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Perfect Measures and Maps

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This paper is designed to provide a solid introduction to the theory of non-measurable calculus. In this context basic concepts are reviewed and fundamental facts are presented. Perfect measures and maps are shown to of a vital importance to support the theory, and therefore these concepts are treated separately as well. In particular, many equivalent statements to the property of being perfect are given. The last part of the paper deals with a setting arising in the investigation on extension of measures in the case of perturbations of their σ-algebras.

1. Introduction

This paper is designed to provide a solid introduction to the theory of non-measurable calculus. The content is divided into three parts. Section 2 deals with the non-measurable calculus. Together with some new definitions of concepts which are shown useful, it contains several new results which are mainly generalizations or completions of the corresponding old ones. For additional information in this context we refer to [2]. Section 3 deals with perfect measures and maps. From the historical point of view the concept of perfectness relies upon the following two well-known facts:

(1.1) If μ is a finite, finitely additive measure defined on the algebra $\mathcal{A}$ of subsets of a set X such that for each decreasing sequence $\{ A_n \mid n \geq 1 \}$ in $\mathcal{A}$ with empty intersection we have $\lim_{n\to\infty} \mu(A_n) = 0$, then μ is countably additive and admits a unique extension to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$.

(1.2) If $\{ K_n \mid n \geq 1 \}$ is a decreasing sequence of compact sets in a topological space X with empty intersection, then there is $N \geq 1$ such that $\bigcap_{n=1}^{N} K_n$ is empty.

These two facts in a certain way form a base of the general axiomatic theory of Kolmogorov, see [10]. Marczewski in [12] generalized this idea and introduced *compact* measures. In [4] Gnedenko and Kolmogorov were defined *perfect* measures, while Ryll-Nardzewski in [20] introduced and studied the same concept of *quasi-compact* measures. Since that time the properties of compactness and perfectness have been subject of a number of studies, see [6], [11], [13], [14], [15], [16], [19], [21], [22], [23]. It is shown that by the aid of these properties we avoid many pathological phenomena that arise within the framework of Kolmogorov's probability theory (*independence*, see [21]; *the existence of the regular conditional probability*, see [15], [16], [21]; *countable additivity of the product measures*, see [12], [13], [15], [20]). Also, using mainly this idea Alexandroff in [1] has given an extensive development of the theory of bounded regular finitely additive measures in topological spaces, which is at the background of many later studies. In this section we present many equivalent statements to the property of being perfect, as well as many general properties

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of perfect measures and closely related perfect maps. Many of these facts have its own origin in Høffmann-Jørgensen's unpublished notes [7]. For simplicity, most of the proofs are usually omitted or briefly sketched. Section 4 deals with an algebraic setting arising in the investigation on extension of measures in the case of perturbations of their σ-algebras, see [17] and [18]. The approach presented could be suitable for investigations carried out in the framework of the general vector space theory, with the given examples in mind.

2. Non-measurable calculus

2.1. Completions. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Then μ^* and μ_* denote *the outer* and *the inner μ-measure*, $\mathcal{A}^\mu$ denotes the σ-algebra of *μ-measurable subsets* of X , i.e.

$$\mathcal{A}^\mu = \{\ B \in 2^X \mid \exists A, A' \in \mathcal{A}\ ,\ A \subset B \subset A' \text{ such that } \mu(A' \setminus A) = 0\ \}$$

and $\mathcal{A}^*$ denotes the σ-algebra of *universally measurable subsets* of X , i.e.

$$\mathcal{A}^* = \bigcap\nolimits_\mu \mathcal{A}^\mu$$

where μ ranges over the family of all finite measures on $(X, \mathcal{A})$. Let us recall that the σ-algebra $\mathcal{A}^\mu$ is called *the completion of* $\mathcal{A}$ with respect to μ , that $A \in \mathcal{A}^\mu$ if and only if $\mu^*(A) = \mu_*(A)$, and that the restriction of μ^* or μ_* on $\mathcal{A}$ is the only measure on $\mathcal{A}^\mu$ that agrees on $\mathcal{A}$ with μ . This measure is called *the completion of* μ and is denoted by $\bar{\mu}$. In particular, for each finite measure ν on $(X, \mathcal{A})$ there is a unique measure on $(X, \mathcal{A}^*)$ that agrees with μ on $\mathcal{A}$. Let us recall that the given unique finite measure space $(X, \mathcal{A}^\mu, \bar{\mu})$ is called *the completion of* $(X, \mathcal{A}, \mu)$, and that a finite measure space $(X, \mathcal{A}, \mu)$ (or a measure μ) is called *complete* if $\mathcal{A}^\mu = \mathcal{A}$ and $\bar{\mu} = \mu$.

2.2. Kernels and hulls. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. If $C \in 2^X$ is an arbitrary subset of X , then C^* denotes *the μ-hull* of C , and C_* denotes *the μ-kernel* of C , i.e. $C^*, C_* \in \mathcal{A}$ and $\mu(C^*) = \mu^*(C)$, $\mu(C_*) = \mu_*(C)$. It is easily seen that each subset of X has the μ-hull and the μ-kernel, uniquely determined up to a μ-nullset, and we will without loss of generality always assume that $C_* \subset C \subset C^*$ for all $C \in 2^X$.

2.3. Restrictions, traces and extensions. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $\mathcal{A}'$ be a sub-σ-algebra of $\mathcal{A}$, and let $C \in 2^X$ be an arbitrary set. *The restriction of* μ *on the σ-algebra* $\mathcal{A}'$ is a measure $r(\mu, \mathcal{A}') : \mathcal{A}' \to [0, \infty\rangle$ defined by:

$$r(\mu, \mathcal{A}')(A') = \mu(A')$$

for all $A' \in \mathcal{A}'$. *The trace of* $\mathcal{A}$ *on the set* C is a σ-algebra defined by:

$$tr(\mathcal{A}, C) = \{A \cap C \mid A \in \mathcal{A}\}\ .$$

If $C \in \mathcal{A}$, then *the trace of* μ *on the set* C is a measure $tr(\mu, C) : \mathcal{A} \to [0, \infty\rangle$ defined by:

$$tr(\mu, C)(A) = \mu(A \cap C)$$

for all $A \in \mathcal{A}$. If C is not necessarily from $\mathcal{A}$, then we distinguish the following two possibilities:

– *the outer trace of* μ *on the set* C is a measure $tr^*(\mu, C) : \sigma(\mathcal{A} \cup \{C\}) \to [0, \infty>$ defined by:

$$tr^*(\mu, C)([A, B]) = \mu(A \cap C^*)$$

where $[A, B] = (A \cap C) \cup (B \cap C^c)$, with $A, B \in \mathcal{A}$, is a general representation of an element from $\sigma(\mathcal{A} \cup \{C\})$;

– *the inner trace of* μ *on the set* C is a measure $tr_*(\mu, C) : \sigma(\mathcal{A} \cup \{C\}) \to [0, \infty)$ defined by:

$$tr_*(\mu, C)([A, B]) = \mu(A \cap C_*)$$

where $[A, B]$ is as above.

It is easily seen that the definitions of the outer and inner traces are good, i.e. they do not depend on the choice of the μ-hull and the μ-kernel of the set C , as well of the representation of elements from the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$. Moreover, the following properties are satisfied:

(2.1) $tr^*(\mu, C)([A, B]) = \mu^*(A \cap C)$, $\forall [A, B] \in \sigma(\mathcal{A} \cup \{C\})$

(2.2) $tr_*(\mu, C)([A, B]) = \mu_*(A \cap C)$, $\forall [A, B] \in \sigma(\mathcal{A} \cup \{C\})$

(2.3) $tr^*(\mu, C)(A) + tr_*(\mu, C^c)(A) = \mu(A)$, $\forall A \in \mathcal{A}$

(2.4) $tr^*(\mu, C)(C) = \mu^*(C)$

(2.5) $tr_*(\mu, C)(C) = \mu_*(C)$

(2.6) $tr^*(\mu, C)(C^c) = tr_*(\mu, C)(C^c) = 0$

(2.7) If $C \in \mathcal{A}$, then $tr^*(\mu, C) = tr_*(\mu, C) = tr(\mu, C)$

(2.8) If $C \in \mathcal{A}^\mu$, then $tr^*(\mu, C) = tr_*(\mu, C) = r\big(tr(\bar{\mu}, C), \sigma(\mathcal{A} \cup \{C\})\big)$.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\hat{X} \supset X$ be a set. *The extension of* $\mathcal{A}$ *on the set* $\hat{X}$ is a σ-algebra defined by:

$$\hat{\mathcal{A}} = \{ \ \hat{A} \in 2^{\hat{X}} \mid \hat{A} \cap X \in \mathcal{A} \ \} \ .$$

The extension of the measure μ *on the set* $\hat{X}$ is a measure $ext(\mu, \hat{X}) : \hat{\mathcal{A}} \to [0, \infty)$ defined by:

$$ext(\mu, \hat{X})(\hat{A}) = \mu(\hat{A} \cap X)$$

for all $\hat{A} \in \hat{\mathcal{A}}$. It is easily checked that the following properties are satisfied:

(2.9) $\mathcal{A} \subset \hat{\mathcal{A}}$ and $r\big(ext(\mu, \hat{X}), \mathcal{A}\big) = \mu$

(2.10) $ext(\mu, \hat{X})(\hat{A}) = 0$, whenever $\hat{A} \subset \hat{X} \setminus X$

(2.11) $tr\{ext(\mu, \hat{X}), X\} = ext(\mu, \hat{X})$

(2.12) $r\Big(ext\big((tr(\mu, C), \hat{X}\big), \mathcal{A}\Big) = tr(\mu, C)$, whenever $C \in \mathcal{A}$

(2.13) $\{ext(\mu, \hat{X})\}^*(\hat{C}) = \mu^*(\hat{C} \cap X)$, whenever $\hat{C} \in 2^{\hat{X}}$

(2.14) $\{ext(\mu, \hat{X})\}_*(\hat{C}) = \mu_*(\hat{C} \cap X)$, whenever $\hat{C} \in 2^{\hat{X}}$.

2.4. Lower and upper integrals. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $L^1(\mu)$ be the set of all μ-integrable $\bar{\mathbf{R}}$-valued functions on X . Then by:

$$\int^* f \ d\mu = \inf \{ \ \int g \ d\mu \mid g \in L^1(\mu), f \le g \ \}$$
$$\int_* f \ d\mu = \sup \{ \ \int g \ d\mu \mid g \in L^1(\mu), g \le f \ \}$$

the upper and *lower μ-integral* of an arbitrary $\bar{\mathbf{R}}$-function f on X are defined. It is well-known that the upper and lower μ-integrals satisfy the following properties:

(2.15) $\int_* f \ d\mu + \int_* \ g \ d\mu \le \int_*(f+g) \ d\mu \le \int_* f \ d\mu + \int^* g \ d\mu$

(2.16) $\int_* f \ d\mu + \int^* g \ d\mu \le \int^*(f+g) \ d\mu \le \int^* f \ d\mu + \int^* g \ d\mu$

(2.17) $\int^*(af) \ d\mu = a \int^* f \ d\mu \ , \ \forall a \in \mathbf{R}_+$

(2.18) $\int_*(af) \ d\mu = a \int_* f \ d\mu \ , \ \forall a \in \mathbf{R}_+$

(2.19) $\int^*(af) \ d\mu = a \int_* f \ d\mu \ , \ \forall a \in \mathbf{R}_-$

(2.20) $\int_*(af) \ d\mu = a \int^* f \ d\mu \ , \ \forall a \in \mathbf{R}_-$

(2.21) If $f \le g$, then $\int^* f \ d\mu \le \int^* g \ d\mu$

(2.22) If $f \le g$, then $\int_* f \ d\mu \le \int_* g \ d\mu$

(2.23) If $f \le g$, then $\int_* f \ d\mu \le \int^* g \ d\mu$

(2.24) If $f_n \uparrow f$ and $\int^* f_1 \ d\mu > -\infty$, then $\int^* f \ d\mu = \lim \ \int^* f_n \ d\mu$

(2.25) If $f_n \downarrow f$ and $\int_* f_1 \ d\mu < +\infty$, then $\int_* f \ d\mu = \lim \ \int_* f_n \ d\mu$

(2.26) If $\int^*(\text{ inf } f_n) \ d\mu > -\infty$, then $\int^*(\text{ lim inf } f_n) \ d\mu \le \text{lim inf} \int^* f_n \ d\mu$

(2.27) If $\int_*(\text{ sup } f_n) \ d\mu < +\infty$, then $\int_*(\text{ lim sup } f_n) \ d\mu \ge \text{lim sup} \int_* f_n \ d\mu$

(2.28) $\int^* 1_C \ d\mu = \mu^*(C) \ , \ \forall C \in 2^X$

(2.29) $\int_* 1_C \ d\mu = \mu_*(C) \ , \ \forall C \in 2^X$

(2.30) $\int f \ d\mu$ exists if and only if $\int_* f \ d\mu = \int^* f \ d\mu$, and in this case we have $\int f \ d\mu = \int^* f d\mu = \int_* f d\mu$

provided that all relations are well-defined. All preceding statements remain valid for an arbitrary measure space $(X, \mathcal{A}, \mu)$, where in the definition of the upper and inner μ-integral one can use:

$$L(\mu) = \{ \ f \in \bar{\mathbf{R}}^X \mid \int f \ d\mu \ \text{ exists in } \ \bar{\mathbf{R}} \ \}$$

instead of $L^1(\mu)$, but with the same meaning if μ is finitely founded, see below.

Remark 2.1 Let H be a Hamel basis for $\mathbf{R}$ as a vector space over $\mathbf{Q}$, and let $\{ \ H_n \mid n \ge 1\}$ be a strictly increasing sequence of subsets of H such that $\bigcup_{n=1}^{\infty} H_n = H$. Let L_n be the smallest subspace of $\mathbf{R}$ over $\mathbf{Q}$ which contains H_n for all $n \geq 1$. Then $L_n \uparrow \mathbf{R}$ and $L_n \neq \mathbf{R}$ for all $n \geq 1$. If $B \in \mathcal{B}(\mathbf{R})$ and $B \subset L_n$, then $\lambda(B) > 0$ implies $<-\varepsilon\ ,\varepsilon> \subset \mathrm{diff(B)} = \{\ \mathrm{x-y} \mid \mathrm{x,y} \in \mathrm{B}\ \} \subset \mathrm{L_n}$, for some $\varepsilon > 0$ and hence the smallest subspace of $\mathbf{R}$ over $\mathbf{Q}$ which contains $<-\varepsilon\ ,\varepsilon>$ is contained in L_n . But this subspace is actually equal to $\mathbf{R}$. Thus $\lambda_*(L_n) = 0$ and hence $\lambda^*([0,1] \cap L_n^c) = 1$ for all $n \geq 1$, but $A_n = [0,1] \cap L_n^c \downarrow \emptyset$. Now putting $f_n = 1_{A_n}$ or $1_{A_n^c}$ on the finite measure space $([0,1], tr\big(\mathcal{B}(\mathbf{R}), [\mathbf{0},\mathbf{1}]\big), \lambda)$, it is easily verified that (2.24)-(2.27) do not hold if we replace the upper integral $\int^*$ with the lower $\int_*$, or reversely.

