holds for the equation (3.3) if for any two solutions (X^1, B) and (X^2, B) defined on the same filtered probability space we have $X_t^1 = X_t^2$ outside a set of probability measure zero for all $t \geq 0$. It is evident that strong existence implies weak existence and it is well known that pathwise uniqueness implies uniqueness in law. For further details on all these claims including references see the paragraph containing (3.24) below.

1. *Weak existence and uniqueness.* Note that the diffusion coefficient $x \mapsto I(x\!>\!0)$ in the equations (3.1) and (3.3) is both discontinuous and degenerate (taking value zero) so that the standard results on the existence and uniqueness of solutions are not directly applicable (see e.g. [12, Chapter IV]). A basic existence and uniqueness result for the system (3.1)+(3.2) and the equation (3.3) may now be stated as follows.

Theorem 5. *The system* (3.1)+(3.2) *is equivalent to the equation* (3.3) *and they both have a jointly unique weak solution which is non-negative.*

Proof. Clearly if X solves (3.1)+(3.2) then X solves (3.3) so that it is enough to show the converse. This can be done in two steps as follows.

1. We show that if X solves (3.3) then $X_t \geq 0$ almost surely for all $t \geq 0$. For this, apply the Itô–Tanaka formula to $F(x) = x^-$ composed with X solving (3.3) and note that $F'_+(x) = -I(x\!<\!0)$ for $x \in \mathbb{R}$. This yields

$$\begin{aligned} X_t^- &= -\int_0^t I(X_s\!<\!0)\, dX_s + \tfrac{1}{2}\ell_t^{0-}(X) \\ &= -\int_0^t I(X_s\!<\!0)\Big(\mu I(X_s\!=\!0)\, ds + I(X_s\!>\!0)\, dB_s\Big) + \tfrac{1}{2}\ell_t^{0-}(X) = 0 \end{aligned} \tag{3.5}$$

where we use that the left local time $\ell_t^{0-}(X) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0}(1/\varepsilon)\int_0^t I(-\varepsilon \leq X_s \leq 0)\, d\langle X, X\rangle_s$ equals 0 since $d\langle X, X\rangle_s = I(X_s\!>\!0)\, ds$ for $0 \leq s \leq t$. Thus $X_t^- = 0$ and hence $X_t \geq 0$ (both almost surely) for all $t \geq 0$ as claimed.

2. We show that if X solves (3.3) then X satisfies (3.2). For this, apply the Itô–Tanaka formula to $F(x) = x^+$ composed with X solving (3.3) and note that $F'_-(x) = I(x>0)$ for $x \in I\!R$. This yields

$$X_t^+ = x + \int_0^t I(X_s>0)\, dX_s + \tfrac{1}{2}\,\ell_t^0(X) \tag{3.6}$$
$$= x + \int_0^t I(X_s>0)\Big(\mu I(X_s=0)\, ds + I(X_s>0)\, dB_s\Big) + \tfrac{1}{2}\,\ell_t^0(X)$$
$$= x + \int_0^t I(X_s>0)\, dB_s + \tfrac{1}{2}\,\ell_t^0(X)$$

for $t \geq 0$. Recalling that $X_t = X_t^+$ and comparing (3.6) with (3.3) we see that (3.2) holds as claimed. But then X also solves (3.1) showing that (3.1)+(3.2) and (3.3) are equivalent. In the rest of the proof we only refer to (3.1)+(3.2) but the existence and uniqueness claims also hold for (3.3) by the equivalence just established.

3. We show that the system (3.1)+(3.2) has a weak solution. For this, let X^1 be the weak solution to the system (2.1)+(2.2) constructed in the proof of Theorem 1 with $X_0^1 = x$ in $I\!R_+$. Set $X = |X^1|$ and recall from (2.8) that $X = |x + B_T^1|$ where T is the time change given by (2.6) and (2.7). Using that X^1 solves (2.1) (with B defined in (2.11) above) we find by the Itô–Tanaka formula that

$$X_t = |X_t^1| = x + \int_0^t \operatorname{sign}(X_s^1)\, dX_s^1 + \ell_t^0(X^1) \tag{3.7}$$
$$= x + \int_0^t \operatorname{sign}(X_s^1)\, I(X_s^1 \neq 0)\, dB_s + \tfrac{1}{2}\,\ell_t^0(|X^1|)$$
$$= x + \int_0^t I(X_s>0)\, d\hat{B}_s + \tfrac{1}{2}\,\ell_t^0(X)$$

where in the third equality we use that the right and symmetric local times for the continuous martingale X^1 coincide (cf. footnote 1 above) and $\hat{B}_t = \int_0^t \operatorname{sign}(X_s^1)\, dB_s$ for $t \geq 0$ is a standard Brownian motion (by Lévy's characterisation theorem). From (3.7) we see that X solves (3.1) with B equal to $\hat{B}$. Moreover, using that X^1 satisfies (2.2) we find that

$$\int_0^t I(X_s=0)\, ds = \int_0^t I(X_s^1=0)\, ds = \tfrac{1}{\mu}\,\ell_t^0(X^1) = \tfrac{1}{2\mu}\,\ell_t^0(|X^1|) = \tfrac{1}{2\mu}\,\ell_t^0(X) \tag{3.8}$$

for $t \geq 0$. This shows that X satisfies (3.2) and the proof of weak existence is complete.

