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On Separability of Families of Reversed Submartingales

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Certain statistical models are described by a family of reversed submartingales. The main problem under consideration is to estimate maximum points of the associated information function. It turns out that some maximal inequalities are important to be established in this direction. In this paper we show that these inequalities remain valid for suitable chosen modifications of those families. However this result has no practical meaning for statistics. For that reason we introduce and investigate a concept of separability of the family of reversed submartingales which is according to [3] indexed by an analytic metric space. We prove the existence of separable modifications for these families and show that semicontinuity implies separability. Our method in these considerations relies upon results of the classical theory of stochastic processes. Finally we introduce conditional S-regular families of reversed submartingales as those which are as close as possible to satisfy the maximal inequalities. We show that separable families belong to this class. Moreover we present one significant class of this type which includes all U-processes and cover a large number of random functions occurring in probability and statistics.

1. Introduction

Some statistical models are formed by the family $\mathcal{H} = \big(\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0 \,\big)$ of reversed submartingales defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an analytic metric space Θ_0 (see [2], [3], [4], [5], [6], [7]). The following two inequalities appear to be important in the proof of consistency for these models:

$$Eh_n^*(B) \geq Eh_{n+1}^*(B) \tag{1}$$

$$E\bar{h}_n(\theta) \geq E\bar{h}_{n+1}(\theta) \tag{2}$$

for some $B \subset \Theta_0$ and $\theta \in \Theta_0$, and all $n \geq k$ with some $k \geq 1$, where $h_n^*(\omega, B) = \sup_{\theta\in B} h_n(\omega,\theta)$ and $\bar{h}_n(\omega,\cdot) = \lim_{r\downarrow 0} h_n^*(\omega, b(\cdot, r))$ is the upper semicontinuous envelope of $h_n(\omega,\cdot)$ for $\omega \in \Omega$ and $n \in \mathbf{N}$. The next two examples show that inequalities (1) and (2) may fail in the general reversed submartingale case.

Example 1.

Let $(\Omega, \mathcal{F}, P) = (\,[0,1], \mathcal{B}([0,1]), \lambda\,)$ and $\Theta_0 = [0,1] \times \mathbf{N}$, and let $g : \Omega \to \bar{\mathbf{R}}$ be an arbitrary function. Then for every $\omega \in \Omega$ there exists a sequence $\{\, g_n(\omega) \mid n \geq 1 \,\}$ in $\bar{\mathbf{R}}$ such that $g_n(\omega) \uparrow$

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$g(\omega)$. Given $\omega \in \Omega$ and $\theta = (x, m) \in \Theta_0$, define:

$$h_2(\omega, \theta) = g_m(\omega) \cdot \delta_{\{\omega = x\}}$$

where $\delta_{\{\omega=x\}} = 1$ if $\omega = x$ and $\delta_{\{\omega=x\}} = 0$ if $\omega \neq x$. Put $h_n(\omega, \theta) = 0$ for all $\omega \in \Omega$, $\theta \in \Theta_0$, $n \neq 2$. Then $\mathcal{H} = \big(\{h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0 \big)$ is a family of reversed martingales whenever $\{ \mathcal{S}_n \mid n \geq 1 \}$ is a decreasing sequence of σ-algebras on Ω , for which $h_n(\,\cdot\,, \theta)$ is $\mathcal{S}_n$-measurable when $\theta \in \Theta_0$ and $n \geq 1$. Note that:

$$\sup_{\theta \in \Theta_0} h_2(\omega, \theta) = g(\omega)$$

for all $\omega \in \Omega$. Hence we see, first of all, that $Eh_2^*(\Theta_0)$ is not well-defined in general. Moreover, if we suppose that h_n is $\mathcal{S}_n \times \mathcal{B}(\Theta_0)$-measurable, then g_n (and thus g) is $\mathcal{F}$-measurable, and if $g \in L(P)$, then from (1) we get:

$$0 = Eh_1^*(\Theta_0) \geq Eh_2^*(\Theta_0) = Eg \ .$$

However this contradicts the fact that g is arbitrary. □

Example 2.

Let $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$ and $\Theta_0 = [0,1] \times [0,1]$, and let $g : \Omega \rightarrow \bar{\mathbf{R}}$ be an arbitrary function. Then for every $\omega \in \Omega$ there exists a sequence $\{ g_n(\omega) \mid n \geq 1 \}$ in $\bar{\mathbf{R}}$ such that $g_n(\omega) \uparrow g(\omega)$. Given $\omega \in \Omega$ and $\theta = (x, y) \in \Theta_0$, $y \neq 0$, define:

$$\tilde{h}_2(\omega, \theta) = g_{[\frac{1}{y}]}(\omega) \cdot \delta_{\{\omega = x\}} \ .$$

Put $\tilde{h}_2(\omega, (x, 0)) = 0$ for $\omega \in \Omega$ and $x \in [0,1]$. Let $A_n = (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$, and let $f_n : (0,1] \rightarrow A_n$ be a homeomorphism for $n \geq 1$. Given $\omega \in \Omega$ and $\theta = (x, y) \in \Theta_0$, define:

$$h_2(\omega, \theta) = \sum_{n=1}^{\infty} \tilde{h}_2\Big(\omega, ({f_n}^{-1}(x), y)\Big) \cdot 1_{A_n}(x)$$

with ${f_n}^{-1}(0) = 0$. Put $h_n(\omega, \theta) = 0$ for all $\omega \in \Omega$, $\theta \in \Theta_0$, $n \neq 2$. Then $\mathcal{H} = \big(\{h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0 \big)$ is a family of reversed martingales whenever $\{ \mathcal{S}_n \mid n \geq 1 \}$ is a decreasing sequence of σ-algebras on Ω , for which $h_n(\,\cdot\,, \theta)$ is $\mathcal{S}_n$-measurable when $\theta \in \Theta_0$ and $n \geq 1$. Note that:

$$\sup_{\theta \in A_n \times A_m} h_2(\omega, \theta) = \sup_{\theta \in (0,1] \times (0,1]} \tilde{h}_2(\omega, \theta) = g(\omega)$$

for all $n, m \geq 1$ and $\omega \in (0,1]$. It shows that for $\theta_0 = (0,0)$ we have:

$$\bar{h}_2(\omega, \theta_0) = \lim_{r \downarrow 0} \sup_{\xi \in b(\theta_0, r)} h_2(\omega, \xi) = g(\omega)$$

for all $\omega \in (0,1]$. However $\bar{h}_1(\omega, \theta) = h_1(\omega, \theta) = 0$ for all $\omega \in \Omega$ and $\theta \in \Theta_0$. Hence if $g \in L(P)$, then from (2) we get:

$$0 = E\bar{h}_1(\theta) \geq E\bar{h}_2(\theta) = Eg \ .$$

However this contradicts the fact that g is arbitrary. □

In this context it appears worthy of consideration to find out when (1) and (2) are satisfied. The purpose of the present paper is to obtain conditions for (1) and (2) in the general reversed submartingale case. For this reason we introduce and investigate a concept of separability of the family $\mathcal{H}$. Roughly speaking, this concept says that the P-almost all trajectories $\theta \mapsto h_n(\omega,\theta)$ are countably determined for $n \geq 1$, and in turn implies that $(\{ h_n^*(\omega,B), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0)$ is a family of reversed submartingales for $B \subset \Theta_0$. In other words:

$$E\{ h_n^*(B) \mid \mathcal{S}_{n+1} \} \geq h_{n+1}^*(B) \tag{3}$$

for all $n \geq 1$, and thus (1) and (2) may follow straightforward. The method in these considerations relies upon results of the classical theory of stochastic processes (see [1]). In addition, we discover that there is another natural property which leads to (3). It describes a symmetric recurrent relation between $h_n(\omega,\theta)$ and $h_{n+1}(\omega,\theta)$, and in this way admits the existence of versions of the conditional expectations $E\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}$ for $B \subset \Theta_0$, which are nice enough to imply (3). Families of reversed submartingales satisfying this property are called *conditionally S-regular*, and one large and significant class of this type is presented in example 4.4 below. In particular we obtain that *all U-processes are conditionally S-regular*. We think that this fact is by itself of theoretical (and practical) interest in the context explained above. However we will not pursue this here, but instead postpone for further research.

2. Existence of S-regular modifications

Let $\mathcal{H} = \big(\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0 \big)$ be a family of reversed submartingales defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an analytic metric space Θ_0 , and let $\mathcal{B}_0 = \mathcal{B}(\Theta_0)$ denote the Borel σ-algebra on Θ_0 . Then we say that $\mathcal{H}$ is:

(i) *measurable*, if h_n is $\mathcal{S}_n \times \mathcal{B}_0$-measurable for all $n \geq 1$

(ii) *$P \times \mu$-measurable*, if h_n is $(\mathcal{S}_n \times \mathcal{B}_0)^{P\times\mu}$-measurable for all $n \geq 1$, where μ is a measure on $(\Theta_0, \mathcal{B}_0)$

(iii) *degenerated*, if the σ-algebra $\mathcal{S}_\infty = \bigcap_{n=1}^\infty \mathcal{S}_n$ is degenerated $(P(A) \in \{0,1\}$, $\forall A \in \mathcal{S}_\infty)$

(iv) *separable relative to* $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C} \subset 2^{\mathbf{R}}$, if $\forall B \in \mathcal{S}$ there exists a sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ_0 such that $\forall C \in \mathcal{C}$ we have:

$$P^*\Big(\bigcup_{n=1}^{\infty} \{ h_n(\theta_i) \in C \mid \theta_i \in B \} \triangle \{ h_n(\theta) \in C \mid \theta \in B \} \Big) = 0 \ .$$

In this case we will say that the sequence $\{ \theta_i \mid i \geq 1 \}$ *satisfies the conditions of the separability definition relative to* B *and* $\mathcal{C}$. If this is true simultaneously for every $B \in \mathcal{S}$, we shall simply say that the sequence $\{ \theta_i \mid i \geq 1 \}$ *satisfies the conditions of the separability definition* (relative to $\mathcal{S}$ and $\mathcal{C}$).