2.5. Lower and upper envelopes. The upper and lower μ-integral can be much easier handled by using so-called upper and lower μ-envelope of the integrand. These envelopes are built in order to establish a simple connection with the ordinary μ-integral. The central point in the construction of the envelopes plays the following well-known property of a finite measure.

Lemma 2.2 (Segal's localization principle)

Let $(X,\mathcal{A},\mu)$ be a finite measure space, and let $\mathcal{F}$ be an arbitrary family of $\mathcal{A}$-measurable $\bar{\mathbf{R}}$-valued functions on X . Then there exists a sequence $\{\ f_n \mid n \geq 1\} \subset \mathcal{F}$ such that the $\mathcal{A}$–measurable function $S : X \to \bar{\mathbf{R}}$, defined by:

$$S(x) = \sup_{n\geq 1} f_n(x)$$

for $x \in X$, satisfies the following properties:

(2.31) *$f \leq S$ μ-a.a. for all $f \in \mathcal{F}$*

(2.32) *If $S' : X \to \bar{\mathbf{R}}$ is an another $\mathcal{A}$-measurable function such that $f \leq S'$ μ-a.a. for all $f \in \mathcal{F}$, then $\mu\{S' < S\} = 0$, i.e. $S \leq S'$ μ-a.s.*

Proof. Since $x \mapsto (2/\pi)\arctan x$ is a strictly increasing function from $\bar{\mathbf{R}}$ onto $[-1,1]$, it is no restriction to assume that:

$$|f(x)| \leq 1\ ,\ \forall x \in X \ \text{ and } \ \forall f \in \mathcal{F}\ .$$

Define $\mathcal{G} = \{\ g \in \mathbf{R}^X \mid g = \sup_{n\geq 1} f_n$, for some $\{f_n \mid n \geq 1\} \subset \mathcal{F}\ \}$ and put:

$$G = \sup\{\ \textstyle\int g\ d\mu \mid g \in \mathcal{G}\ \}\ .$$

By assumption, we see that $G \in \mathbf{R}$, and by definition of G , we can choose a sequence $\{\ g_n \mid n \geq 1\ \} \subset \mathcal{G}$ such that $G = \sup_{n\geq 1} \int g_n\ d\mu$. Put:

$$S = \sup_{n\geq 1}\ g_n$$

and note that $S \in \mathcal{G}$ actually. Since $S \geq g_n$ for all $n \geq 1$, hence it follows $\int S\ d\mu = G$. Furthermore, since $\mathcal{F} \subset \mathcal{G}$ we have $S \vee f \in \mathcal{G}$. This implies $\int (S \vee f)\ d\mu = G = \int S\ d\mu$ for all $f \in \mathcal{F}$, i.e. $S = S \vee f$ μ-a.a. for all $f \in \mathcal{F}$ which implies (i), while (ii) is an easy consequence of (i) and definition of S . □

Such a function S , which satisfies (2.31) and (2.32) and which is uniquely determined up to a μ-nullset actually, is called *the μ-essential supremum* of $\mathcal{F}$ and is denoted by $S = \mu$-ess sup $\mathcal{F}$. Similarly, *the μ-essential infimum* of $\mathcal{F}$ is a function $T : X \to \bar{\mathbf{R}}$ denoted and defined by $T = \mu$-ess inf $\mathcal{F} = -\{\ \mu$-ess sup $(-\mathcal{F})\ \}$, where $-\mathcal{F} = \{-f \mid f \in \mathcal{F}\}$.

If $(X, \mathcal{A}, \mu)$ is a finite measure space, then $\mathcal{M}(\mathcal{A})$ denotes the set of all $\mathcal{A}$-measurable real valued function on X , and $\bar{\mathcal{M}}(\mathcal{A})$ denotes the set of all $\mathcal{A}$-measurable $\bar{\mathbf{R}}$-valued functions on X . Then by:

$$f^* = \mu\text{-ess inf } \{\ g \in \bar{\mathcal{M}}(\mathcal{A}) \mid f \le g\ \}$$

$$f_* = \mu\text{-ess sup } \{\ g \in \bar{\mathcal{M}}(\mathcal{A}) \mid g \le f\ \}$$

the upper and lower μ-envelope of an arbitrary $\bar{\mathbf{R}}$-valued function f on X are defined. The following properties show that f^* and f_* are "the smallest" and "the largest" $\mathcal{A}$-measurable $\bar{\mathbf{R}}$-valued function on X which is greater or less than f , respectively:

(2.33) f^* and f_* are $\mathcal{A}$-measurable $\bar{\mathbf{R}}$-valued functions

(2.34) $f_*(x) \le f(x) \le f^*(x)$, $\forall x \in X$

(2.35) $g \le f_* \le f^* \le h$ μ-a.s. for all $g, h \in \bar{\mathcal{M}}(\mathcal{A})$ such that $g \le f \le h$ μ-a.s.

(2.36) $\mu_*\{f_* < g \le f\} = \mu_*\{f \le g < f^*\} = 0$, $\forall g \in \bar{\mathcal{M}}(\mathcal{A})$

(2.37) f is μ-measurable if and only if $f_* = f^*$ μ-a.s.

For (2.36) suppose that $\mu_*\{\ f_* < g \le f\ \} > 0$ for some $g \in \mathcal{M}(\mathcal{A})$. Then there exists $A \in \mathcal{A}$, $A \subset \{\ f_* < g \le f\ \}$ such that $\mu(A) > 0$. Put:

$$f_+ = g \cdot 1_A + f_* \cdot 1_{A^c}\ .$$

Then we have $f_+ \in \bar{\mathcal{M}}(A)$, $f_+ \le f$ and $\mu\{\ f_* < f_+\ \} = \mu(A) > 0$ which contradicts the definition of the lower μ-envelope f_* of f and proves (5.4).

Actually, by (2.33)-(2.35) we see that the upper and lower μ-envelope f^* and f_* of a function f are uniquely determined up to a μ-nullset, while (2.36) is equivalent to (2.37).

More generally, let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space and let $\bar{\mathcal{M}}(\mu)$ be the set of all μ-measurable $\bar{\mathbf{R}}$-valued functions on X . Then $\bar{\mathcal{M}}(\mu)$ is a lattice ordered by $\le$ μ-a.s., and we say that μ satisfies *Segal's localization principle* if $(\bar{\mathcal{M}}(\mu)\ , \le \mu\text{-a.s.})$ is a complete lattice. We obviously have:

(2.38) If μ satisfies Segal's localization principle, then f^* and f_* exist for all functions $f : X \to \bar{\mathbf{R}}$.

Using the preceding result on a finite measure and standard extension arguments we easily find:

(2.39) Every σ-finite measure satisfies Segal's localization principle. Moreover, if there exists a disjoint family $\mathcal{C} \subset \mathcal{A}^\mu$ such that:

(1) $\bar{\mu}(C) < \infty$, for all $C \in \mathcal{C}$

(2) If $A \cap C \in \mathcal{A}$ for all $C \in \mathcal{C}$, then $A \in \mathcal{A}^\mu$

(3) If $A \in \mathcal{A}$ and $\bar{\mu}(A \cap C) = 0$ for all $C \in \mathcal{C}$, then $\mu(A) = 0$

then μ satisfies Segal's localization principle.

Let $\mathcal{N}(\mu) = \{ N \in 2^X \mid \mu^*(N) = 0 \}$ be the family of all μ-null sets in X, and let ν be a measure on $(X, \mathcal{A})$. Then we obviously have:

(2.40) If $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ and μ satisfies Segal's localization principle, then so does ν.

The measure μ is said to be Σ-*finite*, if there is a sequence of finite measures $\{ \mu_n \mid n \geq 1 \}$ on $(X, \mathcal{A})$ such that $\mu = \sum_{n=1}^{\infty} \mu_n$. It is clear that each σ-finite measure is Σ-finite. Moreover:

(2.41) Every Σ-finite measure satisfies Segal's localization principle.

For this, let $\mu = \sum_{n=1}^{\infty} \mu_n$ be a Σ-finite measure. It is no restriction to assume $\mu_n \neq 0$ for all $n \geq 1$. Then:

$$\nu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu_n(X)} \ \mu_n(A)$$

is a finite measure on $(X, \mathcal{A})$ such that $\mathcal{N}(\nu) = \mathcal{N}(\mu)$. Thus (2.41) follows from (2.40).

Using the definition properties of the μ-envelopes (2.33)-(2.35) one can easily check that the following statements are satisfied:

(2.42) $f_* + g_* \leq (f+g)_* \leq f^* + g_* \leq (f+g)^* \leq f^* + g^*$

(2.43) $(\alpha f)^* = \alpha f^* \ , \ (\alpha f)_* = \alpha f_* \ , \ \forall \alpha \geq 0$

(2.44) $(\alpha f)^* = \alpha f_* \ , \ (\alpha f)_* = \alpha f^* \ , \ \forall \alpha < 0$

(2.45) $-|f|^* \leq f_* \leq f^* \leq |f|^*$

(2.46) $|f_*| \leq |f|^* \ , \ |f^*| \leq |f|^*$

(2.47) $|f^* - g^*| \leq |f - g|^*$

(2.48) $|f_* - g_*| \leq |f - g|^*$

(2.49) $f_* g_* \leq (fg)_* \leq f_* g^* \leq (fg)^* \leq f^* g^*$, if $f, g \geq 0$

provided that all relations are well-defined.

We shall now state the basic connection between upper and lower μ-integrals, upper and lower μ-envelopes, and outer and inner μ-measures. Namely, if $(X, \mathcal{A}, \mu)$ is a finite measure space, then for each $\bar{\mathbf{R}}$-valued function f on X, and for each subset C of X, we have:

(2.50) $$\int^* f \ d\mu = \begin{cases} \int f^* \ d\mu \ , & \text{if } f^* \in L(\mu) \\ +\infty \ , & \text{otherwise} \end{cases}$$

(2.51) $$\int_* f \ d\mu = \begin{cases} \int f_* \ d\mu \ , & \text{if } f_* \in L(\mu) \\ -\infty \ , & \text{otherwise} \end{cases}$$

(2.52) $(1_C)^* = 1_{C^*}$ and $(1_C)_* = 1_{C_*}$

(2.53) $\mu^*(C) = \mu(C^*) = \int^* 1_C \ d\mu$

(2.54) $\mu_*(C) = \mu(C_*) = \int_* 1_C \ d\mu$.

More generally, if an arbitrary measure space $(X, \mathcal{A}, \mu)$ satisfies Segal's localization principle, then (2.52)-(2.54) hold. We shall now characterize those measures μ for which (2.50) and (2.51) remain valid. Let us recall that a measure μ is called *finitely founded* if for every $A \in \mathcal{A}$ with $\mu(A) = \infty$, there is $B \in \mathcal{A}$, $B \subset A$ such that $0 < \mu(B) < \infty$, i.e. if μ has no infinite atoms. It is easily verified that any finite, σ-finite or Σ-finite measure is finitely founded, resp.

Theorem 2.3

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the following three statements are equivalent:

(2.55) μ *is finitely founded*

(2.56) $\mu(A) = \sup \{ \mu(B) \mid B \in \mathcal{A}, B \subset A \, , \, \mu(B) < \infty \}$ *for all* $A \in \mathcal{A}$

(2.57) *For each μ-measurable function $f : X \to \bar{\mathbf{R}}_+$ such that $\int f \, d\mu = \infty$, there exists a sequence $\{ f_n \mid n \geq 1 \}$ in $L^1(\mu)$ such that $0 \leq f_n \leq f$ for all $n \geq 1$ and $\int f_n \, d\mu \to \infty$ as $n \to \infty$.*

Moreover, if μ satisfies Segal's localization principle, then (2.55)-(2.57) is equivalent to each of the following two equivalent statements:

(2.58) $$\int^* f \, d\mu = \begin{cases} \int f^* \, d\mu \ , & \text{if } f^* \in L(\mu) \\ +\infty \ , & \text{otherwise} \end{cases} \quad , \quad \textit{for all } f \in \bar{\mathbf{R}}^X$$

(2.59) $$\int_* f \, d\mu = \begin{cases} \int f_* \, d\mu \ , & \text{if } f_* \in L(\mu) \\ -\infty \ , & \text{otherwise} \end{cases} \quad , \quad \textit{for all } f \in \bar{\mathbf{R}}^X \ .$$

Proof. (2.55) $\Rightarrow$ (2.56): Take $A \in \mathcal{A}$ with $\mu(A) = \infty$ and suppose that $S = \sup \{ \mu(B) \mid B \in \mathcal{C} \} < \infty$, where $\mathcal{C} = \{ C \in \mathcal{A} \mid C \subset A \, , \, \mu(C) < \infty \}$. Then there is a sequence $B_n \in \mathcal{C}$, $n \geq 1$ such that $\mu(B_n) > S - \frac{1}{n}$, $\forall n \geq 1$. Hence $B = \bigcup_{n=1}^\infty B_n \in \mathcal{C}$ with $\mu(B) = S$. Since $\mu(A) = \infty$, we have $\mu(A \setminus B) = \infty$, and by (2.55) there is $C \subset A \setminus B$ such that $0 < \mu(C) < \infty$. Then $B \cup C \in \mathcal{C}$, and hence $S \geq \mu(B \cup C) = \mu(B) + \mu(C) > \mu(B) = S$, which is a contradiction. This completes the proof of (2.56).

(2.56) $\Rightarrow$ (2.57): Let $f : X \to \bar{\mathbf{R}}_+$ be a μ-measurable function such that $\int f \, d\mu = \infty$. Let:

$$f_n = \sum_{i=1}^{m_n} x_i^n \, 1_{A_i{}^n} \ , \quad n \geq 1$$

be an increasing sequence of simple μ-measurable $\mathbf{R}_+$-valued functions on X such that $f_n \uparrow f$ as $n \to \infty$. For fixed $n \in \mathbf{N}$, and each $i = 1, 2, \dots, m_n$ such that $\mu(A_i^n) = \infty$, by (2.56) we can find an increasing sequence $B_{p,i}^n \in \mathcal{A}$, $p \geq 1$ such that $B_{p,i}^n \subset A_i$, $0 < \mu(B_{p,i}^n) < \infty$ and $\mu(B_{p,i}^n) \to \infty$ for $p \to \infty$. For each $i = 1, 2, \dots, m_n$ such that $\mu(A_i^n) < \infty$ put $B_{p,i}^n = A_i^n$, $\forall p \geq 1$. Define:

$$g_p^n = \sum_{i=1}^{m_n} x_i^n \, 1_{B_{p,i}^n} \ , \quad p \geq 1 \ .$$

Then by definition of g_p^n we easily find:

$$0 \leq g_p^n \leq f_n \ , \ \int g_p^n d\mu < \infty \ , \ \forall n, p \geq 1 \ \text{ and } \ \int g_p^n d\mu \to \int f_n d\mu$$

as $p \to \infty$, for all $n \geq 1$. Since by the monotone convergence theorem $\int f_n \, d\mu \to \int f \, d\mu = \infty$, it is easily checked that the sequence $\{ \, g_n^n \mid n \geq 1 \, \}$ satisfies:

$$0 \leq g_n^n \leq f \ , \ \int g_n^n \, d\mu < \infty \ , \ \forall n \geq 1 \ \text{ and } \ \int g_n^n \, d\mu \to \infty$$

as $n \to \infty$, which completes the proof of (2.57).