4. We show that uniqueness in law holds for the system (3.1)+(3.2). For this, we will undo the time change from the previous part of the proof starting with the notation afresh. Suppose that X and B solve (3.1) subject to (3.2). Consider the additive functional

$$T_t = \int_0^t I(X_s>0)\, ds \tag{3.9}$$

for $t \geq 0$ and note that $T_t \uparrow \infty$ as $t \uparrow \infty$. Indeed, to verify this claim set

$$M_t = \int_0^t I(X_s>0)\, dB_s \tag{3.10}$$

for $t \geq 0$. Then M is a continuous martingale with $\langle M, M \rangle_t = T_t \uparrow T_\infty$ as $t \uparrow \infty$. It follows therefore that $M_t \to M_\infty$ in $I\!R$ almost surely on $\{T_\infty < \infty\}$ as $t \to \infty$. Setting further $T_t^0 = \int_0^t I(X_s \!=\! 0)\, ds$ and noting that $T_t + T_t^0 = t$ for $t \geq 0$ we see that $T_t^0 \uparrow \infty$ on $\{T_\infty < \infty\}$ as $t \uparrow \infty$. But then $X_t = x + M_t + \mu T_t^0 \to \infty$ almost surely on $\{T_\infty < \infty\}$ as $t \to \infty$ contradicting the fact that $T_t^0 \uparrow \infty$ on $\{T_\infty < \infty\}$ unless its probability is zero. This shows that $T_\infty = \infty$ (with probability one) as claimed.

Since $T_t \uparrow \infty$ as $t \uparrow \infty$ it follows that its (right) inverse $t \mapsto A_t$ defined by

$$A_t = \inf \{\, s \geq 0 \mid T_s > t \,\} \tag{3.11}$$

is finite for all $t \geq 0$. Note that $t \mapsto A_t$ is increasing and right-continuous on $I\!R_+$. Moreover, since $T = (T_t)_{t \geq 0}$ is adapted to $I\!F$ it follows that each A_t is a stopping time with respect to $I\!F$, so that $A = (A_t)_{t \geq 0}$ defines a time change with respect to $I\!F$.

Consider the time-changed process

$$R_t = X_{A_t} \tag{3.12}$$

for $t \geq 0$. Note that (3.1) yields

$$R_t = x + \tfrac{1}{2} \ell_{A_t}^0(X) + \int_0^{A_t} I(X_s \!>\! 0)\, dB_s = x + \tfrac{1}{2} \ell_t^0(R) + M_{A_t} \tag{3.13}$$

where we use that $t \mapsto T_t$ is constant on each $[A_{s-}, A_s]$ and therefore the same is true for $t \mapsto M_t$ so that $\langle X, X \rangle_{A_t} = \langle M, M \rangle_{A_t} = T_{A_t} = t$ and $\langle R, R \rangle_t = \langle M_A, M_A \rangle_t = \langle M, M \rangle_{A_t} = T_{A_t} = t$ for $t \geq 0$. Since M is a continuous martingale with respect to $I\!F$ we know that

$$B_t^1 := M_{A_t} \tag{3.14}$$

is a continuous martingale with respect to $I\!F_A$ satisfying $\langle B^1, B^1 \rangle_t = t$. It follows therefore by Lévy's characterisation theorem that B^1 is a standard Brownian motion. Using Skorokhod's lemma (see e.g. [19, p. 239]) we find that (3.13) can be rewritten as follows

$$R_t = B_t^1 - \Big((-x) \wedge \inf_{0 \leq s \leq t} B_s^1 \Big) \tag{3.15}$$

for $t \geq 0$ showing that R is a reflecting Brownian motion starting at x in $I\!R_+$. Moreover, recalling (3.2) we see that

$$\begin{aligned} t = T_{A_t} &= \int_0^{A_t} I(X_s \!>\! 0)\, ds = A_t - \int_0^{A_t} I(X_s \!=\! 0)\, ds \\ &= A_t - \tfrac{1}{2\mu} \ell_{A_t}^0(X) = A_t - \tfrac{1}{2\mu} \ell_t^0(X_A) = A_t - \tfrac{1}{2\mu} \ell_t^0(R) \end{aligned} \tag{3.16}$$