(v) *separable*, if it is separable relative to $\mathcal{G}(\Theta_0)$ and $\mathcal{C}(\mathbf{R})$, where $\mathcal{G}(\Theta_0)$ and $\mathcal{C}(\mathbf{R})$ denote the family of all open sets in Θ_0 and all closed sets in $\mathbf{R}$ respectively.

We will say that $\mathcal{H}$ *admits a modification*, if there exists a family of reversed submartingales $\tilde{\mathcal{H}} = \big(\{\tilde{h}_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0 \big)$ satisfying:

$$P\Big(\bigcup_{n=1}^{\infty} \{ \tilde{h}_n(\theta) \neq h_n(\theta) \} \Big) = 0$$

for all $\theta \in \Theta_0$. In this case we say that $\tilde{\mathcal{H}}$ is *a modification* of $\mathcal{H}$. In considerations concerning the statistical models mentioned above it is convenient to embed the analytic metric space Θ_0 in a compact metric space (see [3]). Following this practice we will in the next suppose that (Θ, d) is a compact metric space containing Θ_0 , and for every function $h : \Theta_0 \to \mathbf{R}$ we put $h(\theta) = -\infty$ for $\theta \in \Theta \setminus \Theta_0$. The Borel σ-algebra on Θ will be denoted by $\mathcal{B}$. The next theorem shows that (1.1) and (1.2) remain valid in the general reversed submartingale case provided that $\mathcal{H}$ is replaced by a suitable modification $\tilde{\mathcal{H}}$. Unfortunately this result has no practical meaning for the statistical models mentioned above.

Theorem 1.

Let $\mathcal{H} = \big(\{ h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$ be a given family of reversed submartingales. Suppose that $\mathcal{H}$ is measurable, and let $\mathcal{A}$ be a finite or countable family of Borel sets in Θ_0 such that $Eh_k^*(A) < \infty$ for all $A \in \mathcal{A}$ with some $k \geq 1$. Then there exists a measurable modification $\tilde{\mathcal{H}} = \big(\{ \tilde{h}_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$ of $\mathcal{H}$ such that $\{ \tilde{h}_n^*(A), \mathcal{S}_n^P \mid n \geq k \}$ is a reversed submartingale for every $A \in \mathcal{A}$.

Proof. Let $\mathcal{A} = \{ A_i \mid i \in I \}$, then by hypotheses we have $card\ I \leq \aleph_0$. Put $\tilde{h}_n(\omega, \theta) = h_n(\omega, \theta)$, $\forall \omega \in \Omega$, $\forall \theta \in \Theta_0$ and $\forall n = 1, 2, \dots, k$. Then $\tilde{h}_n$ is $\mathcal{S}_n \times \mathcal{B}_0$-measurable $\forall n = 1, 2, \dots, k$ and by the projection theorem (see [3]) $\tilde{h}_k^*(A_i)$ is $\mathcal{S}_k^P$-measurable for all $i \in I$. Since $\{ h_n(\theta), \mathcal{S}_n \mid n \geq k \}$ is a reversed submartingale and $\tilde{h}_k(\theta) = h_k(\theta)$ P-$a.s.$ for all $\theta \in \Theta_0$, then we have:

$$E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1}^P \} = E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1} \} \geq E\{ \tilde{h}_k(\theta) \mid \mathcal{S}_{k+1} \} = \tag{1}$$

$$= E\{ h_k(\theta) \mid \mathcal{S}_{k+1} \} \geq h_{k+1}(\theta) \quad P\big|_{\mathcal{S}_{k+1}}\text{-}\ a.s.$$

for all $\theta \in A_i$ and all $i \in I$. Hence for given versions $E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1}^P \}(\omega)$ and $E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1} \}(\omega)$ of the conditional expectations $E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1}^P \}$ and $E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1} \}$ there exists a P-nullset $N \in \mathcal{S}_{k+1}$ such that:

$$E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1}^P \}(\omega) = E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1} \}(\omega) \tag{2}$$

for all $\omega \in \Omega \setminus N$ and all $i \in I$. Define:

$$\tilde{h}_{k+1}(\omega, \theta) = \inf_{i \in I} \Big\{ \frac{1}{1_{A_i}(\theta)} E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1} \}(\omega) \wedge h_{k+1}(\omega, \theta) \Big\} \tag{3}$$

for all $\omega \in \Omega$ and all $\theta \in \Theta_0$, where $\frac{1}{0} = +\infty \cdot 0 = +\infty$. Hence we see that $\tilde{h}_{k+1}$ is $\mathcal{S}_{k+1} \times \mathcal{B}_0$-measurable, and thus by the projection theorem (see [3]) $\tilde{h}_{k+1}^*(A)$ is $\mathcal{S}_{k+1}^P$-measurable for every analytic subset A of Θ_0 . By (1) we easily find that $\tilde{h}_{k+1}(\theta) = h_{k+1}(\theta)$ P-$a.s.$ for all $\theta \in \Theta_0$, and hence by (2) and (3) we have:

$$E\{ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1}^P \}(\omega) \geq \tilde{h}_{k+1}^*(\omega, A_i)$$

for all $\omega \in \Omega \setminus N$ and all $i \in I$. This shows:

$$E\{\ \tilde{h}_k^*(A_i) \mid \mathcal{S}_{k+1}^P\ \} \ge \tilde{h}_{k+1}^*(A_i)$$

for all $i \in I$. Since $\tilde{h}_{k+1}$ is $\mathcal{S}_{k+1} \times \mathcal{B}_0$-measurable and $\tilde{h}_{k+1}(\theta) = h_{k+1}(\theta)$ P-$a.s.$ for all $\theta \in \Theta_0$, the preceding construction can be repeated and by the induction we get:

$$E\{\ \tilde{h}_n^*(A_i) \mid \mathcal{S}_{n+1}^P\ \} \ge \tilde{h}_{n+1}^*(A_i)$$

for all $n \ge k$ and all $i \in I$. Hence we see that $\mathcal{H}_i = \{\ \tilde{h}_n^*(A_i)\ ,\ \mathcal{S}_n^P \mid n \ge k\ \}$ is a reversed submartingale $\forall i \in I$, and the proof is complete. □

Corollary 2.

Let $\mathcal{H} = \big(\{h_n(\omega,\theta), \mathcal{S}_n \mid n \ge 1\} \mid \theta \in \Theta_0\ \big)$ *be a given family of reversed submartingales. Suppose that* $\mathcal{H}$ *is measurable, then we have:*

(i) *If* A *is a Borel subset of* Θ_0 *such that* $Eh_k^*(A) < \infty$ *for some* $k \ge 1$ *, then there exists a measurable modification* $\tilde{\mathcal{H}} = \big(\{\tilde{h}_n(\omega,\theta), \mathcal{S}_n \mid n \ge 1\} \mid \theta \in \Theta_0\ \big)$ *of* $\mathcal{H}$ *such that* $\{\ \tilde{h}_n^*(A), \mathcal{S}_n^P \mid n \ge k\ \}$ *is a reversed submartingale. In particular, we have:*

(1) $$E\tilde{h}_n^*(A)\ \ge\ E\tilde{h}_{n+1}^*(A)$$

for all $n \ge k$.

(ii) *If* $\theta \in \Theta_0$ *such that* $E\bar{h}_k(\theta) < \infty$ *for some* $k \ge 1$ *, then there exists a measurable modification* $\mathcal{H}^\circ = \big(\{h_n^\circ(\omega,\theta), \mathcal{S}_n \mid n \ge 1\} \mid \theta \in \Theta_0\ \big)$ *of* $\mathcal{H}$ *such that* $\{\ \bar{h}_n^\circ(\theta), \mathcal{S}_n^P \mid n \ge k\ \}$ *is a reversed submartingale. In particular, we have:*

(2) $$E\bar{h}_n^\circ(\theta)\ \ge\ E\bar{h}_{n+1}^\circ(\theta)$$

for all $n \ge k$.

Proof. (i): It follows by applying theorem 1 to the family $\mathcal{A} = \{A\}$.

(ii): Since $b(\theta,r)$ is an analytic subset of Θ_0 for all $r > 0$, then by the projection theorem (see [3]) we have that the upper semicontinuous envelope:

$$\bar{h}_n(\theta) = \lim_{r\downarrow 0} \sup_{\xi \in b(\theta,r)} h_n(\xi)\ =\ \lim_{r\downarrow 0, r\in\mathbf{Q}} \sup_{\xi \in b(\theta,r)} h_n(\xi)$$

is $\mathcal{S}_n^P$-measurable for all $n \ge 1$. Since $E\bar{h}_k(\theta) < \infty$, there exists $r_0 \in \mathbf{Q}_+$ such that $\forall r \in \mathbf{Q}_+$, $r \le r_0$ we have $Eh_k^*\big(b(\theta,r)\big) < \infty$. Hence we see that $\mathcal{A} = \{\ b(\theta,r) \mid r \in \mathbf{Q}_+\ ,\ r \le r_0\ \}$ is a countable family of Borel sets in Θ which satisfies the conditions of theorem 1, and therefore there exists a measurable modification $\mathcal{H}^\circ = \big(\{h_n^\circ(\omega,\theta), \mathcal{S}_n \mid n \ge 1\} \mid \theta \in \Theta_0\ \big)$ of $\mathcal{H}$ such that $\{\ \sup_{\xi\in b(\theta,r)} h_n^\circ(\omega,\xi), \mathcal{S}_n^P \mid n \ge k\ \}$ is a reversed submartingale $\forall r \in \mathbf{Q}_+$, $r \le r_0$. Now by the conditional monotone convergence theorem we find:

$$E\{\ \bar{h}_n^\circ(\theta) \mid \mathcal{S}_{n+1}^P\ \} = E\{\ \lim_{r\downarrow 0} \sup_{\xi \in b(\theta,r)} h_n^\circ(\xi) \mid \mathcal{S}_{n+1}^P\ \} =$$

$$= \lim_{r\downarrow 0} E\{\ \sup_{\xi \in b(\theta,r)} h_n^\circ(\xi) \mid \mathcal{S}_{n+1}^P\ \} \ge \lim_{r\downarrow 0} \sup_{\xi \in b(\theta,r)} h_{n+1}^\circ(\xi) = \bar{h}_{n+1}^\circ(\theta)$$

.