(2.57) $\Rightarrow$ (2.55): Let $A \in \mathcal{A}$ with $\mu(A) = \infty$. Then by (2.57) there is a sequence $\{ \, f_n \mid n \geq 1 \}$ in $L^1(\mu)$ such that $0 \leq f_n \leq 1_A$, $\int f_n \, d\mu < \infty$, $\forall n \geq 1$ and $\int f_n \, d\mu \to \infty$ as $n \to \infty$. It is no restriction to assume that each f_n , $n \geq 1$ is $\mathcal{A}$-measurable with $0 \leq f_n \leq 1_A$ μ-a.s. Then each $B_n = \{f_n > 0\}$ is in $\mathcal{A}$ and $B_n \subset A$ μ-a.s. Moreover $\int f_n \, d\mu = \int_{B_n} f_n \, d\mu \leq \mu(B_n)$, which shows $\mu(B_n) < \infty$, $n \geq 1$ and $\mu(B_n) \to \infty$ for $n \to \infty$. This completes the proof of (2.55).

Since $(-f)^* = -f_*$ and $(-f)_* = -f^*$, the equivalence of (2.58) and (2.59) is obvious. Hence it is enough to show that (2.57) and (2.59) are equivalent.

(2.57) $\Rightarrow$ (2.59): Since $f_* \leq f$, we have $\int_* f_* \, d\mu \leq \int_* f \, d\mu$. If there exists $g \in L^1(\mu)$ such that $g \leq f$, then $g \leq f_*$ μ-a.s., and hence $\int g \, d\mu \leq \int f_* \, d\mu$, which implies $\int_* f \, d\mu \leq \int f_* \, d\mu = \int_* f_* \, d\mu$. If there is no such $g \in L^1(\mu)$, then by definition we have $\int_* f \, d\mu = -\infty$ which together with the preceding statements in both cases implies:

(2.60)
$$\int_* f \, d\mu = \int_* f_* \, d\mu \ .$$

Now we shall show that $\int_* f_* \, d\mu$ is equal to the right side in (2.59). If $f_* \in L^1(\mu)$ this is obvious, and if $\int f_* \, d\mu = -\infty$ then this holds by definition of the lower μ-integral. In the case where $f_* \notin L(\mu)$, by (2.60) and definition of the lower μ-integral we get $\int_* f \, d\mu = -\infty$. So, let us suppose that $\int f_* \, d\mu = \infty$. Then $\int (f_*)^+ \, d\mu = \infty$ and by (2.57) we can find a sequence $\{ \, f_n \mid n \geq 1 \, \}$ in $L^1(\mu)$ such that $0 \leq f_n \leq (f_*)^+$, $n \geq 1$ and $\int f_n \, d\mu \to \infty$ as $n \to \infty$. Since $\int (f_*)^- \, d\mu < \infty$, by $g_n = f_n - (f_*)^-$, $n \geq 1$ a sequence from $L^1(\mu)$ is defined such that $g_n = f_n - (f_*)^- \leq (f_*)^+ - (f_*)^- = f_* \leq f$ and such that $\int g_n \, d\mu = \int f_n \, d\mu - \int (f_*)^- \, d\mu \to \infty$ as $n \to \infty$. This shows $\int_* f \, d\mu = \infty$ and completes the proof of (2.59).

(2.59) $\Rightarrow$ (2.57): Let $f : X \to \bar{\mathbf{R}}_+$ be a given μ-measurable function such that $\int f \, d\mu = \infty$. Since $f^* = f$ μ-a.s. hence by (2.59) it follows $\int^* f \, d\mu = \infty$. Then by definition of the upper μ-integral there is a sequence $\{ \, f_n \mid n \geq 1 \, \}$ in $L^1(\mu)$ satisfying $f_n \leq f$, $n \geq 1$ and $\int f_n \, d\mu \to \infty$ as $n \to \infty$. Since it is no restriction to assume $f_n \geq 0$, $n \geq 1$, the proof of (2.57) is complete. □

Next we extend some well-known results for the ordinary integral to the upper integral case. In this direction the following proposition (see [7]) is going to play a central role.

Proposition 2.4

Let $(X, \mathcal{A}, \mu)$ be a measure space such that μ satisfies Segal's localization principle, and let $f : X \to \bar{\mathbf{R}}$ and $G : \bar{\mathbf{R}} \to \bar{\mathbf{R}}$ be given functions. Then we have:

(2.61) *If* G *is increasing and left continuous, then* $G(f)^* = G(f^*)$.

(2.62) *If* G *is increasing and right continuous, then* $G(f)_* = G(f_*)$.

(2.63) *If* G *is decreasing and left continuous, then* $G(f)^* = G(f_*)$.

(2.64) *If* G *is decreasing and right continuous, then* $G(f)_* = G(f^*)$.

Proof. (2.61): Since G is increasing we have $G(f) \le G(f^*)$, and hence $G(f)^* \le G(f^*)$. Conversely, take $g \in \bar{\mathcal{M}}(\mathcal{A})$ such that $G(f) \le g$, and define:

$$G^{(-1)}(y) = \sup \{ \, x \in \mathbf{R} \mid G(x) \le y \, \}$$

for all $y \in \mathbf{R}$ with $\sup \emptyset = -\infty$. Put $G^{(-1)}(-\infty) = -\infty$ and $G^{(-1)}(+\infty) = +\infty$. Then $G^{(-1)} : \bar{\mathbf{R}} \to \bar{\mathbf{R}}$ is an increasing function and hence Borel measurable. Since G is increasing and left continuous we have:

(2.65) $$G(x) \le y \ \text{ if and only if } \ G^{(-1)}(y) \ge x$$

for all $x, y \in \bar{\mathbf{R}}$. Hence we find $G^{(-1)}(g) \ge f$, and since $G^{(-1)}$ belongs to $\bar{\mathcal{M}}(\mathcal{A})$, it follows $G^{(-1)}(g) \ge f^*$. Moreover, by (2.65) we get $G(f^*) \le g$. This implies $G(f^*) \le G(f)^*$, and the proof of (2.61) is complete. The proof of (2.62)-(2.64) is quite similar. □

Theorem 2.5
Let $(X, \mathcal{A}, \mu)$ *be a measure space, and let us for every* $1 \le p < \infty$ *define:*

$$L^{<p>}(\mu) = \left\{ \, f \in \mathbf{R}^X \;\; \middle| \;\; \int^* |f|^p \, d\mu < \infty \, \right\} .$$

Suppose that μ *satisfies Segal's localization principle. Then we have:*

(2.66) **(Markov's inequality)**
If $f : X \to \bar{\mathbf{R}}$ *is an arbitrary function, then:*

$$\mu^*\{ \, |f| > \varepsilon \, \} \le \frac{1}{\varepsilon^p} \int^* |f|^p \, d\mu$$

for all $p > 0$ *and all* $\varepsilon > 0$.

(2.67) *(***Holder's inequality)**
If $f \in L^{<p>}(\mu)$ *and* $g \in L^{<q>}(\mu)$ *for* $1 < p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *, then* $f \cdot g \in L^{<1>}(\mu)$ *and we have:*

$$\int^* |f \cdot g| \, d\mu \le \left\{ \int^* |f|^p \, d\mu \right\}^{\frac{1}{p}} \cdot \left\{ \int^* |g|^q \, d\mu \right\}^{\frac{1}{q}} .$$

(2.68) **(Minkowski's inequality)**
If $f, g \in L^{<p>}(\mu)$ *for* $1 \le p < \infty$ *, then* $f + g \in L^{<p>}(\mu)$ *and we have:*

$$\left\{ \int^* |f+g|^p \, d\mu \right\}^{\frac{1}{p}} \le \left\{ \int^* |f|^p \, d\mu \right\}^{\frac{1}{p}} + \left\{ \int^* |g|^p \, d\mu \right\}^{\frac{1}{p}} .$$

(2.69) **(Jensen's inequality)**
If $\mu(X) = 1$ *and* $C : \mathbf{R} \to \mathbf{R}$ *is convex, then for every* $f \in L^{<1>}(\mu)$ *we have:*

$$C\Big(\int^* f\ d\mu\Big) \le \int^* C \circ f\ d\mu\ .$$

Proof. The statement (2.66) follows by (2.50) and (***) below from the classic Markov's inequality. The statement (2.67) follows by (2.49), (2.50) and (2.61) from the classic Holder's inequality. The statement (2.68) follows by (2.50) and (2.61) from the classic Minkovski's inequality. For (2.69) note if C is decreasing, then $C(f^*) \le C(f)$ and hence measurability of C implies $C(f^*) \le C(f)^*$. If C is increasing, then by (2.61) we have $C(f^*) = C(f)^*$. Combining these two facts, using (i) of lemma 4.4 in [19] and piecewise monotonicity of a convex function, the statement (2.69) follows by (2.50) from the classic Jensen's inequality. $\square$

Let us remark that all relations (equalities, inequalities, ...) concerning the hulls and kernels of sets and envelopes of functions should be understood in the a.s.-sense, i.e. like ordinary relations with properly chosen representant from the given equivalence classes. Also, to avoid technical difficulties and to be concerned with main ideas only, we shall restrict ourselves to the finite case, i.e. we shall mainly deal with finite measure spaces and real valued functions in question.

We will now show that some known results on the envelopes (see [2]) remain valid if we replace Euclidean topology by Sorgenfrey topology on the real line. For this, if X is a set and $\mathcal{F}$ a family of subsets of X , then $\tau(\mathcal{F})$ denotes the smallest topology on X which contains $\mathcal{F}$. Let us further introduce the following notation for families of intervals on the real line: $\mathcal{I}_l(\mathbf{R}) = \{<a,b] \mid a,b \in \mathbf{R}\}$, $\mathcal{I}_r(\mathbf{R}) = \{[a,b> \mid a,b \in \mathbf{R}\}$, and $\mathcal{I}(\mathbf{R}) = \{<a,b> \mid a,b \in \mathbf{R}\}$. Then by $\mathcal{L}(\mathbf{R}) = \tau(\mathcal{I}_l(\mathbf{R}))$, $\mathcal{R}(\mathbf{R}) = \tau(\mathcal{I}_r(\mathbf{R}))$ and $\mathcal{E}(\mathbf{R}) = \tau(\mathcal{I}(\mathbf{R}))$ *left, right Sorgenfrey* and *Euclidean topology* on the real line are defined, resp. It is well-known that $(\mathbf{R}, \mathcal{L}(\mathbf{R}))$ and $(\mathbf{R}, \mathcal{R}(\mathbf{R}))$ are separable perfectly normal non-metrizable topological spaces which do not allow countable basis for their topology. Families $\mathcal{I}_l(\mathbf{R})$ and $\mathcal{I}_r(\mathbf{R})$ are the basis of the corresponding topologies, and their elements are open and closed at the same time. The corresponding weights of $\mathbf{R}$ equals continuum, i.e. $\mathrm{w}\{(\mathbf{R}, \mathcal{L}(\mathbf{R}))\} = \mathrm{w}\{(\mathbf{R}, \mathcal{R}(\mathbf{R}))\} = c$. Note that $\mathcal{E}(\mathbf{R}) = \mathcal{L}(\mathbf{R}) \cap \mathcal{R}(\mathbf{R})$.

Although topologies $\mathcal{L}(\mathbf{R})$ and $\mathcal{R}(\mathbf{R})$ do not allow countable basis, let us note if $L \in \mathcal{L}(\mathbf{R})$ is an open set in the left Sorgenfrey topology, then there exists a countable family of disjoint intervals $\{\, I_n \mid n \ge 1\,\} \subset \mathcal{I}_l(\mathbf{R})$ such that $L = \bigcup_{n=1}^\infty I_n$. Of course, the analogous conclusion holds for right Sorgenfrey topology $\mathcal{R}(\mathbf{R})$ as well. Indeed, if L is $\mathcal{L}(\mathbf{R})$-open, then there exists a family $\{\, I_\alpha \mid \alpha \in A\,\} \subset \mathcal{I}_l(\mathbf{R})$ such that $L = \bigcup_{\alpha \in A} I_\alpha$. Put $L_Q = L \cap \mathbf{Q}$ and for each $q \in L_Q$ look at the family $\mathcal{I}(g) = \{\, I_\alpha \mid q \in I_\alpha\ ,\ \alpha \in A\,\}$. Then it is easily checked that $I_q := \bigcup_{I \in \mathcal{I}(q)} I \in \{\mathcal{I}_l(\mathbf{R}) \cup \mathcal{I}(\mathbf{R})\}$ and $L = \bigcup_{q \in L_Q} I_q$. Since each $I \in \mathcal{I}(\mathbf{R})$ is a countable union of elements from $\mathcal{I}_l(\mathbf{R})$, hence we find that there exists a countable subfamily $\{\, J_n \mid n \ge 1\}$ of $\mathcal{I}_l(\mathbf{R})$ such that $L = \bigcup_{n=1}^\infty J_n$. Now take $I_1 = J_1$, $I_2 = J_2 \setminus J_1\ \dots\ I_n = J_n \setminus (\bigcup_{i=1}^{n-1} J_i)$ for $n \ge 2$, and then one can easily verify that $\{\, I_n \mid n \ge 1\,\}$ is a family of disjoint elements from $\mathcal{I}_l(\mathbf{R})$ such that $L = \bigcup_{n=1}^\infty I_n$. This completes the claim.

Proposition 2.6

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let f be an arbitrary real valued function on X .

(2.70) *If $G \in \mathcal{L}(\mathbf{R})$, then* $\mu\{f^* \in G\} = \mu^*\{f^* \in G\,,\, f \in G\,\}$.

(2.71) *If $G \in \mathcal{R}(\mathbf{R})$, then* $\mu\{f_* \in G\} = \mu^*\{f_* \in G\,,\, f \in G\,\}$.

Proof. (2.70): First note that it is enough to show:

$$\mu_*\{\, f^* \in G\,,\, f \notin G \,\} = 0\ .$$

For this, take $A \in \mathcal{A}$ such that $A \subset \{\, f^* \in G\,,\, f \notin G \,\}$. Since $G \in \mathcal{L}(\mathbf{R})$, we can find a family of disjoint intervals $\{\, <a_n, b_n] \mid n \geq 1 \,\}$ such that $G = \bigcup_{n=1}^{\infty} <a_n, b_n]$. Put $A_n = A \cap \{\, f^* \in <a_n, b_n] \,\}$ for $n \geq 1$, and $f^+ = f^* \cdot 1_{A^c} + \sum_{n=1}^{\infty} a_n 1_{A_n}$. Then $f^+ \in \mathcal{M}(\mu)$ and since $A \subset \{\, f \notin G \,\}$, by definition of f^* we find $f \leq f^+ \leq f^*$ and $A = \{\, f^+ < f^* \,\}$. This implies $\mu(A) = 0$, and the proof of (2.70) is complete. The proof of (268) follows similarly if we take on the right place $f_+ = f_* \cdot 1_{A^c} + \sum_{n=1}^{\infty} b_n 1_{A_n}$. □

Corollary 2.7

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let f be an arbitrary real valued function on X .