from where it follows that

$$A_t = t + \tfrac{1}{2\mu} \ell_t^0(R) \tag{3.17}$$

for $t \geq 0$. This shows that $t \mapsto A_t$ is strictly increasing (and continuous) and hence

$$T_t = A_t^{-1} \tag{3.18}$$

is the proper inverse for $t \ge 0$ (implying also that $t \mapsto T_t$ is strictly increasing and continuous). It follows in particular that $A_{T_t} = t$ so that

$$X_t = X_{A_{T_t}} = R_{T_t} \tag{3.19}$$

for $t \ge 0$. From (3.15) and (3.17)–(3.19) we see that X is a well-determined measurable functional of the standard Brownian motion B^1. This shows that the law of X solving (3.1)+(3.2) is uniquely determined and this completes the proof of weak uniqueness.

5. We show that joint uniqueness in law holds for the system (3.1)+(3.2). Note that this fact can be deduced from the general result on the equivalence between uniqueness in law and joint uniqueness in law for (3.3) reviewed in the paragraph containing (3.24) below upon recalling that the system (3.1)+(3.2) is equivalent to the equation (3.3) as established above. To describe the underlying structure in fuller detail we will present different arguments as follows. For this, we will continue our considerations starting with (3.9) above upon assuming that X and B solve (3.1) subject to (3.2). Note from (3.1) that

$$B_t = X_t - \tfrac{1}{2}\ell_t^0(X) - x + M_t^0 \tag{3.20}$$

where the process M^0 is defined by

$$M_t^0 = \int_0^t I(X_s\!=\!0)\, dB_s \tag{3.21}$$

for $t \ge 0$. Note further that M^0 is a continuous martingale with respect to $I\!\!F$ such that $\langle M^0, M^0\rangle_t = \int_0^t I(X_s\!=\!0)\, ds = T_t^0 \uparrow \infty$ almost surely as $t \uparrow \infty$. Indeed, to see this recall that $T_t = \langle M, M\rangle_t \uparrow \infty$ almost surely as $t \uparrow \infty$ so that $\liminf_{t\to\infty} M_t = -\infty$ and $\limsup_{t\to\infty} M_t = +\infty$ almost surely from where the claim follows from (3.1) using (3.2) since otherwise the left-hand side of (3.1) could not remain non-negative. Defining the (right) inverse of $t \mapsto T_t^0$ by $A_t^0 = \inf\{\, s \ge 0 \mid T_s^0 > t \,\}$ it then follows by the Dambis–Dubins–Schwarz theorem that

$$B_t^0 = M_{A_t^0}^0 \tag{3.22}$$

is a standard Brownian motion such that

$$M_t^0 = B_{T_t^0}^0 \tag{3.23}$$

for $t \ge 0$. Moreover, it is clear from (3.9)+(3.11) that (3.14) can also be seen as an application of the Dambis–Dubins–Schwarz theorem for the continuous martingale M, and since M and M^0 are orthogonal in the sense that $\langle M, M^0\rangle_t = \int_0^t I(X_s \ne 0)\, I(X_s = 0)\, ds = 0$ for all $t \ge 0$, it follows by Knight's theorem that the standard Brownian motions B^1 and B^0 are independent. Because T_t^0 is a measurable functional of X we see from (3.15) with (3.17)–(3.19) and (3.20)+(3.23) that (X, B) is a well-determined measurable functional of the two-dimensional Brownian motion (B^1, B^0). This shows that the law of (X, B) solving (3.1)+(3.2) is uniquely determined as claimed. This completes the proof. $\square$

2. *No strong existence.* We continue by proving in Theorem 6 below that the system (3.1)+(3.2) and the equation (3.3) have no strong solution thus verifying Skorokhod's conjecture in

this case as well. This will be done in parallel to the proof of Theorem 3 and in addition to the results and facts reviewed there we shall also make use of the following general facts.

2.1. Recalling that the system (3.1)+(3.2) is equivalent to the equation (3.3) we may focus on the latter equation and recall that *strong existence* (SE) implies *weak existence* (WE) and *pathwise uniqueness* (PU) implies *uniqueness in law* (UiL) of the solution. In Theorem 5 above we have shown that both WE and UiL hold for (3.3). In Theorem 6 below we will show that SE fails. To this end recall that the following equivalence

$$\mathrm{WE} + \mathrm{PU} \iff \mathrm{SE} + \mathrm{UiL} \tag{3.24}$$

is valid for (3.3). The right-arrow implication was proved by Yamada and Watanabe [26] (see [20, pp. 151–155]) and the left-arrow implication was established by a number of authors (see Engelbert [6] and the references therein) including Cherny [2] who showed that *uniqueness in law* (UiL) is equivalent to *joint uniqueness in law* (JUiL) for a large class of SDEs including the equation (3.3). These facts hold for all SDEs considered in the present paper and we use them throughout with no explicit mention. Because UiL holds for (3.3) it follows from the equivalence (3.24) that SE would be disproved if we show that PU fails. A closer inspection of the proof below reveals that the latter fact can be established, however, the construction of two pathwise different solutions (driven by the same Brownian motion) is complicated and we will rather disprove SE directly in a somewhat simpler manner. Since WE holds for (3.3) it follows from the equivalence (3.24) that this will also disprove PU for (3.3).