This shows that $\{\ \bar{h}_n^\circ(\theta), \mathcal{S}_n^P \mid n \ge k\ \}$ is a reversed submartingale and completes the proof of (ii). □

3. Separable families of reversed submartingales

In this section we present basic properties of separable families of reversed submartingales. In the process we respect a motivation coming from the asymptotic likelihood theory [3], and use the classical theory of stochastic processes [1] like the main framework. In some sense the results may be viewed like a refinement of the rather classical ones, thus most of the proofs are omitted (see [1]).

Proposition 1.

Let $\mathcal{H} = \big(\{h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\} \mid \theta \in \Theta_0\,\big)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is separable relative to* $\mathcal{S}$ *and* $\mathcal{C}$ *, then it is separable relative to* $\mathcal{S}_\sigma$ *and* $\mathcal{C}_\delta$ *. Moreover, if* $\mathcal{S}$ *is countable, then there exists a sequence* $\{\, \theta_i \mid i \geq 1 \,\}$ *in* Θ_0 *which satisfies the conditions of the separability definition relative to all sets in* $\mathcal{S}_\sigma$ *and* $\mathcal{C}_\delta$ *.* □

Proposition 2.

Let $\mathcal{H} = \big(\{\, h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \,\} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales. Then the following six statements are equivalent:*

(i) $\mathcal{H}$ *is separable*

(ii) $\mathcal{H}$ *is separable relative to* $\mathcal{G}_{b,E}(\Theta_0)$ *and* $\mathcal{C}(\mathbf{R})$ *, where* $\mathcal{G}_{b,E}(\Theta_0) = \{\, b(\theta,r) \mid \theta \in E \,,\ r \in \mathbf{Q}_+ \,\}$ *for some countable dense subset* E *of* Θ_0

(iii) $\mathcal{H}$ *is separable relative to* $\mathcal{G}(\Theta_0)$ *and* $\mathcal{C}^u_\infty(\mathbf{R})$ *, where* $\mathcal{C}^u_\infty(\mathbf{R}) = \{\, (-\infty,p\,] \cup [\,q,+\infty) \mid p,q \in \mathbf{Q} \,,\ p < q \,\}$

(iv) $\mathcal{H}$ *is separable relative to* $\mathcal{G}_{b,E}(\Theta_0)$ *and* $\mathcal{C}^u_\infty(\mathbf{R})$ *for some countable dense subset* E *of* Θ_0

(v) *There exists a countable dense subset* D *of* Θ_0 *and a* P*-null set* $N \in \mathcal{F}$ *such that* $\forall \omega \in \Omega \setminus N$ *,* $\forall \theta \in \Theta_0$ *and* $\forall \varepsilon > 0$ *we have:*

$$h_n(\omega,\theta) \in cl\,\{\, h_n(\omega,\delta) \mid \delta \in b(\theta,\varepsilon) \cap D \,\}$$

for all $n \geq 1$

(vi) *There exists a countable dense subset* D *of* Θ_0 *and a* P*-null set* $N \in \mathcal{F}$ *such that* $\forall \omega \in \Omega \setminus N$ *and* $\forall \theta \in \Theta_0$ *there exists a sequence* $\{\, \delta_i \mid i \geq 1 \,\}$ *in* D *such that:*

$$\delta_i \rightarrow \theta \quad and \quad h_n(\omega,\theta) = \lim_{i\to\infty} h_n(\omega,\delta_i)$$

for all $n \geq 1$ *.*

In this case there exists a single sequence $\{\, \theta_i \mid i \geq 1 \,\}$ *in* Θ_0 *which satisfies the conditions of the separability definition in any of the cases (i), (ii), (iii) and (iv). Furthermore, it is no restriction to assume that any sequence with this property is dense in* Θ_0 *.* □

Proposition 3.

Let $\mathcal{H} = \big(\{\, h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \,\} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales.*

(i) *If* $\mathcal{H}$ *is separable relative to* $\mathcal{S}$ *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *, where* $\mathcal{C}_{-\infty}(\mathbf{R}) = \{\, (-\infty,p\,] \mid p \in \mathbf{Q} \,\}$ *, then for every* $B \in \mathcal{S}$ *and every sequence* $\{\, \theta_i \mid i \geq 1 \,\}$ *in* Θ_0 *which satisfies the condi-*

tions of the separability definition relative to B *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *there exists a* P*-null set* $N \in \mathcal{F}$ *such that* $\forall n \geq 1$ *we have:*

(1) $$\sup_{\theta \in B} h_n(\omega, \theta) = \sup_{\theta_i \in B} h_n(\omega, \theta_i)$$

for all $\omega \in \Omega \setminus N$.

(iii) *If* $\mathcal{H}$ *is separable relative to* $\mathcal{S}$ *and* $\mathcal{C}_{+\infty}(\mathbf{R})$ *, where* $\mathcal{C}_{+\infty}(\mathbf{R}) = \{ [q, +\infty) \mid q \in \mathbf{Q} \}$ *, then for every* $B \in \mathcal{S}$ *and every sequence* $\{ \theta_i \mid i \geq 1 \}$ *in* Θ_0 *which satisfies the conditions of the separability definition relative to* B *and* $\mathcal{C}_{+\infty}(\mathbf{R})$ *there exists a* P*-null set* $N \in \mathcal{F}$ *such that* $\forall n \geq 1$ *we have:*

(2) $$\inf_{\theta \in B} h_n(\omega, \theta) = \inf_{\theta_i \in B} h_n(\omega, \theta_i)$$

for all $\omega \in \Omega \setminus N$.

Conversely, if $\mathcal{S}$ *is a family of subsets of* Θ_0 *such that for every* $B \in \mathcal{S}$ *there exists a sequence* $\{ \theta_i \mid i \geq 1 \}$ *in* Θ_0 *and a* P*-null set* $N \in \mathcal{F}$ *for which (1) resp. (2) holds for all* $\omega \in \Omega \setminus N$ *, then* $\mathcal{H}$ *is separable relative to* $\mathcal{S}$ *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *resp.* $\mathcal{C}_{+\infty}(\mathbf{R})$ *, and the given sequence* $\{ \theta_i \mid i \geq 1 \}$ *satisfies the conditions of the separability definition relative to the given* B *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *resp.* $\mathcal{C}_{+\infty}(\mathbf{R})$ *.* □

Remark 1. Since $\mathcal{C}_{-\infty}(\mathbf{R})$ and $\mathcal{C}_{+\infty}(\mathbf{R})$ are contained in $\mathcal{C}(\mathbf{R})$, then the preceding proposition shows that any family $\mathcal{H}$ of reversed submartingales which is separable relative to $\mathcal{S}$ and $\mathcal{C}(\mathbf{R})$ satisfies (1) and (2) in proposition 3 for all $B \in \mathcal{S}$ and all $\omega \in \Omega \setminus N_B$ with a P-null set $N_B \in \mathcal{F}$, where the sequence $\{ \theta_i \mid i \geq 1 \}$ can be an arbitrary sequence in Θ_0 which satisfies the conditions of the separability definition relative to B and $\mathcal{C}(\mathbf{R})$. Moreover a slight modification of the proof of the last statement in proposition 3 shows that the reverse statement is also true, i.e. if $\mathcal{S}$ is a family of subsets of Θ_0 such that for every $B \in \mathcal{S}$ there exist a sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ_0 and a P-nullset $N_B \in \mathcal{F}$ for which (1) and (2) in proposition 3 hold for all $\omega \in \Omega \setminus N_B$, then $\mathcal{H}$ is separable relative to $\mathcal{S}$ and $\mathcal{C}(\mathbf{R})$, and the given sequence satisfies the conditions of the separability definition relative to B and $\mathcal{C}(\mathbf{R})$.

Remark 2. If $\mathcal{H}$ is separable relative to $\mathcal{S}$ and $\mathcal{C}$ and there exists a countable subfamily $\mathcal{S}_0$ of $\mathcal{S}$ such that for every $B \in \mathcal{S}$ there exists $B_i \in \mathcal{S}_0$ for $i \geq 1$ with $B = \bigcup_{i=1}^{\infty} B_i$, then by proposition 1 there exists a sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ_0 which satisfies the conditions of the separability definition relative to all $B \in \mathcal{S}$ and $\mathcal{C}$. In particular, by remark 1 hence we see that any separable family $\mathcal{H}$ of reversed submartingales satisfies (1) and (2) in proposition 3 for all open sets $B \in \mathcal{G}(\Theta_0)$ and all $\omega \in \Omega \setminus N_B$ with a unique sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ_0 that can be an arbitrary one which satisfies the conditions of the separability definition of $\mathcal{H}$ and where $N_B \in \mathcal{F}$ is a P-null set. Moreover, we have:

Corollary 4.