(2.72) *If $F \in 2^{\mathbf{R}}$ is $\mathcal{L}(\mathbf{R})$-closed, then $\mu\{\, f^* \in F \,\} = \mu_*(\{\, f^* \in F \,\} \cup \{\, f \in F \,\})$.*

(2.73) *If $F \in 2^{\mathbf{R}}$ is $\mathcal{R}(\mathbf{R})$-closed, then $\mu\{\, f_* \in F \,\} = \mu_*(\{\, f_* \in F \,\} \cup \{\, f \in F \,\})$.*

(2.74) *If $F \in 2^{\mathbf{R}}$ is $\mathcal{E}(\mathbf{R})$-closed, then $\mu_*\{\, f \in F \,\} = 1$ implies $\mu\{\, f^* \in F\,,\, f_* \in F \,\} = 1$.*

Proof. Since for an arbitrary subset C of X we have:

$$\mu^*(C) + \mu_*(C^c) \,=\, \mu(X)$$

then by Proposition 2.6 we find $\mu_*(\{\, f^* \in F \,\} \cup \{\, f \in F \,\}) = \mu(X) - \mu^*\{\, f^* \in F^c\,,\, f \in F^c \,\} = \mu(X) - \mu\{\, f^* \in F^c \,\} = \mu\{\, f^* \in F \,\}$ which proves (2.72), and $\mu_*(\{\, f_* \in F \,\} \cup \{\, f \in F \,\}) = \mu(X) - \mu^*\{\, f_* \in F^c\,,\, f \in F^c \,\} = \mu(X) - \mu\{\, f_* \in F^c \,\} = \mu\{\, f_* \in F \,\}$ which implies (2.73), while (2.74) follows directly by (2.72) and (2.73). □

Since Euclidean topology $\mathcal{E}(\mathbf{R})$ equals to $\mathcal{L}(\mathbf{R}) \cap \mathcal{R}(\mathbf{R})$, all the preceding statements from proposition 2.6 and Corollary 2.7 hold in the case when G is $\mathcal{E}(\mathbf{R})$-open and F is $\mathcal{E}(\mathbf{R})$-closed. In particular, we find that for an arbitrary real valued function f on a finite measure space $(X, \mathcal{A}, \mu)$, and for each $t \in \mathbf{R}$, the following relations are satisfied:

(2.75) $$\mu\{\, f^* > t \,\} \,=\, \mu^*\{\, f > t \,\}$$

(2.76) $$\mu\{\, f_* < t \,\} \,=\, \mu^*\{\, f < t \,\}$$

(2.77) $$\mu\{\, f^* \leq t \,\} \,=\, \mu_*\{\, f \leq t \,\}$$

(2.78) $$\mu\{\, f_* \geq t \,\} \,=\, \mu_*\{\, f \geq t \,\}\ .$$

Since for each $f \in \mathcal{M}(\mathcal{A})$ with $f \geq 0$, we have $\int_X f^p \; d\mu = p \int_0^\infty t^{p-1} \mu\{f \geq t\} \; dt = p \int_0^\infty t^{p-1} \mu\{f > t\} \; dt$, then from (2.75) and (2.78) together with (2.50) and (2.51) we obtain:

Corollary 2.8

If $f : X \to \mathbf{R}_+$ is an arbitrary function, then for all $p \geq 1$ we have:

(2.79) $$\int^* f^p \; d\mu = p \int_0^\infty t^{p-1} \mu^*\{f \geq t\} \; dt = p \int_0^\infty t^{p-1} \mu^*\{f > t\} \; dt$$

$$(2.80) \qquad \int_* f^p \, d\mu = p \int_0^\infty t^{p-1} \mu_* \{ f \geq t \} \, dt = p \int_0^\infty t^{p-1} \mu_* \{ f > t \} \, dt \ . \qquad \square$$

Let us remark that (2.75)-(2.80) remain valid if μ is an arbitrary measure on $(X, \mathcal{A})$ which satisfies Segal's localization principle.

2.6. Image and coimage measures. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let Y be a set, and let $f : X \to Y$ be a function. Then *the image measure* μ_f of μ *under* f is defined on the σ-algebra $\mathcal{B} = f^1(\mathcal{A}) = \{ \ B \in 2^Y \mid f^{-1}(B) \in \mathcal{A} \ \}$ as follows:

$$\mu_f(B) = \mu\{ f^{-1}(B) \}$$

for all $B \in \mathcal{B}$. Let X be a set, let $(Y, \mathcal{B}, \nu)$ be a finite measure space, and let $g : X \to Y$ be a function. Then *the outer and inner coimage measure* $\nu_{g^{-1}}$ and $\nu_{g_{-1}}$ *of* ν *under* g are defined on the σ-algebra $\mathcal{A} = g^{-1}(\mathcal{B})$ as follows:

$$\nu_{g^{-1}}(A) = \nu^*\{ g(A) \} \quad \text{and} \quad \nu_{g_{-1}}(A) = \nu_*\{ g(A) \}$$

for all $A \in \mathcal{A}$, resp. It is easily verified that μ_f , $\mu_{g^{-1}}$ and $\nu_{g_{-1}}$ are finite measures, and that the following statements are satisfied:

(2.81) $\quad (\mu_f)_* \leq \mu_* \circ f^{-1} \leq \mu^* \circ f^{-1} \leq (\mu_f)^*$ on 2^Y

(2.82) $\quad \nu_{g_{-1}} \leq \nu_{g^{-1}}$ on $\mathcal{A} = g^{-1}(\mathcal{B})$

(2.83) $\quad \nu_{g_{-1}}(A) = \nu_{g^{-1}}(A)$ if and only if $g(A) \in \mathcal{B}^\nu$, where $A \in \mathcal{A} = g^{-1}(\mathcal{B})$

(2.84) $\quad (\nu_{g^{-1}})^*(C) = \nu^*\{ g(C) \}$, $\forall C \in 2^X$

(2.85) $\quad (\nu_{g^{-1}})_*(C) \leq \nu^*\{ g(C) \}$, $\forall C \in 2^X$

(2.86) $\quad (\nu_{g_{-1}})^*(C) \geq \nu_*\{ g(C) \}$, $\forall C \in 2^X$

(2.87) $\quad (\nu_{g_{-1}})_*(C) \leq \nu_*\{ g(C) \}$, $\forall C \in 2^X$, with equality if g is injective

(2.88) $\quad (\mu_f)_{f^{-1}} = \mu$ on $\mathcal{A} = f^{-1}(\mathcal{B})$

(2.89) $\quad (\mu_f)_{f_{-1}} \leq \mu$ on $\mathcal{A} = f^{-1}(\mathcal{B})$

(2.90) $\quad (\mu_f)_{f_{-1}}(A) = \mu(A)$ if and only if $f(A) \in \mathcal{B}^{\mu_f}$, where $A \in \mathcal{A} = f^{-1}(\mathcal{B})$

(2.91) $\quad (\nu_{g^{-1}})_g = \nu$ if and only if $\nu_*\{ Y \setminus g(X) \} = 0$

(2.92) $\quad (\nu_{g_{-1}})_g = \nu$ if and only if $\nu^*\{ Y \setminus g(X) \} = 0$.

In this context note that for each $A \in \mathcal{A} = f^{-1}(\mathcal{B})$, there is $B \in \mathcal{B}$ such that $A = g^{-1}(B)$. Moreover, it is very useful to observe that we can replace this general set B by the ν-hull $f(A)^*$ of $f(A)$, since without loss of generality we can assume that $f(A)^* \subset B$.

Proposition 2.9

Let X be a set, let $(Y, \mathcal{B}, \nu)$ be a finite measure space, let $f : X \to Y$ be a function, and let $\mathcal{A} = f^{-1}(\mathcal{B})$ be the σ-algebra of subsets of X . For given $C \in 2^X$ let C^* and C_* denote the $(\nu_{g^{-1}})$-hull and kernel of C , and for given $D \in 2^Y$ let D^* and D_* denote the ν-hull and *kernel of* D *, resp. Then we have:*

(2.93) $g(C^*)^* = g(C)^*$, $\forall C \in 2^X$

(2.94) $\nu_*\{g(C^*) \setminus g(C)\} = 0$, $\forall C \in 2^X$

(2.95) $\nu_*\{g(C) \setminus g(C_*)\} = 0$ *if and only if* $g(C_*)_* = g(C)_*$

(2.96) *If* g *is injective, then* $\nu_*\{g(C) \setminus g(C_*)\} = 0$ *, and hence* $g(C_*)_* = g(C)_*$

(2.97) $\nu^*\{g(C^*) \setminus g(C)\} = 0$ *if and only if* $g(C^*)^* = g(C)_*$ *if and only if* $g(C) \in \mathcal{B}^\nu$

(2.98) $\nu^*\{g(C) \setminus g(C_*)\} = 0$ *if and only if* $g(C^*)^* = g(C_*)^*$ *if and only if* $C \in \mathcal{A}^{\nu_{g^{-1}}}$.

Proof. (2.93): Since $g(C) \subset g(C^*)$, it follows $g(C)^* \subset g(C^*)^*$. Further, without loss of generality we can assume $C^* = g^{-1}\{g(C^*)^*\}$. Thus $g^{-1}\{g(C)^*\} \subset C^*$, and hence it is no restriction assume that $C^* = g^{-1}\{g(C)^*\}$. This implies $g(C^*) \subset g(C)^*$ and hence $g(C^*)^* \subset g(C)^*$, which together with the preceding inclusion implies (2.93).

(2.94): Take $B \in \mathcal{B}$ with $B \subset g(C^*) \setminus g(C)$. Since $C^* = g^{-1}\{g(C^*)^*\}$ it follows $g^{-1}(B) \subset C^* \setminus C$, and since $B \subset g(C^*)$ by definition of $\{\nu_{g^{-1}}\}$-hull C^* we find $0 = \nu_{g^{-1}}\{g^{-1}(B)\} = \nu^*\{g(g^{-1}(B))\} = \nu(B)$. This implies (2.94) and completes the proof. (Note that (2.94) follows directly by (2.93) as well.)

(2.95): First suppose that $g(C_*)_* = g(C)_*$. Then $\nu_*\{g(C) \setminus g(C_*)\} \le \nu_*\{g(C) \setminus g(C_*)_*\} = \nu_*\{g(C) \setminus g(C)_*\} = 0$, and hence $\nu_*\{g(C) \setminus g(C_*)\} = 0$. Conversely, without loss of generality suppose $C_* = g^{-1}\{g(C_*)^*\}$. Hence we have $g(C_*) = g(C) \cap g(C_*)^*$. Then $g(C)_* = g(C_*)_* \cup B$, with $B \in \mathcal{B}$, $B \subset g(C) \cap \{g(C_*)^*\}^c$. But, then $\nu_*\{g(C) \setminus g(C_*)\} = 0$ implies $\nu(B) = 0$. Thus $g(C)_* = g(C_*)_*$, and the proof of (2.95) is complete.

(2.96): If g is injective, then $g^{-1}(B) = A \subset C \setminus C_*$, where $g(C)_* = g(C_*)_* \cup B$ with $B \in \mathcal{B}$, $B \subset g(C) \cap \{g(C_*)^*\}^c$. Hence $\nu_{g^{-1}}(A) = 0$ by definition of $(\nu_{g^{-1}})$-kernel C_* . But $\nu_{g^{-1}}(A) = \nu^*\{g(A)\} = \nu(B)$, since $B \subset g(C)$, and thus $\nu(B) = 0$, which implies $g(C)_* = g(C_*)_*$. Now (2.96) follows by (2.95).

(2.97): First suppose that $\nu^*\{g(C^*) \setminus g(C)\} = 0$. Since $g(C) \subset g(C^*)$, then it follows $\nu^*\{g(C^*) \setminus g(C)\} = \nu(\{g(C^*) \setminus g(C)\}^*) = \nu\{g(C^*)^* \setminus g(C)_*\}$. Hence $g(C^*)^* = g(C)_*$. If $g(C^*)^* = g(C)_*$, then by (2.93) we find $g(C)^* = g(C)_*$ and hence $g(C) \in \mathcal{B}^\nu$. If $g(C) \in \mathcal{B}^\nu$, then $g(C)^* = g(C)_*$ and hence by (2.93) we have $\nu^*\{g(C^*) \setminus g(C)\} = \nu(\{g(C^*) \setminus g(C)\}^*) = \nu\{g(C^*)^* \setminus g(C)_*\} = \nu\{g(C)^* \setminus g(C)_*\} = 0$. This completes the proof.

(2.98): It is no restriction to assume $g(C)^* = g(C_*)^* \cup \{g(C) \setminus g(C_*)\}^*$, and now if $\nu^*\{g(C) \setminus g(C_*)\} = \nu(\{g(C) \setminus g(C_*)\}^*) = 0$, then by (1) we have $g(C_*)^* = g(C)^* = g(C^*)^*$. If $g(C_*)^* = g(C^*)^*$, then $C_* = g^{-1}\{g(C_*)^*\} = g^{-1}\{g(C^*)^*\} = C^*$ which implies $C \in \mathcal{A}^{\nu_{g^{-1}}}$. If $C \in \mathcal{A}^{\nu_{g^{-1}}}$, then $\nu_{g^{-1}}\{C^* \setminus C_*\} = 0$. Since $C_* = g^{-1}\{g(C_*)^*\}$ we have $\{g(C) \setminus g(C_*)\} \subset g(C^* \setminus C_*) \subset g(C^*)^* \setminus g(C_*)^*$ which implies $\nu_{g^{-1}}(C^* \setminus C_*) = \nu^*\{g(C^* \setminus C_*)\} = \nu^*\{g(C) \setminus g(C_*)\}$ and hence $\nu^*\{g(C) \setminus g(C_*)\} = 0$. □

Proposition 2.10

Let X *be a set, let* $(Y, \mathcal{B}, \nu)$ *be a finite measure space, let* $f : X \to Y$ *be a function, and* *let* $\mathcal{A} = f^{-1}(\mathcal{B})$ *be the* σ*-algebra of subsets of* X . *For given* $C \in 2^X$ *let* C^* *and* C_* *denote the* $(\nu_{g_{-1}})$*-hull and kernel of* C , *and for given* $D \in 2^Y$ *let* D^* *and* D_* *denote the* ν*-hull and kernel of* D , *resp. Then we have:*

(2.99) $\qquad g(C^*)^* = g(C)^* \quad , \quad \forall C \in 2^X$

(2.100) $\qquad \nu_*\{g(C^*) \setminus g(C)\} = 0 \quad , \quad \forall C \in 2^X$

(2.101) $\qquad \nu_*\{g(C) \setminus g(C_*)\} = 0$ *if and only if* $g(C_*)_* = g(C)_*$

(2.102) *If* g *is injective, then* $\nu_*\{g(C) \setminus g(C_*)\} = 0$, *and hence* $g(C_*)_* = g(C)_*$

(2.103) $\qquad \nu^*\{g(C^*) \setminus g(C)\} = 0$ *if and only if* $g(C^*)^* = g(C)_*$ *if and only if* $g(C) \in \mathcal{B}^\nu$

(2.104) $\qquad \nu^*\{g(C) \setminus g(C_*)\} = 0$ *if and only if* $g(C^*)^* = g(C_*)^*$

(2.105) $\qquad \nu^*\{g(C) \setminus g(C_*)\} = 0 \ \Rightarrow \ C \in \mathcal{A}^{\nu_{g_{-1}}} \ \Rightarrow \ \nu_*\{g(C) \setminus g(C_*)\} = 0$

(2.106) $\qquad C \in \mathcal{A}^{\nu_{g_{-1}}}$ *if and only if* $\bar{\nu}\{g(C) \setminus g(C_*)\} = 0$.