2.2. The main result on strong existence of the stochastic differential system/equation for sticky reflecting Brownian motion may now be stated as follows.

Theorem 6. *The system* (3.1)+(3.2) *and the equation* (3.3) *have no strong solution.*

Proof. The central idea of the proof is the same as in the proof of Theorem 3 above. We begin by defining a regularised equation and constructing the approximating sequence of strong solutions by time change.

1. Fix any sequence $\delta_n \downarrow 0$ as $n \to \infty$ and consider the equation (2.38) where σ_n is given by $\sigma_n(x) = I(|x| > \delta_n) + \sqrt{2\mu\delta_n}\, I(|x| \le \delta_n)$ for $x \in I\!\!R$ and $n \ge 1$. Proceeding as in (2.39)–(2.43) we can conclude as in (2.44) that

$$X_t^{1,n} := B_{T_t^n}^1 \rightrightarrows B_{T_t}^1 =: X_t^1 \tag{3.25}$$

for $t \in [0, N]$ as $n \to \infty$ with any $N > 0$. As in (2.45) we can construct a standard Brownian motion $W^{1,n}$ such that

$$dX_t^{1,n} = \sigma_n(X_t^{1,n})\, dW_t^{1,n} \tag{3.26}$$

with $X_0^{1,n} = 0$ for $n \ge 1$. Recall also that X^1 solves (2.1) and (2.2) with B given by (2.11) with X^1 in place of X. It follows from (3.25) that

$$X_t^n := |X_t^{1,n}| \rightrightarrows |X_t^1| =: X_t \tag{3.27}$$

for $t \in [0,N]$ as $n \to \infty$ with any $N>0$. Using that $X^{1,n}$ solves (3.26) we find by the Itô–Tanaka formula as in (3.7) above that

$$dX^n_t = \tfrac{1}{2}\, d\ell^0_t(X^n) + \sigma_n(X^n_t)\, dW^n_t \tag{3.28}$$

with $X^n_0 = 0$ where $W^n_t = \int_0^t \mathrm{sign}(X^{1,n}_s)\, dW^{1,n}_s$ for $t \ge 0$ is a standard Brownian motion. Similarly, using that X^1 solves (2.1) we find by the Itô–Tanaka formula as in (3.7) above that X solves (3.1) with $X_0 = 0$ and $\hat{B}$ in place of B where $\hat{B}_t = \int_0^t \mathrm{sign}(X^1_s)\, dB_s$ for $t \ge 0$ is a standard Brownian motion. Moreover, using that X^1 satisfies (2.2) we find as in (3.8) above that X satisfies (3.2). Finally, since $t \mapsto \ell^0_t(|B^1|)$ is continuous (uniformly on compacts) we see that (2.43) also implies that

$$\ell^0_t(X^n) = \ell^0_t(|X^{1,n}|) = \ell^0_t(|B^1_{T^n}|) = \ell^0_{T^n_t}(|B^1|) \rightrightarrows \ell^0_{T_t}(|B^1|) = \ell^0_t(|B^1_T|) = \ell^0_t(X) \tag{3.29}$$

for $t \in [0,N]$ as $n \to \infty$ with any $N>0$.

2. From (3.27) and (3.29) we see that $\{\, X^n \mid n \ge 1 \,\}$ and $\{\, \ell^0(X^n) \mid n \ge 1 \,\}$ are tight in C, and since $\{\, W^n \mid n \ge 1 \,\}$ is tight in C, it follows that $\{\, (X^n, W^n, \ell^0(X^n)) \mid n \ge 1 \,\}$ is tight in $C \times C \times C$. Hence by Prohorov's theorem we know that

$$(X^n, W^n, \ell^0(X^n)) \xrightarrow{\;\mathsf{d}\;} (X^\infty, W^\infty, \ell^0(X^\infty)) \tag{3.30}$$

as $n \to \infty$ (possibly over a subsequence which we again denote by n for simplicity) where $(X^\infty, W^\infty, \ell^0(X^\infty))$ is a random element with the given limit law (note that $(X^\infty, \ell^0(X^\infty)) \sim (X, \ell^0(X))$ due to (3.27)+(3.29) and W^∞ is a standard Brownian motion). We will show below that this three-dimensional law coincides with the corresponding three-dimensional law arising from the jointly unique solution law for (3.1)+(3.2) (or equivalently (3.3)) derived in Theorem 5. To this end we will first apply Skorokhod's representation theorem as in the proof of Theorem 3 above and conclude that there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathsf{P}})$ and random elements $(\tilde{X}^n, \tilde{W}^n, \ell^0(\tilde{X}^n)) : \tilde{\Omega} \to C \times C \times C$ such that