Let $\mathcal{H} = \big(\{ h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is separable, then there exist a sequence* $\{ \theta_i \mid i \geq 1 \}$ *in* Θ_0 *and a* P*-null set* $N \in \mathcal{F}$ *such that for every open set* $G \in \mathcal{G}(\Theta_0)$ *, all* $\omega \in \Omega \setminus N$ *and all* $n \geq 1$ *we have:*

(1) $$\sup_{\theta\in G} h_n(\omega,\theta) = \sup_{\theta_i\in G} h_n(\omega,\theta_i)$$

(2) $$\inf_{\theta\in G} h_n(\omega,\theta) = \inf_{\theta_i\in G} h_n(\omega,\theta_i) \ .$$

Furthermore, the given equalities remain true for every sequence $\{ \theta_i \mid i \geq 1 \}$ *in* Θ_0 *which satisfies the conditions of the separability definition of* $\mathcal{H}$ *. Consequently, the functions:*

(3) $$\omega \mapsto \sup_{\theta\in G} h_n(\omega,\theta) \ , \quad \omega \mapsto \inf_{\theta\in G} h_n(\omega,\theta) \ ,$$

$$\omega \mapsto \limsup_{\xi\to\theta} h_n(\omega,\xi) \ , \quad \omega \mapsto \liminf_{\xi\to\theta} h_n(\omega,\xi)$$

are P*-measurable, whenever* $G \in \mathcal{G}(\Theta_0)$ *,* $\theta \in \Theta_0$ *and* $\forall n \geq 1$ *.* □

Proposition 5.

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales.*

(i) *Statement (1) in corollary 4 is equivalent to the following condition:*

(1)" $$h_n(\omega,\theta) \leq \lim_{r\downarrow 0} \sup_{\theta_i\in b(\theta,r)} h_n(\omega,\theta_i)$$

being valid $\forall\omega \in \Omega\setminus N$ *,* $\forall\theta\in\Theta_0$ *and* $\forall n \geq 1$ *.*

(ii) *Statement (2) in corollary 4 is equivalent to the following condition:*

(2)" $$h_n(\omega,\theta) \geq \lim_{r\downarrow 0} \inf_{\theta_i\in b(\theta,r)} h_n(\omega,\theta_i)$$

being valid $\forall\omega \in \Omega\setminus N$ *,* $\forall\theta\in\Theta_0$ *and* $\forall n \geq 1$ *.*

(iii) *Moreover, if* $\mathcal{H}$ *is separable and there exists a sequence* $\{ \theta_i \mid i \geq 1 \}$ *in* Θ_0 *for which (1)" and (2)" are satisfied for all* $\omega \in \Omega \setminus N_\theta$ *, where* $N_\theta \in \mathcal{F}$ *is a* P*-nullset depending on* $\theta \in \Theta_0$ *, then the sequence* $\{ \theta_i \mid i \geq 1 \}$ *satisfies the conditions of the separability definition of* $\mathcal{H}$ *.* □

Proposition 6.

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales.*

(i) *If* $\mathcal{H}$ *is separable relative to* $\mathcal{S}$ *and* $\mathcal{C}$ *and for some* $S \in \mathcal{S}$ *,* $C \in \mathcal{C}$ *and* $n \geq 1$ *we have* $P\{ h_n(\theta) \in C \} = 1$ *for all* $\theta \in S$ *, then* $\bigcap_{\theta\in S}\{ h_n(\theta) \in C \} \in \mathcal{S}_n^P$ *and:*

$$\bar{P}\{ h_n(\theta) \in C \mid \theta \in S \} = 1 \ .$$

(ii) *If* $\mathcal{H}$ *is separable and for some open set* $U \in \mathcal{G}(\Theta_0)$ *and* $n \geq 1$ *we have* $P\{ h_n(\theta) = 0 \} = 1$ *for all* $\theta \in U$ *, then* $\bigcap_{\theta\in U}\{ h_n(\theta) = 0 \} \in \mathcal{S}_n^P$ *and:*

$$\bar{P}\{ h_n(\theta) = 0 \mid \theta \in U \} = 1 \ .$$ □

Proposition 7.

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is separable and for all* $\theta \in \Theta_0$ *we have:*

$$P\text{-}\lim_{\xi\to\theta} h_n(\xi) = h_n(\theta)$$

for all $n \geq 1$ *, then every countable dense subset of* Θ_0 *satisfies the conditions of the separability definition.* □

Proposition 8.

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales such that we have:*

$$P\text{-}\lim_{\xi\to\theta} h_n(\xi) = h_n(\theta)$$

for all $\theta \in \Theta_0$ *and all* $n \geq 1$ *. Let* B *be a subset of* Θ_0 *and suppose that for every* $k \in \mathbf{N}$ *there exists a family* $D_k = \{ \delta_1^{(k)}, \ldots, \delta_{m_k}^{(k)} \}$ *in* B *such that* D_k *become dense in* B *when* $k \to \infty$ *, i.e. such that* $\forall \theta \in B$ *we have:*

$$\lim_{k\to\infty} \min_{1\leq i\leq m_k} d(\delta_i^{(k)}, \theta) = 0 \ .$$

Then we have:

(i) *If* $\mathcal{H}$ *is separable relative to* B *and* $\mathcal{C}_{-\infty}(\mathbf{R})$ *, then there exists a* P*-nullset* $N \in \mathcal{F}$ *such that:*

(1) $$\lim_{k\to\infty} \max_{1\leq i\leq m_k} h_n(\omega, \delta_i^{(k)}) = \sup_{\theta\in B} h_n(\omega,\theta)$$

for all $\omega \in \Omega \setminus N$.

(ii) *If* $\mathcal{H}$ *is separable relative to* B *and* $\mathcal{C}_{+\infty}(\mathbf{R})$ *, then there exists a* P*-nullset* $N \in \mathcal{F}$ *such that:*

(2) $$\lim_{k\to\infty} \min_{1\leq i\leq m_k} h_n(\omega, \delta_i^{(k)}) = \inf_{\theta\in B} h_n(\omega,\theta)$$

for all $\omega \in \Omega \setminus N$.

In particular, if $\mathcal{H}$ *is separable relative to the given* B *and* $\mathcal{C}(\mathbf{R})$ *, then there exists a* P*-nullset* $N \in \mathcal{F}$ *such that (1) and (2) hold for all* $\omega \in \Omega \setminus N$ *.* □

Remark 3. Since Θ_0 is by our assumption metrizable, then for a given subset B of Θ_0 there exists a countable dense set $D = \{ \delta_i \mid i \geq 1 \}$ in B . Putting $D_k = \{ \delta_1, \ldots, \delta_k \}$ for $k \geq 1$ we may easily verify that D_k's become dense in B when $k \to \infty$. Hence we see that the given hypothesis in proposition 8 is automatically satisfied in our setting.

Proposition 9.

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0\big)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is separable, then there exists a* P*-nullset* $N \in \mathcal{F}$ *such that for every* $\theta \in \Theta_0$ *and every* $n \geq 1$ *there exists a sequence* $\{ \xi_i \mid i \geq 1 \}$ *in* Θ_0 *satisfying* $\xi_i \to \theta$ *for* $i \to \infty$ *such that:*

(1) $$\limsup_{\xi\to\theta} h_n(\omega,\xi) = \limsup_{i\to\infty} h_n(\omega,\theta_i)$$

(2) $$\liminf_{\xi\to\theta} h_n(\omega,\xi) = \liminf_{i\to\infty} h_n(\omega,\theta_i)$$

for all $\omega \in \Omega \setminus N$. $\square$

Corollary 10.

Let $\mathcal{H} = \big(\{\ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\ \} \mid \theta \in \Theta_0\ \big)$ *be a given family of reversed submartingales. If* $\mathcal{H}$ *is separable and* $\lim_{i\to\infty} h_n(\theta_i)$ *exists with probability 1 whenever* $\theta_i \to \theta$ *for some* $\theta \in \Theta_0$ *and* $n \geq 1$, *then* $\lim_{\xi\to\theta} h_n(\xi)$ *exists with probability 1.* $\square$

Remark 4. Note that the exceptional set of probability zero may depend on the sequence $\{\ \theta_i \mid i \geq 1\ \}$ in the hypotheses of corollary 10, but nevertheless $\lim_{\xi\to\theta} h_n(\xi)$ exists with probability 1.

The next theorem shows that the assumption of separability on $\mathcal{H}$ is not a restriction to the finite dimensional distributions of the stochastic process $\{\ h_n(\omega,\theta) \mid \theta \in \Theta_0\ \}$ for $n \geq 1$. The following lemma plays a fundamental role in this direction.

Lemma 11.

Let $\mathcal{H} = \big(\{\ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1\ \} \mid \theta \in \Theta_0\ \big)$ *be a given family of reversed submartingales, and let us fix* $n \geq 1$. *Then we have:*

(i) *For every* $B \in \mathcal{B}(\mathbf{R})$, *there exists a sequence* $\{\ \theta_i \mid i \geq 1\ \}$ *in* Θ_0 *such that:*

(1) $$P\{\ h_n(\theta_i) \in B\,,\, h_n(\theta) \notin B \mid i \geq 1\ \} = 0$$

for all $\theta \in \Theta_0$.

(ii) *Moreover, let* $\mathcal{A}_0 \subset \mathcal{B}(\mathbf{R})$ *such that* $card\ \mathcal{A}_0 \leq \aleph_0$ *and let* $\mathcal{A} = (\mathcal{A}_0)_\delta$. *Then there exists a sequence* $\{\ \theta_i \mid i \geq 1\ \}$ *in* Θ_0 *such that for every* $\theta \in \Theta_0$ *there exists a* P*-null set* $N_\theta \in \mathcal{S}_n$ *satisfying:*

(2) $$\{\ h_n(\theta_i) \in A\,,\, h_n(\theta) \notin A \mid i \geq 1\ \} \subset N_\theta$$

for all $A \in \mathcal{A}$.