Proof. The proofs of (2.99)-(2.104) are very similar to the proofs of the facts from the preceding proposition. For (2.105) it is no restriction assume that $C_* = g^{-1}\{g(C_*)^*\}$, $g(C)^* = g(C_*)^* \cup \{g(C) \setminus g(C_*)\}^*$ and $\{g(C) \setminus g(C)_*\}_* \cup g(C_*)_* = g(C)_*$. Note that $g(C) \setminus g(C_*) \subset g(C^* \setminus C_*) \subset g(C)^* \setminus g(C_*)^*$, which implies $\nu_*\{g(C) \setminus g(C_*)\} \le \nu_*\{g(C^* \setminus C_*)\} \le \nu\{g(C)^* \setminus g(C_*)^*\}$. Since $\nu_*\{g(C^* \setminus C_*)\} = \nu_{g_{-1}}(C^* \setminus C_*)$ and $\nu\{g(C)^* \setminus g(C_*)^*\} = \nu^*\{g(C) \setminus g(C_*)\}$, this completes the proof of (2.105). Finally, it remains to note that (2.106) is an easy consequence of (2.105). $\square$

We will conclude this section by noting if $(X, \mathcal{A}, \mu)$ is a finite measure space and $C \in 2^X$, then for the map $g : C \to X$ defined by $g(x) = x$ for $x \in C$ we have:

$$\mu_{g^{-1}} = r\{tr^*(\mu, C), tr(\mathcal{A}, C)\}$$

$$\mu_{g_{-1}} = r\{tr_*(\mu, C), tr(\mathcal{A}, C)\} \ .$$

We remark that all of the preceding results can be applied to this particular case.

3. Perfect measures and maps

3.1. Perfect measures. A finite *measure* μ on a measurable space $(X, \mathcal{A})$ is said to be *perfect*, if $f(X)$ belongs to $\mathcal{B}(\mathbf{R})^{\mu_f}$ whenever f is an $\mathcal{A}$-measurable real valued function on X . A finite *measure space* $(X, \mathcal{A}, \mu)$ is said to be *perfect*, if the measure μ is perfect. A *measurable space* $(X, \mathcal{A})$ is said to be *perfect*, if every finite measure on $(X, \mathcal{A})$ is perfect.

Proposition 3.1

Let $(X, \mathcal{A}, \mu)$ *be a finite measure space. Then the following statements are equivalent:*

(3.1) $\qquad \mu$ *is perfect*

(3.2) $\qquad \bar{\mu}$ *is perfect*

(3.3) $\qquad \forall f \in \mathcal{M}(\mathcal{A}), \ \exists B_f \in \mathcal{B}(\mathbf{R}), \ B_f \subset f(X)$ *such that* $\mu\{f^{-1}(\mathbf{R} \setminus B_f)\} = 0$

(3.4) $\mu\{f^{-1}(\mathbf{R}\setminus f(X)_*)\} = 0$, $\forall f \in \mathcal{M}(\mathcal{A})$ *, where* $f(X)_*$ *is the* μ_f*-kernel of* $f(X)$

(3.5) $(\mu_f)_*\{f(X)\} = \mu(X)$, $\forall f \in \mathcal{M}(\mathcal{A})$

(3.6) $\forall f \in \mathcal{M}(\mathcal{A}),\ \exists N_f \in \mathcal{A},\ \mu(N_f) = 0$ *such that* $f(X \setminus N_f) \in \mathcal{B}(\mathbf{R})$

(3.7) $\forall f \in \mathcal{M}(\mathcal{A}),\ \exists g \in \mathcal{M}(\mathcal{A}),\ g = f$ *μ-a.s. such that* $g(X) \in \mathcal{B}(\mathbf{R})$

(3.8) $\forall f \in \mathcal{M}(\mathcal{A}),\ \exists N_f \in \mathcal{A},\ \mu(N_f) = 0$ *such that* $f(X \setminus N_f) \in \mathcal{B}(\mathbf{R})^{\mu_f}$

(3.9) $\forall f \in \mathcal{M}(\mathcal{A}),\ \exists g \in \mathcal{M}(\mathcal{A}),\ g = f$ *μ-a.s. such that* $g(X) \in \mathcal{B}(\mathbf{R})^{\mu_f}$.

Proof. For (3.8) suppose first that μ is perfect. Then there is $B \in \mathcal{B}(\mathbf{R})$ such that $B \subset f(X)$ and $\mu_f(\mathbf{R} \setminus B) = 0$. Put $N_f = f^{-1}(\mathbf{R} \setminus B)$, then $\mu(N_f) = 0$ and since $B \subset f(X)$, we have $f(X \setminus N_f) = B \in \mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\mathbf{R})^{\mu_f}$. Conversely, for given $f \in \mathcal{M}(\mathcal{A})$, let N_f be a set from $\mathcal{A}$ such that $\mu(N_f) = 0$ and $f(X \setminus N_f) \in \mathcal{B}(\mathbf{R})^{\mu_f}$. Let $M = f(N_f) \setminus f(X \setminus N_f)$, then $f(X) = f(X \setminus N_f) \cup M$. Since $\mu_f\{f(X \setminus N_f)^* \setminus f(X \setminus N_f)_*\} = 0$, it follows $f(X)_* = f(X \setminus N_f)_* \cup M_*$ and hence $\mu_f\{f(X)_*\} = \mu_f\{f(X \setminus N_f)_*\} + \mu_f(M_*)$. But $\mu_f(M_*) = \mu\{f^{-1}(M_*)\} = 0$, since $f^{-1}(M_*) \subset f^{-1}(M) \subset N_f$. Similarly, we find that $f(X)^* = f(X \setminus N_f)^* \cup \{M^* \setminus f(X \setminus N_f)^*\}$ and hence $\mu_f\{f(X)^*\} = \mu_f\{f(X \setminus N_f)^*\} + \mu_f\{M^* \setminus f(X \setminus N_f)^*\}$. But $\mu_f\{M^* \setminus f(X \setminus N_f)^*\} = 0$, since $f^{-1}\{M^* \setminus f(X \setminus N_f)^*\} \subset f^{-1}(M) \subset N_f$. Now, since $\mu_f\{f(X \setminus N_f)^*\} = \mu_f\{f(X \setminus N_f)_*\}$, it follows $\mu_f\{f(X)^*\} = \mu_f\{f(X)_*\}$, i.e. $f(X) \in \mathcal{B}(\mathbf{R})^{\mu_f}$ and the proof is complete. □

Let us recall that a class $\mathcal{K}$ of subsets of X is said to be *compact*, if for each sequence $\{\ K_n \mid n \geq 1\ \}$ in $\mathcal{K}$ such that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, there is $N \geq 1$ such that $\bigcap_{n=1}^{N} K_n = \emptyset$. It is well-known (see [12]) that we have:

(3.10) Every subclass of a compact class is compact.

(3.11) The class consisting of countable intersections of the sets in some compact class is itself compact.

(3.12) The class formed by taking finite unions of the sets in some compact class is itself compact.

A finite measure μ on a measurable space $(X, \mathcal{A})$ is said to be *compact*, if there is a compact class $\mathcal{K}$ of subsets of X such that:

$$\mu(A) = \sup_{K \in \mathcal{K}, K \subset A} \mu_*(K) \ .$$

The compact class $\mathcal{K}$ which approximates the given compact measure μ in the sense of the above relation can always be chosen such that it consists of sets in $\mathcal{A}$, see [12].

If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are finite measure spaces, then *the product of* μ *and* ν is a finitely additive measure λ defined on the smallest algebra $\alpha(\mathcal{A} \boxtimes \mathcal{B})$ generated by the family $\mathcal{A} \boxtimes \mathcal{B} = \{\ A \times B \mid A \in \mathcal{A}\ ,\ B \in \mathcal{B}\ \}$ which satisfies:

$$\lambda(A \times Y) = \mu(A) \quad \text{and} \quad \lambda(X \times B) = \nu(B)$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let us recall that a finite measure μ on a topological space X is said to be *Radon*, if for each Borel set $B \in \mathcal{B}(X)$ we have:

$$\mu(B) = \sup_{K \in \mathcal{K}(X), K \subset B} \mu^*(K)$$

where $\mathcal{K}(X)$ is the family of all compact sets in X. It is well-known that the following statements (see [16], [20], [21]) are also equivalent to (3.1)-(3.9):

(3.13) $\forall \varepsilon > 0$, $\forall \{ A_n \mid n \geq 1 \} \subset \mathcal{A}$, $\exists A \in \mathcal{A}$ such that $\mu(A) \geq \mu(X) - \varepsilon$ and $\{ A \cap A_n \mid n \geq 1 \}$ is compact.

(3.14) Any product of μ with any finite measure is countably additive.

(3.15) If $f : X \to S$ is a Borel measurable function into a separable metric space S, then the image measure μ_f is Radon.

(3.16) If $(Y, \mathcal{B}, \nu)$ is a finite measure space and λ is a measure on the product σ-algebra $\sigma(\mathcal{A} \boxtimes \mathcal{B})$ such that its restriction to the algebra $\alpha(\mathcal{A} \boxtimes \mathcal{B})$ is the product of μ and ν , then $\lambda_*(X \times C) = \nu_*(C)$ for every C subset of Y .

(3.17) For each countably generated σ-algebra $\mathcal{B}$ contained in the σ-algebra $\mathcal{A}$, the restriction $r(\mu, \mathcal{B})$ of μ on $\mathcal{B}$ is compact.

Furthermore, if we extend the definition of the product of two finite measures to an arbitrary family of finite measures naturally, then we have (see [20]):

(3.18) Any product of an arbitrary family of perfect measures is countably additive and its countably additive extension to the product σ-algebra is perfect.

In [20] we can also find that:

(3.19) Every compact measure is perfect.

Combining this result with (3.17) we get:

(3.20) If the σ-algebra $\mathcal{A}$ is countably generated, then a finite measure μ on $\mathcal{A}$ is perfect, if and only if it is compact.

Remark 3.2 Let us take $C \in 2^{[0,1]}$ such that $\lambda_*(C) = 0$ and $\lambda^*(C) = 1$, where as usual λ denotes the Lebesgue measure on the Borel σ-algebra $\mathcal{B}([0,1]) = tr\{\mathcal{B}(\mathbf{R}), [0,1]\}$. Put $X = C$, $\mathcal{A} = tr\{\mathcal{B}([0,1]), C\}$ and $\mu = tr^*(\lambda, C)$. Then $(X, \mathcal{A}, \mu)$ is a finite measure space, but the measure μ is not perfect. Indeed, for $f \in \mathcal{M}(\mathcal{A})$ defined by $f(x) = x$, $\forall x \in X$, and for $B \in \mathcal{B}(\mathbf{R})$ such that $B \subset f(X) = C$ we have $\mu_f(B) = \mu\{f^{-1}(B)\} = \mu(B \cap C) = \mu(B) = tr^*(\lambda, C)(B) = \lambda(B \cap C^*) = \lambda(B) = 0$. Hence we can conclude $(\mu_f)_*\{f(X)\} = 0 \neq 1 = \mu(X)$ which by (3.5) shows that μ is not perfect.

3.2. Blackwell spaces. Last example is dealing with a very pathological measurable space as we shall see now introducing a very wide and powerful class of perfect measurable spaces.

A measurable space $(X, \mathcal{A})$ is called *Blackwell space*, if $f(X)$ is an analytic subset of the real line, whenever f is a measurable real valued function on X . Blackwell spaces have a lot of nice properties. Let us recall some of them. For this, if $\mathcal{P}$ is a given paving on a set X , let $S(\mathcal{P})$ denote the paving of all $\mathcal{P}$-Souslin sets in X . Then we have:

(The projection theorem)

Let $(X,\mathcal{A})$ be a Blackwell space, and let $(Y,\mathcal{B})$ be a measurable space. If p_Y is the projection of $X \times Y$ onto Y , then $p_Y(A) \in S(\mathcal{B})$ for all $A \in S(\mathcal{A}\times\mathcal{B})$.

(The image theorem)

Let $(X,\mathcal{A})$ be a Blackwell space, and let $(Y,\mathcal{B})$ be a countably separated measurable space. If $f: X \to Y$ is a measurable function, then $f(A) \in S(\mathcal{B})$ for all $A \in S(\mathcal{A})$.

(The first separation theorem)

Let $(X,\mathcal{A})$ be a Blackwell space. If $A, B \in S(\mathcal{A})$ are disjoint sets, then there are disjoint sets $A', B' \in \mathcal{A}$ such that $A \subset A'$ and $B \subset B'$.

Let us recall that *an analytic space* is a Hausdorff space which is a continuous image of some Polish space. It is well-known that if X is an analytic space and $\mathcal{P} = \mathcal{F}(X)$ is the paving of all closed subsets of X , then $S(\mathcal{P}) = S\{\mathcal{F}(X)\} = \mathcal{A}(X)$ is the paving of all analytic subsets of X . Furthermore, the main point in our considerations have the following well-known statements:

(3.21) Every analytic space together with its Borel σ-algebra is a Blackwell space.

(3.22) Every Blackwell space is perfect.

(3.23) If X is analytic, then $(X,\mathcal{B}(X))$ is perfect.

Note that (3.23) follows straightforward by (3.21) and (3.22), and that (3.22) is a trivial consequence of our definition of the Blackwell space. This shows that there is a lot of nice Blackwell spaces, and that the class of Blackwell spaces (and thus of the perfect ones as well) is very wide. Moreover, let us recall that a Borel or Baire measure μ on a topological space X is said to be *tight*, if we have:

$$\mu(X) = \sup_{K\in\mathcal{K}(X), K\subset X} \mu^*(K) \ .$$

It is well-known (see [9]) that we have:

(3.24) Every tight Borel or Baire measure is perfect.

For separable metric spaces the converse statement is also true, see [21]:

(3.25) Every perfect Borel measure on a separable metric space is tight.

The following proposition answers the question when the image and coimage measures of a perfect measure are perfect.

Proposition 3.3

(3.26) *Let $(X,\mathcal{A},\mu)$ be a perfect measure space, let Y be a set, let $f: X \to Y$ be a function, and let $\mathcal{B} = f^1(\mathcal{A}) = \{\, B \in 2^Y \mid f^{-1}(B) \in \mathcal{A}\,\}$ be the σ-algebra of subsets of Y . Then*

$(Y, \mathcal{B}, \mu_f)$ *is perfect.*

(3.27) *Let* X *be a set, let* $(Y, \mathcal{B}, \nu)$ *be a perfect measure space, let* $g : X \to Y$ *be a function, and let* $\mathcal{A} = g^{-1}(\mathcal{B})$ *be the* σ*-algebra of subsets of* X *. If* g *is injective and* $g(\mathcal{A}) \subset \mathcal{B}^\nu$ *, then* $(X, \mathcal{A}, \nu_{g^{-1}})$ *and* $(X, \mathcal{A}, \nu_{g_{-1}})$ *are perfect.*

Proof. (3.26): Let $g \in \mathcal{M}(\mathcal{B})$, then $g \circ f \in \mathcal{M}(\mathcal{A})$ and since μ is perfect, we have $\{(\mu_f)_g\}_*\{g(Y)\} = (\mu_{g \circ f})_*\{g(Y)\} \geq (\mu_{g \circ f})_*\{(g \circ f)(X)\} = \mu(X)$. On the other hand, for each $B \in \mathcal{B}(\mathbf{R})$ we have $\{(\mu_f)_g\}(B) = \mu_{g \circ f}(B) = \mu\{(g \circ f)^{-1}(B)\} \leq \mu(X)$, and hence together with the preceding relation it follows that $\{(\mu_f)_g\}_*\{g(Y)\} = \mu(X) = \mu_f(Y)$. This by (3.5) implies that μ_f is perfect.