$$(\tilde{X}^n, \tilde{W}^n, \ell^0(\tilde{X}^n)) \sim (X^n, W^n, \ell^0(X^n)) \;\text{ for all }\; n \in I\!N \cup \{\infty\}\,; \tag{3.31}$$

$$(\tilde{X}^n, \tilde{W}^n, \ell^0(\tilde{X}^n)) \xrightarrow{\;\text{a.s.}\;} (\tilde{X}^\infty, \tilde{W}^\infty, \ell^0(\tilde{X}^\infty)) \;\text{ as }\; n \to \infty\,. \tag{3.32}$$

Note that X^n, W^n and $\ell^0(X^n)$ are continuous semimartingales (with respect to the filtration $I\!F^1_{T^n}$ defined as following (2.8) above) for all $n \ge 1$. By (3.26)–(3.28) we see that $\mathsf{E}\ell^0_t(X^n) = 2\,\mathsf{E}\,X^n_t = 2\,\mathsf{E}|X^{1,n}_t| \le 2\,(\mathsf{E}|X^{1,n}_t|^2)^{1/2} = 2\,(\mathsf{E}\int_0^t \sigma^2_n(X^{1,n}_s)\, ds)^{1/2} \le 2\sqrt{t}$ for all $n \ge 1$ and $t \ge 0$. Taking supremum over all $n \ge 1$ in the previous inequality we see that both X^n and $\ell^0(X^n)$ satisfy (2.29) and since this is also true for W^n for all $n \ge 1$, by the vector version of (2.30) applied to the continuous semimartingale $S^n = (X^n, W^n, \ell^0(X^n))$ (with $H^n \equiv 0$ for $n \in I\!N$ given and fixed as well as when tending to ∞) we see that $(\tilde{X}^n, \tilde{W}^n, \ell^0(\tilde{X}^n))$ is a continuous semimartingale with respect to its natural filtration for all $n \in I\!N \cup \{\infty\}$. It follows in particular that the stochastic integral $U(\tilde{X}^n) \cdot \tilde{W}^n$ is well defined for all $n \in I\!N \cup \{\infty\}$ whenever U is a continuous function. This fact will be used in the rest of the proof with no explicit mention. It also follows that $\ell^0(\tilde{X}^n)$ is the local time of $\tilde{X}^n$ at 0 for all $n \in I\!N \cup \{\infty\}$ as suggested by the notation (in (3.30) above as well). Indeed, since $(X^n, W^n, \ell^0(X^n))$ satisfies the identity (3.28) we see by (3.31) that $(\tilde{X}^n, \tilde{W}^n, \ell^0(\tilde{X}^n))$ satisfies the same identity for all

$t \geq 0$. Likewise, since $(X^n, \ell^0(X^n))$ satisfies the identity $\int_0^t I(X_s^n > 0)\, d\ell_s^0(X^n) = 0$ we see by (3.31) that $(\tilde{X}^n, \ell^0(\tilde{X}^n))$ satisfies the same identity for all $t \geq 0$. Applying the Itô–Tanaka formula to $F(x) = x^+$ composed with $\tilde{X}^n$ (as in the first equality of (3.6) above upon recalling that $\tilde{X}^n$ is non-negative) and making use of the second identity above it follows by comparison with the first identity above that $\ell^0(\tilde{X}^n)$ is the local time of $\tilde{X}^n$ at 0 for all $n \in I\!N$ as claimed. Moreover, we will show below that $(\tilde{X}^\infty, \tilde{W}^\infty, \ell^0(\tilde{X}^\infty))$ satisfies the identity (3.1) for all $t \geq 0$ and since $(X, \ell^0(X))$ satisfies the identity $\int_0^t I(X_s > 0)\, d\ell_s^0(X) = 0$ we see from $(\tilde{X}^\infty, \ell^0(\tilde{X}^\infty)) \sim (X, \ell^0(X))$ that $(\tilde{X}^\infty, \ell^0(\tilde{X}^\infty))$ satisfies the same identity for all $t \geq 0$. The same Itô–Tanaka argument as above then implies that $\ell^0(\tilde{X}^\infty)$ is the local time of $\tilde{X}^\infty$ at 0 as claimed. This shows that the local time notation in (3.30)–(3.32) is justified (using the local time symbol from start should cause no ambiguity).