Proof. (i): Let $\theta_1 \in \Theta_0$ be an arbitrary point. For $\theta_1, \theta_2, \dots, \theta_k$ already chosen define:

$$b_k = \sup_{\theta\in\Theta_0} P\{\ h_n(\theta_i) \in B\,,\, h_n(\theta) \notin B \mid i \leq k\ \}$$

with given $\theta \in \Theta_0$. Then obviously $b_1 \geq b_2 \geq \dots \geq 0$. If $b_k = 0$, then $\theta_1, \theta_2, \dots, \theta_k, \theta_k, \theta_k, \dots$ may form a desired sequence. If $b_k > 0$, then choose $\theta_{k+1} \in \Theta_0$ such that:

$$P\{\ h_n(\theta_i) \in B\,,\, h_n(\theta_{k+1}) \notin B \mid i \leq k\ \} > b_k \cdot (1 - 1/(k{+}1))\ .$$

If $b_k > 0$ for all $k \geq 1$, then we have:

$$P\{\ h_n(\theta_i) \in B\,,\, h_n(\theta) \notin B \mid i \geq 1\ \} \leq \lim_{k\to\infty} b_k\ .$$

Since the sets $\{\ h_n(\theta_i) \in B\,,\, h_n(\theta_{k+1}) \notin B \mid i \leq k\ \}$ are disjoint for $k \geq 1$, then their probabilities form a convergent series. Therefore we have:

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} b_k \cdot (1 - 1/(k{+}1)) \leq \lim_{k\to\infty} P\{\ h_n(\theta_i) \in B\,,\, h_n(\theta_{k+1}) \notin B \mid i \leq k\ \} = 0\ .$$

This fact combined with the preceding inequality completes the proof of (i).

(ii): By (i) for every $B \in \mathcal{A}_0$ there exists a sequence $\{ \theta_i^B \mid i \geq 1 \}$ in Θ_0 such that we have $P(N_\theta^B) = 0$ with $N_\theta^B = \{ h_n(\theta_i^B) \in B , h_n(\theta) \notin B \mid i \geq 1 \} \in \mathcal{S}_n$ for all $\theta \in \Theta_0$. Let us now define:

$$N_\theta = \cup_{B \in \mathcal{A}_0} N_\theta^B \quad \text{and} \quad \{ \theta_i \mid i \geq 1 \} = \cup_{B \in \mathcal{A}_0} \{ \theta_i^B \mid i \geq 1 \} .$$

Then $N_\theta \in \mathcal{S}_n$ and $P(N_\theta) = 0$ for all $\theta \in \Theta_0$. Now if $A = \bigcap_{j=1}^\infty B_j \in \mathcal{A}$ with $B_j \in \mathcal{A}_0$ for $j \geq 1$, then we have:

$$\begin{aligned}
&\{ h_n(\theta_i) \in A , h_n(\theta) \notin A \mid i \geq 1 \} = \\
&\{ h_n(\theta_i) \in \cap_{j=1}^\infty B_j , h_n(\theta) \notin \cap_{j=1}^\infty B_j \mid i \geq 1 \} \subset \\
&\cup_{j=1}^\infty \{ h_n(\theta_i) \in B_j , h_n(\theta) \notin B_j \mid i \geq 1 \} \subset \\
&\cup_{j=1}^\infty \{ h_n(\theta_i^{B_j}) \in B_j , h_n(\theta) \notin B_j \mid i \geq 1 \} \subset \cup_{j=1}^\infty N_\theta^{B_j} \subset N_\theta .
\end{aligned}$$

Thus (ii) follows, and the proof of the lemma is complete. □

Theorem 12.

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$ *be a given family of reversed submartingales. Then* $\mathcal{H}$ *admits a separable* $\bar{\mathbf{R}}$*-valued modification* $\tilde{\mathcal{H}} = \big(\{ \tilde{h}_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$.

Proof. Let D be a countable dense subset of Θ_0, let $\mathcal{A}_0 = \{ \Theta_0 \backslash b(\theta, r) \mid \delta \in D , r \in \mathbf{Q}_+ \}$, and let $\mathcal{A} = (\mathcal{A}_0)_\delta$. Then $\mathcal{A}$ is the family of all closed subsets of Θ_0. Fix $n \geq 1$ and apply lemma 11 to the family of reversed submartingales:

$$\big(\{ h_n(\omega, \delta') , \mathcal{S}_n \mid n \geq 1 \} \mid \delta' \in b(\delta, r) \big)$$

for some $\delta \in D$ and $r \in \mathbf{Q}_+$ with $\mathcal{A}_0$ and $\mathcal{A}$ just defined. According to lemma 11 there exists a countable set $C_{\delta,r} \subset b(\delta, r)$ such that for every $\theta \in b(\delta, r)$ there exists a P-null set $N_{\theta;\delta,r} \in \mathcal{S}_n$ satisfying:

$$\{ h_n(\delta') \in A , h_n(\theta) \notin A \mid \delta' \in C_{\delta,r} \} \subset N_{\theta;\delta,r}$$

for every $A \in \mathcal{A}$. Let us put:

$$C = \bigcup_{\delta \in D , r \in \mathbf{Q}_+} C_{\delta,r} \quad \text{and} \quad N_\theta = \bigcup_{\delta \in D , r \in \mathbf{Q}_+ , b(\delta,r) \ni \theta} N_{\theta;\delta,r} .$$

Let $A_{\delta,r}(\omega) = cl_{\bar{\mathbf{R}}} \{ h_n(\omega, \delta') \mid \delta' \in b(\theta, r) \cap C \}$ for all $\omega \in \Omega$. Note that $A_{\delta,r}(\omega)$ is non-empty closed set in $\bar{\mathbf{R}}$ which may include the values $-\infty$ and $+\infty$. Moreover the set:

$$A_\theta(\omega) = \bigcap_{\delta \in D , r \in \mathbf{Q}_+ , b(\delta,r) \ni \theta} A_{\delta,r}(\omega)$$

is the intersection of a sequence of compact sets satisfying the finite intersection property in the compact space $\bar{\mathbf{R}}$, and thus it is closed and non-empty. Since:

$$h_n(\omega, \theta) \in A_{\delta,r}(\omega)$$

for all $\theta \in b(\delta, r)$ and all $\omega \notin N_{\theta;\delta,r}$, then we have:

$$h_n(\omega,\theta) \in A_\theta(\omega)$$

for all $\theta \in \Theta_0$ and all $\omega \notin N_\theta$. Now for every $\omega \in \Omega$ and $\theta \in \Theta_0$ define:

$$\tilde{h}_n(\omega,\theta) = h_n(\omega,\theta) \text{ , if } \theta \in C \text{ or } \theta \notin C \text{ but } \omega \notin N_\theta \text{ , and}$$
$$\tilde{h}_n(\omega,\theta) = a(\omega,\theta) \text{ , if } \theta \notin C \text{ and } \omega \in N_\theta \text{ ,}$$

where $a(\omega,\theta)$ is an arbitrary point in $A_\theta(\omega)$. With this definition we shall show that the family:

$$\tilde{\mathcal{H}} = \Big(\{ \tilde{h}_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \Big)$$

is a separable modification of $\mathcal{H}$. We obviously have:

$$P\{ \tilde{h}_n(\theta) = h_n(\theta) \} = 1$$

for all $\theta \in \Theta_0$ and all $n \geq 1$. Since $N_\theta \in \mathcal{S}_n$, then every $h_n(\cdot,\theta)$ is $\mathcal{S}_n$-measurable for all $\theta \in \Theta_0$ and all $n \geq 1$. This shows that $\tilde{\mathcal{H}}$ is a modification of $\mathcal{H}$. In addition we shall verify that $\tilde{\mathcal{H}}$ is separable. Let for this $B = b(\delta,r)$ for some $\delta \in D$ and $r \in \mathbf{Q}_+$ and let K be a closed subset of $\bar{\mathbf{R}}$. Suppose that for some $\omega \in \Omega$ we have:

$$\tilde{h}_n(\omega,\delta') \in K$$

for all $\delta' \in B \cap C$, that is $A_{\delta,r}(\omega) \subset K$. By the definition of $\tilde{h}_n(\omega,\theta)$ we have:

$$\tilde{h}_n(\omega,\theta) = h_n(\omega,\theta) \in A_{\delta,r}(\omega) \text{ , if } \theta \in C \text{ or } \theta \notin C \text{ but } \omega \notin N_\theta \text{ , and}$$
$$\tilde{h}_n(\omega,\theta) = a(\omega,\theta) \in A_\theta(\omega) \subset A_{\delta,r}(\omega) \text{ , if } \theta \notin C \text{ and } \omega \in N_\theta \text{ .}$$

for all $\theta \in b(\delta,r)$. This shows that:

$$\{ \tilde{h}_n(\delta') \in K \mid \delta' \in b(\delta,r) \cap C \} = \{ \tilde{h}_n(\theta) \in K \mid \theta \in b(\delta,r) \}$$

for all $\delta \in D$, all $r \in \mathbf{Q}_+$ and every closed set K in $\bar{\mathbf{R}}$. Hence we see that $\tilde{\mathcal{H}}$ is separable relative to $\mathcal{G}_{b,D}(\Theta_0)$ and $\mathcal{C}(\bar{\mathbf{R}})$, and by proposition 2 we find that $\tilde{\mathcal{H}}$ is separable. □

Remark 5. Note that submartingales $\{ \tilde{h}_n(\omega,\theta) \mid n \geq 1 \}$, $\theta \in \Theta_0$ from the given separable modification $\tilde{\mathcal{H}}$ of $\mathcal{H}$ in theorem 12 may attain the values $-\infty$ and $+\infty$ although the submartingales $\{ h_n(\omega,\theta) \mid n \geq 1 \}$, $\theta \in \Theta_0$ from $\mathcal{H}$ can be strictly real valued. Clarify that the separability of a $\bar{\mathbf{R}}$-valued family of reversed submartingales is defined in the analogous way as for the $\mathbf{R}$-valued ones. Unfortunately the result in theorem 12 has no practical meaning for the statistical models mentioned in the beginning of the paper.