(3.27): Let $f \in \mathcal{M}(\mathcal{A})$. Since g is injective and $g(\mathcal{A}) \subset \mathcal{B}^\nu$, for given $B \in \mathcal{B}(\mathbf{R})$ we find that $(\nu_{g^{-1}})_f(B) = \nu_{g^{-1}}\{f^{-1}(B)\} = \nu^*\{g(f^{-1}(B))\} = \bar{\nu}\{g \circ f^{-1}(B)\} = \bar{\nu}_{g \circ f^{-1}}(B)$. Thus $\{(\nu_{g^{-1}})_f\}_*\{f(X)\} = (\bar{\nu}_{f \circ g^{-1}})_*\{f(X)\} \geq (\bar{\nu}_{f \circ g^{-1}})_*\{f \circ g^{-1}(Y)\}$. Since ν is perfect, then by (3.2) we have that $\bar{\nu}$ is also perfect. Then $f \circ g^{-1} \in \mathcal{M}(\mathcal{B})$ implies $(\bar{\nu}_{f \circ g^{-1}})_*\{(f \circ g^{-1})(Y)\} = \bar{\nu}(Y) = \nu(Y) \geq \nu^*\{g(X)\} = \nu_{g^{-1}}(X)$. On the other hand we have $\{(\nu_{g^{-1}})_f\}(B) = \nu_{g^{-1}}\{f^{-1}(B)\} \leq \nu_{g^{-1}}(X)$, and hence together with the preceding relations it follows $\{(\nu_{g^{-1}})_f\}_*\{f(X)\} = \nu_{g^{-1}}(X)$. This by (3.5) implies that $\nu_{g^{-1}}$ is a perfect measure. The proof that the inner coimage measure $\nu_{g_{-1}}$ is perfect is similar. □

Corollary 3.4

(3.28) *If* $(X, \mathcal{A}, \mu)$ *is a perfect measure space and* $\mathcal{A}'$ *is a sub-*σ*-algebra of* $\mathcal{A}$ *, then the restriction measure space* $\big(X, \mathcal{A}', r(\mu, \mathcal{A}')\big)$ *is perfect.*

(3.29) *If* $(X, \mathcal{A}, \mu)$ *is a perfect measure space, and if* $A \in \mathcal{A}^\mu$ *, then the trace measure space* $\big(A, tr(\mathcal{A}^\mu, A), r(tr(\bar{\mu}, A), tr(\mathcal{A}^\mu, A))\big)$ *is perfect.*

(3.30) *If* $(X, \mathcal{A})$ *is a perfect measurable space, and if* $A \in \mathcal{A}^*$ *, then the trace measurable space* $(A, tr(\mathcal{A}^*, A))$ *is perfect.*

(3.31) *If* $(X, \mathcal{A}, \mu)$ *is a perfect measure space and* $\hat{X} \supset X$ *is a set, then the extension measure space* $(\hat{X}, \hat{\mathcal{A}}, ext(\mu, \hat{X}))$ *is perfect.*

(3.32) *If* $(X, \mathcal{A})$ *is a perfect measurable space and* $\hat{X} \supset X$ *is a set, then the extension measurable space* $(\hat{X}, \hat{\mathcal{A}})$ *is perfect.*

Proof. (3.28): It follows directly by definition of perfectness.

(3.29): If $(X, \mathcal{A}, \mu)$ is perfect, then by (3.2) $(X, \mathcal{A}^\mu, \bar{\mu})$ is perfect. For given $A \in \mathcal{A}^\mu$, define $g : A \to X$ by $g(x) = x$, $\forall x \in A$. Then g is injective, $g^{-1}(\mathcal{A}^\mu) = tr(\mathcal{A}^\mu, A)$ and $g\{g^{-1}(\mathcal{A}^\mu)\} = g\{tr(\mathcal{A}^\mu, A)\} = tr(\mathcal{A}^\mu, A) \subset \mathcal{A}^\mu$. Hence we see that (3.27) can be applied, and since $\mu_{g^{-1}} = \mu_{g_{-1}} = r(tr(\bar{\mu}, A), tr(\mathcal{A}^\mu, A))$, it follows that the trace measure space $\big(A, tr(\mathcal{A}^\mu, A), r(tr(\bar{\mu}, A), tr(\mathcal{A}^\mu, A))\big)$ is perfect.

(3.30): Suppose that $(X, \mathcal{A})$ is perfect. Let $A \in \mathcal{A}^*$ be given, and consider any finite measure μ_0 on $(A, tr(\mathcal{A}^*, A))$. Let $\mu = r(ext(\mu_0, X), \mathcal{A})$, i.e. let $\mu(B) = \mu_0(B \cap A)$ for all $B \in \mathcal{A}$. Then μ is a finite measure on $(X, \mathcal{A})$, and hence μ is perfect. Since $A \in \mathcal{A}^* \subset \mathcal{A}^\mu$, by (3.29) it follows that the trace measure space $\big(A, tr(\mathcal{A}^\mu, A), r(tr(\bar{\mu}, A), tr(\mathcal{A}^\mu, A))\big)$ is perfect, while by (3.28) the restriction measure space $\big(A, tr(\mathcal{A}^*, A), r(tr(\bar{\mu}, A), tr(\mathcal{A}^*, A))\big)$ is perfect. But, since $r(\mu_0, tr(\mathcal{A}, A)) = r(tr(\mu, A), tr(\mathcal{A}, A))$, it follows $r(tr(\bar{\mu}, A), tr(\mathcal{A}^*, A)) = \mu_0$, and the proof of (3.30) is complete.

(3.31): Let $f \in \mathcal{M}(\hat{\mathcal{A}})$, then $f^{-1}\{\mathcal{B}(\mathbf{R})\} \subset \hat{\mathcal{A}}$ and hence $f^{-1}(B) \cap X \in \mathcal{A}$, whenever $B \in \mathcal{B}(\mathbf{R})$. This shows that the restriction g of f to X is an $\mathcal{A}$-measurable function on X , and since $(X, \mathcal{A}, \mu)$ is perfect, there is $B_g \in \mathcal{B}(\mathbf{R})$ such that $\mu\{g^{-1}(\mathbf{R} \setminus B)\} = 0$. Since $g(X) = f(X) \subset f(\hat{X})$ and $g^{-1}(B) \subset f^{-1}(B)$, it follows $ext(\mu, \hat{X})\{f^{-1}(\mathbf{R} \setminus B_g)\} = 0$ and hence $(\hat{X}, \hat{\mathcal{A}}, ext(\mu, \hat{X}))$ is perfect by definition. This proves (3.31).

(3.32): It is a direct consequence of (3.31). □

3.3. Perfect maps. Perfectness of a given finite measure is defined globally, that is, by using the whole space, and one can be interested to see how much has this property local influence to measurability properties. For this, let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $(Y, \mathcal{B})$ be a measurable space, and let $f : X \to Y$ be a measurable map. Then f is said to be *μ-perfect*, if $\forall A \in \mathcal{A},\ \exists B \in \mathcal{B},\ B \subset f(A)$ such that $\mu\{A \setminus f^{-1}(B)\} = 0$.

Proposition 3.5

Under the hypotheses stated above, we have:

(2.33) *f is μ-perfect, if and only if $\mu\{A \setminus f^{-1}\{f(A)_*\}\} = 0$ for all $A \in \mathcal{A}$, where $f(A)_*$ is the μ_f-kernel of $f(A)$*

(2.34) *f is μ-perfect, if and only if $\forall A \in \mathcal{A},\ \exists A' \in \mathcal{A},\ A' \subset A$ such that $\mu(A \setminus A') = 0$ and $f(A') \in \mathcal{B}$*

(2.35) *If $f(A) \in \mathcal{B}^{\mu_f}$ for all $A \in \mathcal{A}$, then f is μ-perfect*

(2.36) *If $f : X \to Y$ is μ-perfect and injective, then $f(A) \in \mathcal{B}^{\mu_f}$ for all $A \in \mathcal{A}$.*

Proof. It follows easily by definition. □

The set of all real valued μ-perfect maps on $(X, \mathcal{A}, \mu)$ is denoted by $\mathcal{M}_p(\mathcal{A})$. If $\mathcal{M}(\mu)$ is the set of all μ-measurable (i.e. $\mathcal{A}^\mu$-measurable) real valued functions on X , then we clearly have:

$$\mathcal{M}_p(\mathcal{A}) \subset \mathcal{M}(\mathcal{A}) \subset \mathcal{M}(\mu) \ .$$

Moreover, it is easily seen that:

(3.37) The measure μ is perfect, if and only if every $\mathcal{A}$-measurable real valued function on X is μ-perfect, i.e. if and only if $\mathcal{M}_p(\mathcal{A}) = \mathcal{M}(\mathcal{A})$

(3.38) The measure μ is perfect and complete, if and only if $\mathcal{M}_p(\mathcal{A}) = \mathcal{M}(\mu)$.

Theorem 3.6

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $(Y, \mathcal{B})$ be a measurable space, let $f : X \to Y$ be a measurable function. Then for any function $\alpha : Y \to \mathbf{R}$ we have:

(3.39) $$\alpha_* \circ f \le (\alpha \circ f)_* \le (\alpha \circ f)^* \le \alpha^* \circ f$$

(3.40) $$\int_* \alpha \ d\mu_f \le \int_* \alpha \circ f \ d\mu \le \int^* \alpha \circ f \ d\mu \le \int^* \alpha \ d\mu_f \ .$$

Moreover, the following statements are equivalent:

(3.41) f *is* μ*-perfect*

(3.42) $(\mu_f)^* = \mu^* \circ f^{-1}$

(3.43) $(\mu_f)_* = \mu_* \circ f^{-1}$

(3.44) $\int^* \alpha \; d\mu_f = \int^* \alpha \circ f \; d\mu$ *, for all functions* $\alpha : Y \to \mathbf{R}$

(3.45) $\int_* \alpha \; d\mu_f = \int_* \alpha \circ f \; d\mu$ *, for all functions* $\alpha : Y \to \mathbf{R}$

(3.46) $\alpha^* \circ f = (\alpha \circ f)^*$ *, for all functions* $\alpha : Y \to \mathbf{R}$

(3.47) $\alpha_* \circ f = (\alpha \circ f)_*$ *, for all functions* $\alpha : Y \to \mathbf{R}$

(3.48) $\mu^*\{f^{-1}(C)\} = 1 \; , \; \forall C \in 2^Y$ *such that* $(\mu_f)^*(C) = 1$

(3.49) $\mu_*\{f^{-1}(C)\} = 0 \; , \; \forall C \in 2^Y$ *such that* $(\mu_f)_*(C) = 0$

(3.50) $\mu\{ A \mid f = y \} = 0 \; , \; \forall y \in Y \setminus f(A) \; , \; A \in \mathcal{A}$

(3.51) $\mu\{ A \mid f = y \} = 1 \; , \; \forall y \in Y \setminus f(X \setminus A) \; , \; A \in \mathcal{A}$

(3.52) $E\{ (\alpha \circ f)_* \mid f \} = \alpha_*$ *, for all functions* $\alpha \; : \; Y \to \mathbf{R}$

(3.53) $E\{ (\alpha \circ f)^* \mid f \} = \alpha^*$ *, for all functions* $\alpha \; : \; Y \to \mathbf{R}$.

Furthermore, for a given map $g : X \to \mathbf{R}$ *define:*

$$M(f,g)(y) = \sup_{x \in f^{-1}(y)} g(x) \; , \quad y \in Y$$

$$m(f,g)(y) = \inf_{x \in f^{-1}(y)} g(x) \; , \quad y \in Y.$$

Then the following statements are also equivalent to (3.41)-(3.53):

(3.54) $E\{ g_* \mid f \} \leq M(f,g)$ *, for all functions* $g : X \to \mathbf{R}$

(3.55) $E\{ g^* \mid f \} \geq m(f,g)$ *, for all functions* $g : X \to \mathbf{R}$

(3.56) $\forall h : X \to \mathbf{R} \; , \; \exists g \in \mathcal{M}(\mu) \; , \; g = h_*$ μ*-a.s. such that* $M(f,g)$ *is* $\mathcal{A}$*-measurable*

(3.57) $\forall h : X \to \mathbf{R} \; , \; \exists g \in \mathcal{M}(\mu) \; , \; g = h^*$ μ*-a.s. such that* $m(f,g)$ *is* $\mathcal{A}$*-measurable*

(3.58) $\int_* g \; d\mu \leq \int_* M(f,g) \; d\mu_f$ *, for all functions* $g : X \to \mathbf{R}$

(3.59) $\int^* g \; d\mu \geq \int^* m(f,g) \; d\mu_f$ *, for all functions* $g : X \to \mathbf{R}$

(3.60) $g_* \leq M(f,g)_* \circ f$ *, for all functions* $g : X \to \mathbf{R}$

(3.61) $g^* \geq m(f,g)^* \circ f$ *, for all functions* $g : X \to \mathbf{R}$.

Proof. The proof of (3.39) and (3.40) follows by definitions of the lower and inner μ-envelopes and integrals. The rest of the proof can be reconstructed from [2] and [7]. □

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\mathcal{N}(\mu)$ be the set of all μ-null sets, i.e.

$$\mathcal{N}(\mu) = \{ C \in 2^X \mid \mu^*(C) = 0 \} \ .$$

Then $f^{-1}\{\mathcal{N}(\mu_f)\} \subset \mathcal{N}(\mu)$ and $f^{-1}(\mathcal{B}^{\mu_f}) \subset \mathcal{A}^\mu$, and if $g \in \mathcal{M}(\mu_f)$ is an arbitrary function, then $g \circ f \in \mathcal{M}(\mu)$. Furthermore, if f is μ-perfect, then we have:

(3.62) $\mathcal{N}(\mu_f) = f^1(\mathcal{N}(\mu)) = \{ B \in 2^Y \mid f^{-1}(B) \in \mathcal{N}(\mu) \}$

(3.63) $\mathcal{B}^{\mu_f} = f^1(\mathcal{A}^\mu) = \{ B \in 2^Y \mid f^{-1}(B) \in \mathcal{A}^\mu \}$

(3.64) $g \in \mathcal{M}(\mu_f)$ if and only if $g \circ f \in \mathcal{M}(\mu)$, where $g : Y \to \mathbf{R}$ is an arbitrary function.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ be measurable spaces, and let $f : X \to Y$, $g : Y \to Z$ be measurable functions. It is well-known (see [2]) that we have:

(3.65) If f is μ-perfect and g is (μ_f)-perfect, then $g \circ f$ is μ-perfect

(3.66) If $g \circ f$ is μ-perfect, then g is (μ_f)-perfect

(3.67) If $g \circ f$ is μ-perfect, and g is injective, then f is μ-perfect.