3. We show that $(\tilde{X}^\infty, \tilde{W}^\infty, \ell^0(\tilde{X}^\infty))$ solves the system (3.1)+(3.2). For this, choose a continuous function $U_m : I\!R_+ \to I\!R$ such that $U_m(x) = 0$ for $0 \leq x \leq 1/m$ and $U_m(x) = 1$ for $x \geq 1/2m$ where $m \geq 1$ is given and fixed. Proceeding as in (2.52)–(2.54) and making use of (3.32) above we find that

$$(\tilde{X}^n, U_m(\tilde{X}^n) \cdot \tilde{W}^n, \ell^0(\tilde{X}^n)) \xrightarrow{\mathsf{P}} (\tilde{X}^\infty, U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty, \ell^0(\tilde{X}^\infty)) \tag{3.33}$$

as $n \to \infty$. From (3.31) we see that $(\tilde{X}^n, U_m(\tilde{X}^n) \cdot \tilde{W}^n, \ell^0(\tilde{X}^n)) \sim (X^n, U_m(X^n) \cdot W^n, \ell^0(X^n))$ and hence by (3.33) it follows that

$$(X^n, U_m(X^n) \cdot W^n, \ell^0(X^n)) \xrightarrow{\mathsf{P}} (\tilde{X}^\infty, U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty, \ell^0(\tilde{X}^\infty)) \tag{3.34}$$

as $n \to \infty$. By the continuous mapping theorem this implies that

$$X^n - U_m(X^n) \cdot W^n - \tfrac{1}{2}\ell^0(X^n) \xrightarrow{\mathsf{d}} \tilde{X}^\infty - U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty - \tfrac{1}{2}\ell^0(\tilde{X}^\infty) \tag{3.35}$$

as $n \to \infty$. Using (3.26) and taking $\delta_n \leq 1/m$ for $n \geq n_m$ large enough upon recalling definitions of σ_n and U_m we find by (2.43) and (3.25)–(3.29) that

$$\begin{aligned} X_t^n - \int_0^t U_m(X_s^n)\, dW_s^n - \tfrac{1}{2}\ell_t^0(X^n) \\ &= |X_t^{1,n}| - \int_0^t U_m(|X_s^{1,n}|)\, \mathrm{sign}(X_s^{1,n})\, \frac{dX_s^{1,n}}{\sigma_n(X_s^{1,n})} - \tfrac{1}{2}\ell_t^0(|X^{1,n}|) \\ &= |B_{T_t^n}^1| - \int_0^{T_t^n} U_m(|B_s^1|)\, \mathrm{sign}(B_s^1)\, dB_s^1 - \tfrac{1}{2}\ell_{T_t^n}^0(|B^1|) \\ &\rightrightarrows\ |B_{T_t}^1| - \int_0^{T_t} U_m(|B_s^1|)\, \mathrm{sign}(B_s^1)\, dB_s^1 - \tfrac{1}{2}\ell_{T_t}^0(|B^1|) \\ &= |B_{T_t}^1| - \int_0^t U_m(|B_{T_s}^1|)\, \mathrm{sign}(B_{T_s}^1)\, dB_{T_s}^1 - \tfrac{1}{2}\ell_t^0(|B_T^1|) \\ &= X_t - \int_0^t U_m(X_s)\, d\hat{B}_s - \tfrac{1}{2}\ell_t^0(X) \end{aligned} \tag{3.36}$$

for $t \in [0, N]$ as $n \to \infty$ with any $N > 0$. Combining this with (3.35) we see that

$$\tilde{X}^\infty - U_m(\tilde{X}^\infty) \cdot \tilde{W}^\infty - \tfrac{1}{2}\ell^0(\tilde{X}^\infty) \sim X - U_m(X) \cdot \hat{B} - \tfrac{1}{2}\ell^0(X)\,. \tag{3.37}$$

By Markov's inequality and Doob's maximal inequality we find that

$$\begin{aligned}(3.38)\qquad \mathsf{P}\Big(\sup_{0\le t\le N}\Big|X_t-\int_0^t I(X_s\!>\!0)\,d\hat{B}_s-\tfrac{1}{2}\ell_t^0(X)\\ -\Big(X_t-\int_0^t U_m(X_s)\,d\hat{B}_s-\tfrac{1}{2}\ell_t^0(X)\Big)\Big|>\varepsilon\Big)\\ \le\frac{4}{\varepsilon^2}\,\mathsf{E}\Big[\int_0^N\big(I(X_s\!>\!0)-U_m(X_s)\big)^2\,d\langle\hat{B},\hat{B}\rangle_s\Big]\\ =\frac{4}{\varepsilon^2}\,\mathsf{E}\Big[\int_0^N\big(I(X_s\!>\!0)-U_m(X_s)\big)^2\,ds\Big]\longrightarrow 0\end{aligned}$$

as $m\to\infty$ by the dominated convergence theorem for any $N>0$. This shows that

$$(3.39)\qquad X-U_m(X)\cdot\hat{B}-\tfrac{1}{2}\ell^0(X)\xrightarrow{\mathsf{P}}X-I(X\!>\!0)\cdot\hat{B}-\tfrac{1}{2}\ell^0(X)=0$$