4. Conditionally S-regular families of reversed submartingales

Let $\mathcal{H} = \big(\{ h_n(\omega,\theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$ be a family of reversed submartingales defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by an analytic metric space Θ_0 , and let (Θ, d) be a compact metric space containing Θ_0 . Then by our convention $h_n(\omega,\theta) = -\infty$ for all $\omega \in \Omega$ and all $\theta \in \Theta \setminus \Theta_0$. Let $\mathcal{M}$ be a family of sets B in Θ satisfying:

(i) $\omega \mapsto h_n^*(\omega,B)$ is $\mathcal{S}_n^P$-measurable

(ii) $Eh_n^*(B) < \infty$

for all $n \geq k$ with some given and fixed $k \geq 1$. Then $\mathcal{H}$ is said to be *conditionally* S-*regular* relative to $\mathcal{M}$, if for every $B \in \mathcal{M}$ there exist a P-null set $N \in \mathcal{F}$ and versions $\hat{E}\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}(\omega)$ of the conditional expectations $E\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}$ satisfying:

$$\hat{E}\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}(\omega) \geq h_{n+1}(\omega, \theta)$$

for all $\omega \in \Omega \setminus N$, all $\theta \in B$ and all $n \geq k$. Note that by the projection theorem (see [3]), condition (i) is automatically satisfied whenever $\mathcal{H}$ is measurable and B is an analytic subset of Θ . The family of all analytic sets in Θ will be denoted by $\mathcal{A}(\Theta)$. Let us for given $k \geq 1$ define:

$$\mathcal{L}^*(\mathcal{H}) = \mathcal{L}^*(\mathcal{H}, k) = \{ B \subset \Theta \mid E^* h_n^*(B) < \infty , \forall n \geq k \}$$

where E^* denotes the upper P-integral. Let B be a subset of Θ , then B is said to be:

(i) *countably S-regular* relative to $\mathcal{H}$, if $\forall n \geq 1$ there exist a sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ and a P-nullset $N \in \mathcal{F}$ satisfying:

$$\sup_{\theta \in B} h_n(\omega, \theta) = \sup_{\theta_i \in B} h_n(\omega, \theta_i)$$

for all $\omega \in \Omega \setminus N$

(ii) *conditionally S-regular* relative to $\mathcal{H}$, if $\{ h_n^*(B), \mathcal{S}_n^P \mid n \geq k \}$ is a reversed submartingale for some $k \geq 1$.

The family of all countably S-regular sets in Θ relative to $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$, and the family of all conditionally S-regular sets in Θ relative to $\mathcal{H}$ with $k = 1$ in the definition is denoted by $\mathcal{M}(\mathcal{H})$. By proposition 3.3 we see that $\mathcal{C}_0(\mathcal{H}) = \{ B \cap \Theta_0 \mid B \in \mathcal{C}(\mathcal{H}) \}$ is the greatest family of subsets of Θ_0 with respect to which $\mathcal{H}$ is separable relative to $\mathcal{C}_{-\infty}(\mathbf{R})$. Similarly, it is evident that $\mathcal{M}(\mathcal{H})$ is the greatest family of subsets of Θ with respect to which $\mathcal{H}$ is conditionally S-regular with $k = 1$ in the definition. Moreover, it is easily verified that we have:

(1) $\mathcal{C}(\mathcal{H})$ is closed under the formations of finite and countable unions

(2) If $B_i \in \mathcal{M}(\mathcal{H})$ for $i \in I$, and I is finite, or I is countable and $\bigcup_{i \in I} B_i \in \mathcal{L}^*(\mathcal{H}, 1)$, then $\bigcup_{i \in I} B_i \in \mathcal{M}(\mathcal{H})$

(3) If $B \subset \Theta$ with $card\ B \leq \aleph_0$, then $B \in \mathcal{C}(\mathcal{H})$

(4) If $B \subset \Theta$ with $card\ B < \aleph_0$, then $B \in \mathcal{C}(\mathcal{H}) \cap \mathcal{M}(\mathcal{H})$

(5) If $\mathcal{H}$ is separable relative to $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C}_{-\infty}(\mathbf{R})$, then $\mathcal{S} \subset \mathcal{C}(\mathcal{H})$

(6) If $\mathcal{H}$ is separable, then $\mathcal{G}(\Theta_0) \subset \mathcal{C}(\mathcal{H})$

(7) If $B \subset \Theta$, then $B \in \mathcal{C}(\mathcal{H})$ if and only if $B_0 = B \cap \Theta_0 \in \mathcal{C}(\mathcal{H})$, and $B \in \mathcal{M}(\mathcal{H})$ if and only if $B_0 \in \mathcal{M}(\mathcal{H})$.

The next proposition shows that separable families of reversed submartingales are conditionally S-regular.

Proposition 1.
Let $\mathcal{H} = \big(\{ h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$ *be a given family of reversed submartingales.*

If $\mathcal{H}$ is separable relative to $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C}_{-\infty}(\mathbf{R})$, then it is conditionally S-regular relative to $\mathcal{S} \cap \mathcal{L}^*(\mathcal{H})$. In particular, if $\mathcal{H}$ is separable, then it is conditionally S-regular relative to $\mathcal{G}(\Theta_0) \cap \mathcal{L}^*(\mathcal{H})$.

Proof. Let $B \in \mathcal{S} \cap \mathcal{L}^*(\mathcal{H})$, then by (i) in proposition 3.3 there exists a sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ_0 and a P-null set $N \in \mathcal{F}$ satisfying:

(1) $$\sup_{\theta \in B} h_n(\omega, \theta) = \sup_{\theta_i \in B} h_n(\omega, \theta_i)$$

for all $\omega \in \Omega \setminus N$ and all $n \geq 1$. Moreover there exists $k \geq 1$ satisfying $Eh_k^*(B) < \infty$. Now for given versions $\tilde{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega)$ and $\tilde{E}\{ h_k(\theta) \mid \mathcal{S}_{k+1} \}(\omega)$ of the conditional expectations $E\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}$ and $E\{ h_k(\theta) \mid \mathcal{S}_{k+1} \}$ we may easily achieve:

$$\tilde{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega) \geq \tilde{E}\{ h_k(\theta) \mid \mathcal{S}_{k+1} \}(\omega) \geq h_{k+1}(\omega, \theta)$$

for all $\theta \in B$ and all $\omega \notin N_\theta$, where $N_\theta \in \mathcal{S}_{k+1}$ is a P-null set. Then $N_\infty = \bigcup_{\theta_i \in B} N_{\theta_i} \in \mathcal{S}_{k+1}$ is a P-null set. Let us define:

$$\hat{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega) = \tilde{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega) \cdot 1_{\Omega \setminus N_\infty}(\omega) \ + \ \sup_{\theta_i \in B} h_{k+1}(\omega, \theta_i) \cdot 1_{N_\infty}(\omega)$$

for all $\omega \in \Omega$. Then $\hat{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega)$ is a version of the conditional expectation $E\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}$ satisfying:

$$\hat{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega) \geq \sup_{\theta_i \in B} h_{k+1}(\omega, \theta_i) \ .$$

Hence by (1) we obtain:

(2) $$\hat{E}\{ h_k^*(B) \mid \mathcal{S}_{k+1} \}(\omega) \geq h_{k+1}^*(\omega, B)$$

for all $\omega \in \Omega \setminus N$. Moreover by taking expectations in (2) we get $Eh_{k+1}^*(B) < \infty$. By induction the preceding procedure can be repeated and in this way we get versions $\hat{E}\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}(\omega)$ of conditional expectations $E\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}$ satisfying:

$$\hat{E}\{ h_n^*(B) \mid \mathcal{S}_{n+1} \}(\omega) \geq h_{n+1}^*(\omega, B)$$

for all $\omega \in \Omega \setminus N$ and all $n \geq k$. These facts complete the proof. □

Corollary 2.

Let $\mathcal{H} = \big(\{ h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0 \big)$ be a given family of reversed submartingales, and let $\mathcal{M} \subset \mathcal{A}(\Theta) \cap \mathcal{L}^*(\mathcal{H})$ be a family of subsets of Θ. Then we have:

(i) *If $\mathcal{H}$ is measurable and conditionally S-regular relative to $\mathcal{M}$, then we have:*

(1) $$Eh_n^*(B) \geq Eh_{n+1}^*(B)$$

for all $B \in \mathcal{M}$ and all $n \geq k$ with some $k \geq 1$. In particular, if $\mathcal{H}$ is separable relative to $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C}_{-\infty}(\mathbf{R})$, then (1) holds for every $B \in \mathcal{S} \cap \mathcal{L}^*(\mathcal{H})$. And if $\mathcal{H}$ is separable, then (1) holds for every $B \in \mathcal{G}(\Theta_0) \cap \mathcal{L}^*(\mathcal{H})$. In any of these three cases $\{ h_n^*(B), \mathcal{S}_n^P \mid n \geq k \}$ forms a reversed submartingale.