We shall now see that a certain classes of μ-measurable functions are closely related to perfect ones, see [6]. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $(Y, \mathcal{B})$ be a measurable space, and let $f : X \to Y$ be a function. Then f is said to be ***Lusin μ-measurable***, if f is μ-measurable (measurable with respect $\mathcal{A}^\mu$ and $\mathcal{B}$) and $\forall A \in \mathcal{A}, \ \exists A' \in \mathcal{A}^\mu, \ A' \subset A$ such that $\mu(A \setminus A') = 0$ and $f(A') \in \mathcal{B}^{\mu_f}$. A finite measure space $(X, \mathcal{A}, \mu)$ is called *a Lusin space*, if all μ-measurable real valued functions on X are Lusin μ-measurable. Let us first note that Lusin μ-measurability can be expressed, up to μ-measurability, in terms which do not involve completions of the corresponding σ-algebras. Namely, it is easily verified that:

(3.68) f is Lusin μ-measurable, if and only if f is μ-measurable and $\forall A \in \mathcal{A}, \ \exists A' \in \mathcal{A}, \ A' \subset A$ such that $\mu(A \setminus A') = 0$ and $f(A') \in \mathcal{B}$.

Now, one can easily check the following statements:

(3.69) If f is μ-perfect, then f is Lusin μ-measurable.

(3.70) If f is Lusin μ-measurable and g is an $\mathcal{A}$-measurable function such that $f = g$ μ-a.s., then g is μ-perfect.

(3.71) A Lusin μ-measurable function f is μ-perfect, if and only if f is $\mathcal{A}$-measurable.

(3.72) If f is Lusin μ-measurable and $g = f$ μ-a.s., then g is Lusin μ-measurable.

From (3.69), (3.70) and (3.71) we get:

(3.73) A finite measure space $(X, \mathcal{A}, \mu)$ is a Lusin space, if and only if μ is perfect.

According to (3.70) and the well-known fact that for every μ-measurable function f there is an $\mathcal{A}$-measurable function g such that $f = g$ μ-a.s. , we easily find:

(3.74) A μ-measurable function f is Lusin μ-measurable, if and only if either of the equivalent statements (3.42)-(3.53) holds.

3.4. Approximating pavings. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\mathcal{P}$ be a paving on X . Then $\mathcal{P}$ is said to be:

– *the inner approximating paving for* μ , if $\mu(A) = \sup\{\,\mu_*(P) \mid P \subset A\ ,\ P \in \mathcal{P}\,\}\ , \forall A \in \mathcal{A}$

– *the outer approximating paving for* μ , if $\mu(A) = \inf\{\,\mu^*(P) \mid P \supset A\ ,\ P \in \mathcal{P}\,\}\ , \forall A \in \mathcal{A}$

– *the approximating paving for* μ , if it is both, inner approximating and outer approximating.

The paving $\mathcal{P}$ is called *measurable*, if $\mathcal{P} \subset \mathcal{A}$. It is easily verified that we have:

(3.75) $\mathcal{P}$ is the inner approximating paving for μ if and only if:

$$\mu_*(C) = \sup\{\,\mu_*(P) \mid P \subset C\ ,\ P \in \mathcal{P}\,\}$$

whenever $C \in 2^X$

(3.76) $\mathcal{P}$ is the outer approximating paving for μ if and only if:

$$\mu^*(C) = \inf\{\,\mu^*(P) \mid P \supset C\ ,\ P \in \mathcal{P}\,\}$$

whenever $C \in 2^X$

(3.77) If $\mathcal{P}$ is the inner approximating paving for μ , then $\mathcal{P}_* = \{\,P_* \mid P \in \mathcal{P}\,\}$ is the measurable inner approximating paving for μ

(3.78) If $\mathcal{P}$ is the outer approximating paving for μ , then $\mathcal{P}^* = \{\,P^* \mid P \in \mathcal{P}\,\}$ is the measurable outer approximating paving for μ

(3.79) $\mathcal{P}$ is the inner approximating paving for μ , if and only if $\mathcal{P}^c = \{\,P^c \mid P \in \mathcal{P}\,\}$ is the outer approximating paving for μ

(3.80) $\mathcal{A}$ is the approximating paving for $\bar{\mu}$ and $r(\bar{\mu}, \mathcal{A}^*)$.

Recall that the most usual example for the inner approximating paving is the family of compact sets or closed sets, and the most usual example for the outer approximating paving is the family of open sets of a given topological space together with the corresponding Borel σ-algebra.

Proposition 3.7

Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $(Y, \mathcal{B})$ *be a measurable space, let* $f : X \to Y$ *be a function, and let* $\mathcal{P}$ *be an inner approximating paving for* μ *. Then the following statements are equivalent:*

(3.81) f *is* μ*-perfect*

(3.82) $\forall P \in \mathcal{P},\ \exists A \in \mathcal{A},\ A \subset P$ *such that* $\mu(A) = \mu_*(P)$ *and* $f(A) \in \mathcal{B}$

(3.83) $\forall P \in \mathcal{P},\ \exists B \in \mathcal{B},\ B \subset f(P)$ *such that* $\mu\{P_* \setminus f^{-1}(B)\} = 0$

(3.84) $\mu_*(f^{-1}\{f(P)\}) \le (\mu_f)_*\{f(P)\}\ ,\ \forall P \in \mathcal{P}$

(3.85) $\mu_*(P) \le (\mu_f)_*\{f(P)\}\ ,\ \forall P \in \mathcal{P}$

(3.86) $\mu\{P_* \,|\, f = y\} = 0 \ , \ \forall y \in Y \setminus f(P) \ , \ \forall P \in \mathcal{P}$.

Proof. It can be reconstructed from [7]. □

In particular, under the same hypotheses as in Proposition 3.7, by (3.82) we get:

(3.87) If $f(P) \in \mathcal{B}^{\mu_f}$, $\forall P \in \mathcal{P}$, then f is μ-perfect.

In addition, under the same hypotheses as in Proposition 3.7, let $\mathcal{Q}$ be an outer approximating paving for μ . Then by (3.81)-(3.86) it follows easily that the following statements are equivalent:

(3.88) f is μ-perfect

(3.89) $\forall Q \in \mathcal{Q}, \ \exists A \in \mathcal{A}, \ A \subset X \setminus Q$ such that $\mu(A) = \mu_*(X \setminus Q)$ and $f(A) \in \mathcal{B}$

(3.90) $\forall Q \in \mathcal{Q}, \ \exists B \in \mathcal{B}, \ B \subset f(X \setminus Q)$ such that $\mu\{f^{-1}(B) \cup Q^*\} = \mu(X)$

(3.91) $(\mu_f)^*\{f(Q^c)^c\} \leq \mu^*(f^{-1}\{f(Q^c)^c\}) \ , \ \forall Q \in \mathcal{Q}$

(3.92) $(\mu_f)^*\{f(Q^c)^c\} \leq \mu^*(Q) \ , \ \forall Q \in \mathcal{Q}$

(3.93) $\mu\{Q^* \,|\, f = y\} = 1 \ , \ \forall y \in Y \setminus f(X \setminus Q) \ , \ \forall Q \in \mathcal{Q}$.

We conclude section by recalling the concept of tightness. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\mathcal{K}$ be a paving on X . Then μ is said to be *$\mathcal{K}$-tight*, if $\mu(X) = \sup\{ \ \mu_*(K) \mid K \in \mathcal{K} \ \}$. It is easily verified that the following two facts are satisfied:

(3.94) If μ is $\mathcal{K}$-tight, then $X = \bigcup_{n=1}^{\infty} K_n$ μ-a.s. for some sequence $\{ \ K_n \mid n \geq 1 \ \}$ in $\mathcal{K}$

(3.95) If μ is $\mathcal{K}$-tight, then $\bigcup_{K \in \mathcal{K}} K \in \mathcal{A}^\mu$ and $\bar{\mu}(\bigcup_{K \in \mathcal{K}} K) = \mu(X)$.

4. On extension of measures

In this section we shall direct our attention to the certain algebraic properties of measures and integrals which are at the background of [17] and [18]. In this process we shall introduce a general setting which could be suitable for further investigation in the framework of the general vector space theory.

Let X be a real linear space. Arbitrary map from X into $\mathbf{R}$ will be called *a functional* on X , and the set of all functionals on X is denoted by $\mathbf{R}^X$. By X^+ we denote the linear space of all *linear* functionals on X . If a norm $|\cdot|$ on X is defined, i.e. if X is a normed space, then X^* denotes the Banach space of all *linear continuous* functionals on X with the usual (operator) norm given by:

$$|f| = \sup\{ \ |f(x)| \mid x \in X \ , \ |x| \leq 1 \ \}.$$

If Y is a subset of X and $f, g \in \mathbf{R}^X$, then by:

$$f \leq_Y g \ \Leftrightarrow \ f(y) \leq g(y) \ , \ \forall y \in Y$$

a partial ordering relation on $\mathbf{R}^X$ is defined, while $sp(Y)$ denotes the smallest linear subspace of X which contains Y . In order to indicate that Y is a subspace of X , we write $Y < X$.

Let $p \in \mathbf{R}^X$ be a functional on X . Then we say that p is *positively homogeneous*, if $p(\alpha x) = \alpha p(x)$, $\forall \alpha \geq 0$, $\forall x \in X$. We say that p *subadditive*, if $p(x+y) \leq p(x)+p(y)$, $\forall x, y \in X$. We say that p is *superadditive*, if $p(x) + p(y) \leq p(x + y)$, $\forall x, y \in X$. The set of all positively homogeneous subadditive (superadditive) functionals on X is denoted by $\check{X}(\hat{X})$. To each $p \in \check{X}$ we can assign the so-called *dual functional* $p_* \in \hat{X}$ defined by:

$$p_*(x) = -p(-x) \ , \ \forall x \in X \ .$$

Proposition 4.1

Let $p \in \check{X}$ *, then we have:*

(4.1) $p(x + y) \leq p(x) + p(y) \ , \ \forall x, y \in X$

(4.2) $p(\alpha x) = \alpha p(x) \ , \ \forall \alpha \geq 0 \ , \ \forall x \in X$

(4.3) $p_*(x) + p_*(y) \leq p_*(x + y) \ , \ \forall x, y \in X$

(4.4) $p_*(\alpha x) = \alpha p_*(x) \ , \ \forall \alpha \geq 0 \ , \ \forall x \in X$

(4.5) $-\alpha p(x) \leq p(-\alpha x) \ , \ \forall \alpha \geq 0 \ , \ \forall x \in X$

(4.6) $p_*(-\alpha x) \leq -\alpha p_*(x) \ , \ \forall \alpha \geq 0 \ , \ \forall x \in X$

(4.7) $p_*(x) \leq p(x) \ , \ \forall x \in X$

(4.8) $(p_*)_* = p$

(4.9) *If* $E = \{x \in X \mid p(x) = p_*(x)\}$ *, then* E *is a linear subspace of* X *and the restriction of* p *to* E *is a linear functional on* E

(4.10) $p = p_*$ *if and only if* p *is linear.*

Proof. It is easily verified by definition. □

Let us recall that the well-known **Hahn-Banach theorem** states:

(4.11) Let X be a real linear space, let $Y < X$, let $f \in Y^+$, and let $p \in \check{X}$ such that $f \leq_Y p$. Then there exists $F \in X^+$ such that the restriction of F to Y is f , and such that $F \leq_X p$.

One of the consequences of this theorem may be stated as follows.

Corollary 4.2

Let X *be a real linear space, let* $p \in \check{X}$ *, and let* $x_1, \ldots, x_n \in X$ *for some* $n \geq 1$ *such that:*

$$p\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) = \sum_{i=1}^{n} p(\alpha_i x_i)$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *. Then there exists* $F \in X^+$ *such that:*

$$F\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) = \sum_{i=1}^{n} \alpha_i p(x_i)$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *, and such that* $F \leq_X p$.

Proof. Let $Y = sp\{ x_1, \ldots, x_n \}$, and let $f \in Y^+$ be defined by $f(\sum_{i=1}^n \alpha_i x_i) = \sum_{i=1}^n \alpha_i p(x_i)$ for all $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$. Then by (4.5) and our assumption we find:

$$f\Big(\sum_{i=1}^n \alpha_i x_i\Big) = \sum_{i=1}^n \alpha_i p(x_i) = \sum_{\alpha \geq 0} \alpha_i p(x_i) + \sum_{\alpha_i < 0} \alpha_i p(x_i)$$

$$= \sum_{\alpha \geq 0} p(\alpha_i x_i) + \sum_{\alpha_i < 0} (-\alpha_i)\{-p(x_i)\} \leq \sum_{\alpha \geq 0} p(\alpha_i x_i) + \sum_{\alpha_i < 0} p((-\alpha_i)(-x_i))$$

$$= \sum_{i=1}^n p(\alpha_i x_i) = p(\sum_{i=1}^n \alpha_i x_i)$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$. This shows that we can apply Hahn-Banach theorem (4.11), and the proof follows straightforward. □

Let us now suppose that a norm $|\cdot|$ on a real linear space X is defined, and let $p \in \check{X}$ be given. If there is a constant $M > 0$ such that:

$$p(x) \leq M|x| \ , \ \forall x \in X$$

then we say that p is $|\cdot|$-*dominated.* In this case by (4.7) we have:

$$-M|x| \leq p_*(x) \leq p(x) \leq M|x| \ , \ \forall x \in X \ .$$

This shows that non-linearity of p is locally controlled by the given norm $|\cdot|$. Moreover, we have:

(4.12) If $F \in X^+$ such that $F(x) \leq p(x)$, $\forall x \in X$, then $p_*(x) \leq F(x) \leq p(x)$, $\forall x \in X$.

Hence we find that the following statement is satisfied:

(4.13) Let X be a real linear space, let $p \in \check{X}$, and let $|\cdot|$ be a norm on X such that p is $|\cdot|$-dominated. Then each $F \in X^+$ satisfying $F \leq_X p$ is $|\cdot|$-continuous.

Let $\leq$ be a partial ordering relation on X . If $x \leq y$ implies $p(x) \leq p(y)$ for $x, y \in X$, then we say that p and $\leq$ are *harmonious*. Let $q : X \to X$ be *a subadditive map with respect to the partial ordering relation* $\leq$, i.e. $q(x+y) \leq q(x) + q(y)$, $\forall x, y \in X$. If $p\big(q(x)\big) \geq 0$ for all $x \in X$, then we say that p is q-*positive*. It is easily verified that we have:

(4.14) If p and $\leq$ are harmonious, and if p is q-positive with $q(0) = 0$, then:

$$d_{p,q}(x,y) = p\big(q(x-y)\big)$$

defines a pseudometric on X .

Let $\tau_{p,q}$ be the topology induced by pseudometric $d_{p,q}$, and let τ be the topology generated by the norm $|\cdot|$. Then we have:

(4.15) Under the hypotheses in (4.14) suppose that p is $|\cdot|$-dominated, and suppose that q is $|\cdot|$-continuous. Then the topology $\tau_{p,q}$ is weaker then τ .