as $m\to\infty$ where we also use the fact that X solves (3.1) with $\hat{B}$ in place of B. Combining this with (3.37) we see that

$$(3.40)\qquad \tilde{X}^\infty-U_m(\tilde{X}^\infty)\cdot\tilde{W}^\infty-\tfrac{1}{2}\ell^0(\tilde{X}^\infty)\xrightarrow{\mathsf{P}}0$$

as $m\to\infty$. The same arguments as in (3.38) also imply that

$$(3.41)\qquad \tilde{X}^\infty-U_m(\tilde{X}^\infty)\cdot\tilde{W}^\infty-\tfrac{1}{2}\ell^0(\tilde{X}^\infty)\xrightarrow{\mathsf{P}}\tilde{X}^\infty-I(\tilde{X}^\infty\!>\!0)\cdot\tilde{W}^\infty-\tfrac{1}{2}\ell^0(\tilde{X}^\infty)$$

as $m\to\infty$. Combining (3.40) and (3.41) we see that

$$(3.42)\qquad \tilde{X}_t^\infty=\int_0^t I(\tilde{X}_s^\infty\!>\!0)\,d\tilde{W}_s^\infty+\tfrac{1}{2}\ell_t^0(\tilde{X}^\infty)$$

for all $t\ge 0$. Moreover, due to $(\tilde{X}^n,\ell^0(\tilde{X}^n))\sim(X^n,\ell^0(X^n))$ for all $n\ge 1$ we see that $(\tilde{X}^n,\ell^0(\tilde{X}^n))\xrightarrow{\mathrm{d}}(\tilde{X}^\infty,\ell^0(\tilde{X}^\infty))$ and $(X^n,\ell^0(X^n))\xrightarrow{\mathrm{d}}(X,\ell^0(X))$ as $n\to\infty$ imply that $(\tilde{X}^\infty,\ell^0(\tilde{X}^\infty))\sim(X,\ell^0(X))$. Since $(X,\ell^0(X))$ satisfies (3.2) hence it follows that

$$(3.43)\qquad \int_0^t I(\tilde{X}_s^\infty\!=\!0)\,ds=\tfrac{1}{2\mu}\,\ell_t^0(\tilde{X}^\infty)$$

for all $t\ge 0$. From (3.42) and (3.43) we see that $(\tilde{X}^\infty,\tilde{W}^\infty,\ell^0(\tilde{X}^\infty))$ solves the system (3.1)+(3.2) as claimed. It follows in particular that the the limit law in (3.30) coincides with the corresponding three-dimensional law arising from the jointly unique solution law for (3.1)+(3.2) (or equivalently (3.3)) derived in Theorem 5 as stated prior to (3.31)+(3.32) above.

4. We can now conclude the proof similarly to the proof of Theorem 3 and Proposition 4 above. For this, suppose that the system (3.1)+(3.2) or equivalently the equation (3.3) has a strong solution (X,B) defined on some probability space $(\Omega,\mathcal{F},\mathsf{P})$. Then there exists a measurable functional $f:C\to C$ such that (2.65) holds and hence there exists a measurable functional $g:C\to C$ such that

$$(3.44)\qquad X-\tfrac{1}{2}\ell^0(X)=g(B)\,.$$

Consider the unique strong solution Y^n to the equation (2.73) and set $\bar{Y}^n = |Y^n|$ for $n \geq 1$ given and fixed. By the Itô–Tanaka formula we find as in (3.28) above that

$$d\bar{Y}^n_t = \tfrac{1}{2}\, d\ell^0_t(\bar{Y}^n) + \sigma_n(\bar{Y}^n_t)\, dB^n_t \tag{3.45}$$

with $\bar{Y}^n_0 = 0$ where $B^n_t = \int_0^t \operatorname{sign}(Y^n_s)\, dB_s$ for $t \geq 0$ is a standard Brownian motion. Since σ_n is (i) uniformly bounded by a strictly positive constant from below and (ii) of bounded variation on compacts, it is well known that the equation (3.45) has a unique strong solution (pathwise uniqueness can be derived in the same way as in Theorem 4.48 of [8] and the right-arrow implication in (3.24) then yields strong existence). This implies that there exists a measurable functional $f_n : C \to C$ such that $\bar{Y}^n = f_n(B^n)$. Since Y^n is a measurable functional of B it follows that B^n is a measurable functional of B and hence $\bar{Y}^n$ is a measurable functional of B. This shows that there exists a measurable functional $g_n : C \to C$ such that

$$\bar{Y}^n - \tfrac{1}{2}\,\ell^0(\bar{Y}^n) = g_n(B) \tag{3.46}$$

for $n \geq 1$. By the joint uniqueness in law for the equation (3.45) (which follows directly from the uniqueness in law since σ_n is strictly positive) we know that