(iii) *If $\mathcal{H}$ is measurable and conditionally S-regular relative to $\mathcal{M}$ and $B(\theta, r_0) = \{ \, b(\theta, r) \mid r \in \mathbf{Q}_+ \, , \, r \leq r_0 \, \} \subset \mathcal{M}$ for some $r_0 > 0$ with a given point $\theta \in \Theta$, then we have:*

$$E\bar{h}_n(\theta) \geq E\bar{h}_{n+1}(\theta) \tag{2}$$

for all $n \geq k$ with some $k \geq 1$. In particular, if $\mathcal{H}$ is separable relative to $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C}_{-\infty}(\mathbf{R})$ and $B_0(\theta, r_o) = \{ \, b(\theta, r) \cap \Theta_0 \mid r \in \mathbf{Q}_+ \, , \, r \leq r_0 \, \} \subset \mathcal{S} \cap \mathcal{L}^*(\mathcal{H})$ for some $r_0 > 0$, then (2) holds. In any of these two cases $\{ \, \bar{h}_n(\theta) \, , \mathcal{S}_n^P \mid n \geq k \, \}$ forms a reversed submartingale.

Proof. (i): If $\mathcal{H}$ is measurable, then by the projection theorem (see [3]) and the hypothesis $\mathcal{M} \subset \mathcal{A}(\Theta)$ we see that the map $\omega \mapsto h_n^*(\omega, B)$ is $\mathcal{S}_n^P$-measurable for all $B \in \mathcal{M}$ and all $n \geq 1$. Hence (i) follows directly by definition and proposition 1.

(ii): Since $b(\theta, r_0) \in \mathcal{M} \subset \mathcal{L}^*(\mathcal{H})$, then there exists $k \geq 1$ such that we have $Eh_k^*(b(\theta, r)) \leq Eh_k^*(b(\theta, r_0)) < \infty$ for all $r \in \mathbf{Q}_+$, $r \leq r_0$. Furthermore, since $\mathcal{H}$ is conditionally S-regular relative to $\mathcal{M}$, then there exist a P-null set $N \in \mathcal{F}$ and a version $\hat{E}\{ \, h_k^*(b(\theta, r_0)) \mid \mathcal{S}_{k+1} \, \}(\omega)$ of the conditional expectation $E\{ \, h_k^*(b(\theta, r_0)) \mid \mathcal{S}_{k+1} \, \}$ satisfying:

$$\hat{E}\{ \, h_k^*(b(\theta, r_0)) \mid \mathcal{S}_{k+1} \, \}(\omega) \geq h_{k+1}^*(\omega, b(\theta, r_0)) \tag{1}$$

for all $\omega \in \Omega \setminus N$. Taking expectations in (1) we find that $Eh_{k+1}^*(b(\theta, r_0)) < \infty$ and by induction we analogously get $Eh_n^*(b(\theta, r)) < \infty$, $\forall n \geq k$, $\forall r \in \mathbf{Q}_+$, $r \leq r_0$. By the conditional monotone convergence theorem and conditional S-regularity of $\mathcal{H}$ relative to $B_0(\theta, r_0)$ we may conclude:

$$E\{ \, \bar{h}_n(\theta) \mid \mathcal{S}_{n+1} \, \} = E\{ \, \lim_{r \downarrow 0} h_n^*(b(\theta, r)) \mid \mathcal{S}_{n+1} \, \} = \lim_{r \downarrow 0} E\{ \, h_n^*(b(\theta, r)) \mid \mathcal{S}_{n+1} \, \} \geq$$
$$\geq \lim_{r \downarrow 0} h_{n+1}^*(b(\theta, r)) = \bar{h}_{n+1}(\theta)$$

for all $n \geq k$. This shows that $\{ \, \bar{h}_n(\theta) \, , \, \mathcal{S}_n^P \mid n \geq k \, \}$ is a reversed submartingale. Hence (2) follows directly. The second statement follows directly by proposition 1. $\square$

The family $\mathcal{H}$ is said to be *a.s.-lower semicontinuous, a.s.-upper semicontinuous* or *a.s.-continuous*, if there exists a P-null set $N \in \mathcal{F}$ such that $\forall \omega \in \Omega \setminus N$ the map $\theta \mapsto h_n(\omega, \theta)$ is lower semicontinuous, upper semicontinuous or continuous on Θ_0 for all $n \geq 1$, respectively.

Proposition 3.

Let $\mathcal{H} = \big(\{ \, h_n(\omega, \theta), \mathcal{S}_n \mid n \geq 1 \, \} \mid \theta \in \Theta_0 \, \big)$ be a given family of reversed submartingales. Then we have:

(i) *If $\mathcal{H}$ is a.s.-continuous, then it is separable relative to 2^{Θ_0} and $\mathcal{C}(\mathbf{R})$. In particular, in this case $\mathcal{H}$ is conditionally S-regular relative to $\mathcal{L}^*(\mathcal{H})$.*

(ii) *If $\mathcal{H}$ is a.s.-lower semicontinuous, then it is separable relative to 2^{Θ_0} and $\mathcal{C}_{-\infty}(\mathbf{R})$. In particular, in this case $\mathcal{H}$ is conditionally S-regular relative to $\mathcal{L}^*(\mathcal{H})$.*

(iii) *If $\mathcal{H}$ is a.s.-upper semicontinuous, then it is separable relative to 2^{Θ_0} and $\mathcal{C}_{+\infty}(\mathbf{R})$.*

In any of the cases (i), (ii) and (iii) every countable dense subset of a given set $B \in 2^{\Theta_0}$ satisfies the *conditions of the separability definition relative to* B *and* $\mathcal{C}(\mathbf{R})$, $\mathcal{C}_{-\infty}(\mathbf{R})$, $\mathcal{C}_{+\infty}(\mathbf{R})$ *, respectively.*

Proof. (i): Let $B \in 2^{\Theta_0}$ be given and fixed, then there exists a countable dense subset $D = \{ \theta_i \mid i \geq 1 \}$ of B . If $C \in \mathcal{C}(\mathbf{R})$ is given and we have:

$$h_n(\omega, \theta_i) \in C$$

for all $\theta_i \in B$ and all $\omega \in \Omega \setminus N$, then by the continuity of $h_n(\omega, \cdot)$ we may conclude:

$$h_n(\omega, \theta) \in C$$

for all $\theta \in B$ and all $\omega \in \Omega \setminus N$, where $N \in \mathcal{F}$ is a P-null set from (i). Hence we see that $\mathcal{H}$ is separable relative to 2^{Θ_0} and $\mathcal{C}(\mathbf{R})$. The last statement in (i) follows directly by proposition 1.

(ii): Let $B \in 2^{\Theta_0}$ be given and fixed, then there exists a countable dense subset $D = \{ \theta_i \mid i \geq 1 \}$ of B . If $C_p = (-\infty, p\,] \in \mathcal{C}_{-\infty}(\mathbf{R})$ with $p \in \mathbf{Q}$ is given and we have:

$$h_n(\omega, \theta_i) \in C_p$$

for all $\theta_i \in B$ and all $\omega \in \Omega \setminus N$, then for every $\theta \in B$ there exists a sequence $\{ \theta_{i_k} \mid k \geq 1 \}$ in D such that $\theta_{i_k} \to \theta$ for $k \to \infty$, and hence by the lower semicontinuity of $h_n(\omega, \cdot)$ we find:

$$h_n(\omega, \theta) \leq \liminf_{k \to \infty} h_n(\omega, \theta_{i_k}) \leq p$$

for all $\theta \in B$ and all $\omega \in \Omega \setminus N$, where $N \in \mathcal{F}$ is a P-null set from (ii). This shows that $\mathcal{H}$ is separable relative to 2^{Θ_0} and $\mathcal{C}_{-\infty}(\mathbf{R})$. The last statement in (ii) follows directly by proposition 1.

(iii): The proof is the very same as the proof of (ii). □

Remark 1. By example 1.1 we see that the given assumptions of a.s.-continuity, a.s.-lower semicontinuity and a.s.-upper semicontinuity in (i), (ii) and (iii) in proposition 3 is not in general possible to replace by weaker *pointwise* a.s.-continuity, a.s.-lower semicontinuity and a.s.-upper semicontinuity in order to deduce separability and regularity.

Example 4. ***(A conditionally*** S***-regular family of reversed submartingales)***

As we have seen in corollary 2 conditional S-regular families of reversed submartingales satisfy inequalities (1.1) and (1.2) for families of sets and points in the index space. However note that example 1.1 describes a family of reversed submartingales which is not conditionally S-regular relative to any non-trivial family of sets in the given index space. Thus in order to convince ourselves in the artless of the conditional S-regularity condition we will now present one significant class of families of reversed submartingales satisfying this condition, which includes all U-processes and covers a large number of random functions occurring in probability and statistics.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(T, \mathcal{B})$ be a measurable space, and let $(T^\infty, \mathcal{B}^\infty)$ be the countable product of $(T, \mathcal{B})$. Let $\mu \in Pr(T^\infty, \mathcal{B}^\infty)$ be a probability measure on $(T^\infty, \mathcal{B}^\infty)$, and let M_μ denote the linear space of all $\mathbf{R}$-valued μ-measurable functions on T^∞ . Let $\{ S_n \mid n \geq 1 \}$ be a family of *idempotent linear operators* on M_μ , that is:

(i) $$S_n^2 = S_n \ , \quad \forall n \geq 1$$

and let $I_n = \{ g \in M_\mu \mid S_n(g) = g \}$ be *the fixed point subspace* associated to S_n for $n \geq 1$.

Let $\mathcal{S}_n = \sigma(I_n)$ be the σ-algebra on T^∞ generated by I_n, and let E_μ^n be the subspace of functions f in M_μ satisfying $\int_{T^\infty} f \, d\mu < \infty$ such that:

(ii) $$\int_B S_n(f) \, d\mu = \int_B f \, d\mu \quad , \quad \forall B \in \mathcal{S}_n$$

for $n \geq 1$. Put $E_\mu = \bigcap_{n=1}^\infty E_\mu^n$ and let $\mathcal{S}_X^n = X^{-1}(\mathcal{S}_n)$ for any map $X : \Omega \rightarrow T^\infty$ and $n \geq 1$.