$$(\bar{Y}^n, B^n, \ell^0(\bar{Y}^n)) \sim (X^n, W^n, \ell^0(X^n)) \tag{3.47}$$

for $n \geq 1$. It follows therefore by (3.31)+(3.32) that

$$(\bar{Y}^n, B^n, \ell^0(\bar{Y}^n)) \xrightarrow{\mathsf{d}} (\tilde{X}^\infty, \tilde{W}^\infty, \ell^0(\tilde{X}^\infty)) \tag{3.48}$$

as $n \to \infty$. Since $(\tilde{X}^\infty, \tilde{W}^\infty, \ell^0(\tilde{X}^\infty))$ solves (3.1)+(3.2) we know by the joint uniqueness in law established in Theorem 5 for this system that

$$(\tilde{X}^\infty, \tilde{W}^\infty, \ell^0(\tilde{X}^\infty)) \sim (X, B, \ell^0(B))\,. \tag{3.49}$$

Combining (3.44)+(3.46) and (3.48)+(3.49) we find by the continuous mapping theorem that

$$(g_n(B), B) \xrightarrow{\mathsf{d}} (g(B), B) \tag{3.50}$$

as $n \to \infty$. By Lemma 2 we can therefore conclude that

$$\bar{Y}^n - \tfrac{1}{2}\,\ell^0(\bar{Y}^n) = g_n(B) \xrightarrow{\mathsf{P}} g(B) = X - \tfrac{1}{2}\,\ell^0(X) \tag{3.51}$$

as $n \to \infty$. Passing to a subsequence if needed (which we again denote by n for simplicity) we can assume that $\delta_{n+1} < \delta_n/4$ for all $n \geq 1$. Setting $Z^n = Y^n - Y^{n+1}$ for $n \geq 1$ given and fixed we see that (2.74) and (2.75) in the proof of Proposition 4 are satisfied. Define

$$\bar{Z}^n = \bar{Y}^n - \tfrac{1}{2}\,\ell^0(\bar{Y}^n) - \big(\bar{Y}^{n+1} - \tfrac{1}{2}\,\ell^0(\bar{Y}^{n+1})\big) \tag{3.52}$$

and redefine (2.76) by setting

$$\tau^n_\varepsilon = \inf\{\, t \geq 0 \mid |\bar{Z}^n_t| = \varepsilon \,\} \tag{3.53}$$

upon noting that $\bar{Z}^n_0 = 0$. Replacing Z^n in (2.77) by $\bar{Z}^n$ we can proceed as in (2.78)–(2.81) from where by taking expectations on both sides of (2.81) and using (2.75)+(3.45) we get

$$\begin{aligned} \mathsf{E}\,\ell^z_{t\wedge\tau^n_\varepsilon}(Z^n) &\leq \mathsf{E}|Z^n_{t\wedge\tau^n_\varepsilon}| \leq \sqrt{\mathsf{E}|Z^n_{t\wedge\tau^n_\varepsilon}|^2} = \sqrt{\mathsf{E}\langle Z^n, Z^n\rangle_{t\wedge\tau^n_\varepsilon}} \\ &\leq \sqrt{\mathsf{E}\langle \bar{Z}^n, \bar{Z}^n\rangle_{t\wedge\tau^n_\varepsilon}} = \sqrt{\mathsf{E}|\bar{Z}^n_{t\wedge\tau^n_\varepsilon}|^2} \leq \varepsilon \end{aligned} \tag{3.54}$$

for all $z \in I\!R$ where in the third inequality we use that $|\sigma_n(x) - \sigma_n(y)| \leq |\text{sign}(x)\sigma_n(x) - \text{sign}(y)\sigma_n(y)|$ for all $x, y \in I\!R$. Proceeding then as in (2.83) and (2.84) and recalling that Z^n in (2.77) is replaced by $\bar{Z}^n$ we see by (3.52) that the conclusion (2.84) reads

$$\liminf_{n\to\infty} \mathsf{P}\Big(\sup_{0\le s\le t} \big|\bar{Y}_s^n - \tfrac{1}{2}\,\ell_s^0(\bar{Y}^n) - \big(\bar{Y}_s^{n+1} - \tfrac{1}{2}\,\ell_s^0(\bar{Y}^{n+1})\big)\big| \geq \varepsilon\Big) > 0 \tag{3.55}$$

for $t > 0$ and any sufficiently small $\varepsilon > 0$ given and fixed. This shows that $\bar{Y}^n - \frac{1}{2}\,\ell^0(\bar{Y}^n)$ does not converge in probability (relative to d_∞) as $n \to \infty$. Since this fact contradicts (3.51) we can conclude that the system (3.1)+(3.2) or equivalently the equation (3.3) has no strong solution as claimed. $\square$

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