(1) If $X : \Omega \rightarrow T^\infty$ is a random function with the law μ, then we have:

$$S_n(f)(X) = E\{ f(X) \mid \mathcal{S}_X^n \}$$

for all $f \in E_\mu^n$ and all $n \geq 1$.

In particular, if we suppose that:

(iii) $$S_n S_{n+1} = S_{n+1} \;\Leftrightarrow\; I_n \supset I_{n+1} \quad , \quad \forall n \geq 1$$

then $\mathcal{S}_n \supset \mathcal{S}_{n+1}$ for all $n \geq 1$, and hence we see that:

$$\{ S_n(f)(X) , \mathcal{S}_X^n \mid n \geq 1 \}$$

is a reversed martingale for $f \in E_\mu \cap L^1(\mu)$, or a reversed submartingale for $f \in E_\mu \setminus L^1(\mu)$. Let $\{ h_\theta \mid \theta \in B \}$ be a family of functions in M_μ indexed by a set B. Suppose that S_{n+1} is *monotone*, that is:

(iv) $$S_{n+1}(f) \geq 0 \quad , \quad \forall f \geq 0 .$$

In addition suppose that:

(v) $$S_{n+1} S_n = S_{n+1} .$$

Then the assumption $\sup_{\theta \in B} S_n(h_\theta) \in E_\mu^{n+1}$ implies:

$$E\{ \sup_{\theta \in B} S_n(h_\theta)(X) \mid \mathcal{S}_X^{n+1} \} = S_{n+1}(\sup_{\theta \in B} S_n(h_\theta))(X) \geq$$
$$\sup_{\theta \in B} S_{n+1} S_n(h_\theta)(X) = \sup_{\theta \in B} S_{n+1}(h_\theta)(X)$$

whenever $n \geq 1$. Counting all these facts we may conclude:

(2) Let $\{ S_n \mid n \geq 1 \}$ be a family of idempotent monotone linear operators defined on the linear space M_μ of all $\mathbf{R}$-valued μ-measurable functions on T^∞ satisfying:

$$S_n S_{n+1} = S_{n+1} S_n = S_{n+1} \quad , \quad \forall n \geq 1$$

where $\mu \in Pr(T^\infty, \mathcal{B}^\infty)$ is a given probability measure on $(T^\infty, \mathcal{B}^\infty)$. Let X be a random function defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in $(T^\infty, \mathcal{B}^\infty)$ and the law μ. Let $H = \{ h(\,\cdot\,, \theta) \mid \theta \in \Theta_0 \}$ be a family of functions in M_μ indexed by an analytic metric space Θ_0, and let $\mathcal{E}_\mu = \{ B \subset \Theta_0 \mid E_\mu^{n+1} \ni \sup_{\theta \in B} S_n(h(\theta))$ is $\mathcal{S}_n^\mu$-measurable for all $n \geq 1 \}$. If $H \subset E_\mu \cap L^1(\mu)$ resp. $H \subset E_\mu \setminus L^1(\mu)$, then:

$$\mathcal{H} = (\{ S_n(h(\theta))(X) , \mathcal{S}_X^n \mid n \geq 1 \} \mid \theta \in \Theta_0)$$

is a family of reversed martingales resp. submartingales that is conditionally S-regular relative to $\mathcal{E}_\mu$.

Furthermore, suppose that:

(vi) $$S_n(1) = 1 \quad , \quad \forall n \geq 1$$

(vii) $$S_n(f_m) \downarrow 0 \ \text{ as } \ f_m \downarrow 0 \ \text{ with } \ f_m \leq 1 \ , \forall n \geq 1 \ .$$

Then there exists *a Markov kernel* $Q_n(B,t)$ on $(T^\infty, \mathcal{B}^\infty)$ satisfying:

$$S_n(f)(t) = \int_{T^\infty} f(s) \ Q_n(ds,t)$$

for all $f \in M_\mu$, all $t \in T^\infty$ and all $n \geq 1$. Moreover, putting:

$$Q_n(\,\cdot\,,t) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \delta_{t_\sigma}(\,\cdot\,)$$

for all $t = (t_1, t_2, \dots) \in T^\infty$ and all $n \geq 1$, where $\mathcal{P}_n$ denotes the set of all permutations of $\{ 1, 2, \dots, n \}$, $t_\sigma = (t_{\sigma_1}, \dots, t_{\sigma_n}, t_{n+1}, \dots)$ for $\sigma \in \mathcal{P}_n$, and δ_t denotes *the Dirac measure* at the point t , it is easily to verify that the family of operators $\{ S_n \mid n \geq 1 \}$ defined by:

$$S_n(f)(t) = \int_{T^\infty} f(s) \ Q_n(ds,t) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} f(t_\sigma)$$

$\forall f \in M_\mu$, $\forall t \in T^\infty$ and $\forall n \geq 1$ satisfies conditions (i), (iii), (iv), (v), (vi) and (vii). Moreover, if $\mu \in Pr(T^\infty, \mathcal{B}^\infty)$ is *an exchangeable probability measure* on $(T^\infty, \mathcal{B}^\infty)$, that is:

$$\mu\,(\prod_{i=1}^{\infty} B_i) = \mu\,(B_{\sigma_1} \times \ldots \times B_{\sigma_n} \times B_{n+1} \times \ldots\,)$$

$\forall \sigma \in \mathcal{P}_n$, $\forall B_i \in \mathcal{B}$, $i \geq 1$ and $\forall n \geq 1$, then (ii) is also satisfied with $E_\mu = \{ f \in M_\mu \mid \int f \ d\mu < \infty \}$. Note that in this case I_n is the linear space of all μ-measurable n-*symmetric* $\mathbf{R}$-valued functions on T^∞ , and $\mathcal{S}_n$ is *the permutation invariant σ-algebra* on T^∞ *of order* n for all $n \geq 1$. And if the function $(t,\theta) \mapsto h(t,\theta)$ is $\mathcal{B}^\infty \times \mathcal{B}_0$-measurable, where $H = \{ h(\,\cdot\,,\theta) \mid \theta \in \Theta_0 \}$ is a given family of functions in M_μ indexed by an analytic metric space Θ_0 , then by the projection theorem (see [3]) the role of the family $\mathcal{E}_\mu$ may play the family $\mathcal{A}(\Theta_0) \cap \mathcal{L}^*(\mathcal{H})$. Hence by (2) and with the notation $S_n(h(\,\cdot\,,\theta))(t) = h^{<n>}(t,\theta)$ we find that:

(3) $$\mathcal{H}(X) = (\{ h^{<n>}(X,\theta) \, , \mathcal{S}_X^n \mid n \geq 1 \} \mid \theta \in \Theta_0\,)$$

is a family of reversed martingales for $H \subset L^1(\mu)$, or a family of reversed submartingales for $H \setminus L^1(\mu) \neq \emptyset$, and $\mathcal{H}(X)$ is conditionally S-regular relative to the family $\mathcal{A}(\Theta_0) \cap \mathcal{L}^*(\mathcal{H})$. Note that if $h(t,\theta) = g(t_1,\theta)$, then we have:

$$h^{<n>}(X,\theta) = \frac{1}{n} \sum_{i=1}^{n} g(X_i,\theta) \ ;$$

if $h(t,\theta) = (g(t_1,\theta) - M(\theta))^2$, then we have:

$$h^{<n>}(X,\theta) = \frac{1}{n} \sum_{i=1}^{n} \big(g(X_i,\theta) - M(\theta) \big)^2 \ ;$$

if $h(t,\theta) = \frac{1}{2}(g(t_1,\theta)-g(t_2,\theta))^2$, then we have:

$$h^{<n>}(X,\theta) = \frac{1}{n-1} \sum_{i=1}^{n} \big(g(X_i,\theta) - \frac{1}{n} \sum_{j=1}^{n} g(X_j,\theta) \big)^2 ;$$

where $X = (X_1, X_2, \dots)$, $t = (t_1, t_2, \dots) \in T^\infty$ and $\theta \in \Theta_0$.

Let us in addition note if $\{ X_i \mid i \geq 1 \}$ is a sequence of independent identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(T, \mathcal{B})$ and a common law π , then the product measure $\mu = \pi^\infty$ on $(T^\infty, \mathcal{B}^\infty)$ is exchangeable and the preceding conclusions can be applied to the random function $X = (X_1, X_2, \dots)$. Moreover, one can easily verify that the following statement is also satisfied:

(4) If $\{ h_n(t,\theta) \mid n \geq 1 \}$ is a sequence of $\mathbf{R}$-valued functions on $T^\infty \times \Theta_0$ satisfying:

(i) $\int h_n(\theta) \; d\mu < \infty$, $\forall \theta \in \Theta_0$ and $\forall n \geq 1$

(ii) The map $(t,\theta) \mapsto h_n(t,\theta)$ is $\mathcal{S}_n \times \mathcal{B}_0$-measurable for all $n \geq 1$

(iii) $h_n^{<n+1>}(\cdot ,\theta) \geq h_{n+1}(\cdot ,\theta)$ μ-$a.s.$, $\forall \theta \in \Theta_0$

then $\mathcal{H}(X) = (\{ h_n(X,\theta), \mathcal{S}_X^n \mid n \geq 1 \} \mid \theta \in \Theta_0)$ is a family of reversed submartingales that is conditionally S-regular relative to $\mathcal{A}(\Theta_0) \cap \mathcal{L}^*(\mathcal{H}(X))$.

The same statement remains valid for exchangeable sequences of random variables as well. We shall leave verification of these facts and the remaining details to the reader.

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