(4.16) Under the hypotheses in (4.14) suppose that q is homogeneous, i.e. $q(\alpha x) = |\alpha| q(x)$ for all $\alpha \in \mathbf{R}$ and all $x \in X$. Then:

$$|x|_{p,q} = p\big(q(x)\big)$$

defines a pseudonorm on X.

Remark 4.3 Let us note that for any fixed $x_1, \ldots, x_n \in X$ and $n \geq 1$ by:

$$p_{x_1,\ldots,x_n}(\alpha_1, \ldots, \alpha_n) = p\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) \quad \text{for} \quad (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$$

a positively homogeneous subadditive (and thus convex and continuous) map from $\mathbf{R}^n$ into $\mathbf{R}$ is defined. In particular, if $x_1 = 1_{C_1}, \ldots, x_n = 1_{C_n}$ for some sets $C_1, \ldots, C_n$, then we shortly write $p_{x_1,\ldots,x_n} = p_{C_1,\ldots,C_n}$.

Let us now turn our attention to the main example of the preceding setting we are interested in. For this, let X be a set and let $\mathcal{A}$ be an algebra of subsets of X. Then $B(X)$ denotes a Banach space of all bounded real valued functions on X with the usual sup-norm defined by:

$$\| \, b \, \| = \sup_{x \in X} |b(x)| \ .$$

Let us define:

$$B_s(X, \mathcal{A}) = \big\{ \ b \in B(X) \mid b = \textstyle\sum_{i=1}^{n} a_i 1_{A_i} \ , \ a_i \in \mathbf{R} \ , \ A_i \in \mathcal{A} \ , \ n \in \mathbf{N} \ \big\}$$

and put:

$$B(X, \mathcal{A}) \ = \ cl^{\infty}\{B_s(X, \mathcal{A})\}$$

where cl^{∞} denotes the closure operator in $B(X)$ with respect to the sup-norm $\| \cdot \|$. Then $B(X, \mathcal{A})$ is a Banach space which is naturally considered in the process of defining the integral with respect to a finitely additive measure on $(X, \mathcal{A})$. In this way we also obtain a satisfactory representation of the dual space $B^*(X, \mathcal{A})$. Let us first introduce some measure spaces.

For a given set $A \in \mathcal{A}$, the family of all finite partitions of A with elements from $\mathcal{A}$ is denoted by $\mathcal{P}_f(A, \mathcal{A})$. The set of all bounded finitely additive real valued functions defined on $\mathcal{A}$ is denoted by $ba(X, \mathcal{A})$, while $ca(X, \mathcal{A})$ denotes the set of all bounded countably additive real valued functions defined on $\mathcal{A}$. It is well-known that $ba(X, \mathcal{A})$ and $ca(X, \mathcal{A})$ are Banach spaces with respect to the each of the following two equivalent norms, see [3]:

$$|\mu|_1 \ = \ v(\mu, X) \quad \text{and} \quad |\mu|_2 \ = \ \sup_{A \in \mathcal{A}} |\mu(A)|$$

where $v(\mu, X) = \sup \{ \sum_{A \in \pi} |\mu(A)| \mid \pi \in \mathcal{P}_f(X, \mathcal{A}) \}$ is *the total variation* of μ on X, for all $\mu \in ba(X, \mathcal{A}) \cup ca(X, \mathcal{A})$. The set of all non-negative elements from $ba(X, \mathcal{A})$ and $ca(X, \mathcal{A})$ is denoted by $ba_+(X, \mathcal{A})$ and $ca_+(X, \mathcal{A})$, resp. Let us also recall that by:

$$\mu^+(A) = \sup_{B \subset A} \mu(B) \quad \text{and} \quad \mu^-(A) = - \inf_{B \subset A} \mu(B)$$

the positive (upper) variation μ^+ and *the negative (lower) variation* μ^- of μ from $ba(X, \mathcal{A}) \cup ca(X, \mathcal{A})$ are defined, resp. It is well-known that we have:

(4.17) $\quad \mu^+,\ \mu^- \in ba(X,\mathcal{A})$, if $\mu \in ba(X,\mathcal{A})$

(4.18) $\quad \mu^+,\ \mu^- \in ca(X,\mathcal{A})$, if $\mu \in ca(X,\mathcal{A})$

(4.19) $\quad \mu = \mu^+ - \mu^-$

(4.20) $\quad v(\mu,\cdot) = \mu^+(\cdot) + \mu^-(\cdot)$

(4.21) $\quad \sup_{A\subset X} |\mu(A)| \leq v(\mu,X) \leq 2 \sup_{A\subset X} |\mu(A)|$.

Definition 4.4 *(The integral of elements in $B(X,\mathcal{A})$ with respect to elements from $ba(X,\mathcal{A})$)* It carries over in the following two steps:

(1) Let us first take $b \in B_s(X,\mathcal{A})$, and note that such a function allows the representation $b = \sum_{i=1}^n a_i 1_{A_i}$, where $a_i \in \mathbf{R}$ and $\pi = \{ A_1,\dots,A_n \} \in \mathcal{P}_f(X,\mathcal{A})$ for some $n \in \mathbf{N}$. For given $\mu \in ba(X,\mathcal{A})$, let us define:

$$\int_X b\ d\mu = \sum_{i=1}^n a_i\mu(A_i) \ .$$

It is easily seen that this definition does not depend on the choice of a given representation of the function b from $B_s(X,\mathcal{A})$, as well as that the following properties are satisfied:

(4.22) $\quad \int_X (\alpha b + \beta c)\ d\mu = \alpha \int_X b\ d\mu + \beta \int_X c\ d\mu$

(4.23) $\quad \left| \int_X b\ d\mu \right| \leq \| b \| \cdot v(\mu,X)$

(4.24) $\quad \int_X b\ d\mu = \int_X b\ d\mu^+ - \int_X b\ d\mu^-$

(4.25) $\quad \int_X b\ d\mu \leq \int_X c\ d\mu$, if $b \leq_X c$ and $\mu \in ba_+(X,\mathcal{A})$

(4.26) $\quad \left| \int_X b\ d\mu \right| \leq \int_X |b|\ d\mu \leq \| b \| \cdot \mu(X)$, if $\mu \in ba_+(X,\mathcal{A})$

whenever $b,c \in B_s(X,\mathcal{A})$, $\mu \in ba(X,\mathcal{A})$ and $\alpha,\beta \in \mathbf{R}$.

(2) Now we shall take $f \in B(X,\mathcal{A})$. Then there is a sequence $\{b_n \mid n \geq 1\}$ from $B_s(X,\mathcal{A})$ such that $\| b - b_n \| \to 0$. Then by (4.23) we have $\left| \int_X b_n\ d\mu - \int_X b_m\ d\mu \right| \leq \| b_n - b_m \| \cdot v(\mu,X)$, which shows that $\{ \int_X b_n\ d\mu \mid n \geq 1 \}$ is a Cauchy sequence of real numbers, so there is a limit, and we define $\int_X b\ d\mu = \lim \int_X b_n\ d\mu$. It is easily verified that this definition does not depend on the choice of the approximating sequence $\{ b_n \mid n \geq 1 \}$, as well as that properties (4.22)-(4.26) hold in this case as well.

In particular, from (4.23) we find the following property:

(4.27) $\quad$ Let $b, b_n \in B(X,\mathcal{A})$ for $n \geq 1$. If $\| b - b_n \| \to 0$, then $\int_X b_n\ d\mu \to \int_X b\ d\mu$.

This fact actually shows that by:

$$b \mapsto \int_X b\ d\mu \tag{4.28}$$

a continuous linear functional on $B(X,\mathcal{A})$ is defined. It is well-known that every continuous linear functionals on $B(X,\mathcal{A})$ is of this type, i.e. the space $B^*(X,\mathcal{A})$ is isometrically isomorphic to the space $ba(X,\mathcal{A})$ equipped with the total variation norm $|\cdot|_1$, where this isometrically isomorphism $i : B^*(X,\mathcal{A}) \to ba(X,\mathcal{A})$ might be defined naturally according to (4.28). Moreover, one can easily check that the following two properties are satisfied:

(4.29) Let $\mathcal{A}'$ be a sub-algebra of $\mathcal{A}$. If $\nu \in ba(X,\mathcal{A})$ and $\nu' \in ba(X,\mathcal{A}')$ such that $\nu' = r(\nu,\mathcal{A}')$, then for all $f' \in B(X,\mathcal{A}')$ we have $\int f'\, d\nu = \int f'\, d\nu'$.

(4.30) Let $\mathcal{A}'$ be a sub-σ-algebra of $\mathcal{A}$. If $\nu \in ba(X,\mathcal{A})$ and $\mu \in ca_+(X,\mathcal{A}')$ such that $\mu = r(\nu,\mathcal{A}')$, then for all $f' \in B(X,\mathcal{A}')$ we have $\int f'\, d\nu = \int f'\, d\mu$, where the integral on the right hand side is the standard one.

Remark 4.5 If $X = \mathbf{N}$, $\mathcal{A} = \{\ A \subset \mathbf{N} \mid A$ is finite or A^c is finite $\}$, and $\mu(A) = 0$, if A is finite, $\mu(A) = 1$, if A^c is finite, and $b_n = 1_{\{\,1,2,\dots,n\,\}}$ for $n \geq 1$, then $\lim_{n\to\infty} \int_X b_n\, d\mu = 0 \neq 1 = \int_X \lim_{n\to\infty} b_n\, d\mu$, which shows that the monotone and dominated convergence theorem (and Fatou's lemma also) do not hold in this case.

Let us now give one more description of the space $B(X,\mathcal{A})$. For this define:

$$B_i(X,\mathcal{A}) = \big\{\, b \in B(X) \ \mid\ b^{-1}\big(<a,b>\big) \in \mathcal{A}\ ,\ \forall a,b \in \mathbf{R} \,\big\}\ .$$

First we shall note that the functions from $B_i(X,\mathcal{A})$ satisfy a somewhat stronger property actually. For this, let $\mathcal{A}(\mathbf{R})$ be the smallest algebra generated by the family of all bounded open intervals in $\mathbf{R}$, i.e. $\mathcal{A}(\mathbf{R}) = \mathcal{A}\big(\mathcal{I}(\mathbf{R})\big)$. Then it is easily seen that $\mathcal{A}(\mathbf{R}) = \mathcal{A}(\mathcal{H})$, where $\mathcal{H} = \{\ <a,b>, [a,b], <a,b], [a,b>, <-\infty,b>, <-\infty,b], <b,\infty>, [b,\infty> \mid a,b \in \mathbf{R}\,\}$, and $\mathcal{A}(\mathcal{H})$ is actually equal to the family of all finite disjoint unions of elements from $\mathcal{H}$. By the well-known fact that $f^{-1}\big(\mathcal{A}(\mathcal{I}(\mathbf{R}))\big) = \mathcal{A}\big(f^{-1}(\mathcal{I}(\mathbf{R}))\big)$, it follows that *every function* b *from* $B_i(X,\mathcal{A})$ *is* $\mathcal{A}(\mathbf{R})$*-measurable*, i.e. $b^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}(\mathbf{R})$.

Proposition 4.6
Under the notation stated above, we have:

$$B(X,\mathcal{A}) = cl^{\infty}\{B_i(X,\mathcal{A})\}\ .$$

Proof. It suffices to show that for each $b \in B_i(X,\mathcal{A})$ there exists a sequence $\{\, b_n \mid n \geq 1\}$ from $B_s(X,\mathcal{A})$, such that $\|\, b - b_n \,\| \to 0$. So, take $b \in B_i(X,\mathcal{A})$. Then there exists $N \in \mathbf{N}$ such that $|b(x)| \leq N$, for all $x \in X$. Put $A_{k,n} = \{\,(kN)/n \leq b < (k+1)N/n\}$ for all $n \in \mathbf{N}$ and all $k = -n,\dots,-1,0,1,\dots,n-1$. Then by:

$$b_n = \textstyle\sum_{k=-n}^{n-1} n^{-1}(kN)\, 1_{A_{k,n}}\ ,\ n \geq 1$$

a sequence from $B_s(X,\mathcal{A})$ is defined such that $\|\, b - b_n \,\| \to 0$, and the proof is complete. $\square$

Remark 4.7 Note that $B_i(X,\mathcal{A})$ *is not a closed subspace of* $B(X)$ *in general.* Indeed, if we take $X = \mathbf{R}_+$ with $\mathcal{A} = \mathcal{A}(\mathbf{R})$, and define a continuous function $b \in B(X)$ by $b(x) = x^{-1}$, if $x \in \mathbf{N}$ and linearly otherwise, then the functions $b_n = b \cdot 1_{[0,n]}$, $n \geq 1$ belong to $B_i(X,\mathcal{A})$ with $\|\, b - b_n \,\| \to 0$, but $b \notin B_i(X,\mathcal{A})$ since $b^{-1}\big(<0,1>\big) \notin \mathcal{A}$, for example.

Remark 4.8 Note that $B(X,\mathcal{A})$ *does not contain all* $\sigma(\mathcal{A})$*-measurable functions from* $B(X)$ *in general*, i.e. functions b from $B(X)$ which satisfy $b^{-1}\big(\mathcal{B}(\mathbf{R})\big) \subset \sigma(\mathcal{A})$. Indeed, for this take $X = \mathbf{R}$, $\mathcal{A} = \mathcal{A}(\mathbf{R})$ and $b = 1_{\mathbf{N}}$. Then b is $\sigma(\mathcal{A}) = \mathcal{B}(\mathbf{R})$-measurable, but

$b \notin B(X, \mathcal{A})$, since in this case for a given $0 < \varepsilon < 1/2$ we can find $c \in B_i(X, \mathcal{A})$ such that $\| b - c \| < \varepsilon < 1/2$, and hence we have $c^{-1}\big(<1-\varepsilon, 1+\varepsilon>\big) = \mathbf{N} \in \mathcal{A}(\mathbf{R})$ which is a contradiction.

Remark 4.9 Note that the example in Remark 4.8 also shows that in general $B(X, \mathcal{A})$ *does not contain all functions from* $B(X)$ *which are measurable with respect to* $\mathcal{A}$ *and* $\mathcal{A}(\mathbf{R})$, i.e. functions b from $B(X)$ which satisfy $b^{-1}\big(\mathcal{A}(\mathbf{R})\big) \subset \mathcal{A}$. Indeed, in the setting of the example above note that $b^{-1}\big(\mathcal{A}(\mathbf{R})\big) \subset \mathcal{A}$ implies that $\sigma(b^{-1}\{\mathcal{A}(\mathbf{R})\} = b^{-1}\{\sigma\mathcal{A}(\mathbf{R})\} = b^{-1}\{\mathcal{B}(\mathbf{R})\} \subset \sigma(\mathcal{A})$, which is a contradiction.

However, let us note if $\mathcal{A}$ is a σ-algebra on X , then $B(X, \mathcal{A})$ equals to the family of all $\mathcal{A}$-measurable functions from $B(X)$, since in this case each of such functions is equal to the uniform limit of a sequence from $B_s(X, \mathcal{A})$, and hence $B(X, \mathcal{A}) = B_i(X, \mathcal{A})$ actually. Also, let us note if $\mathcal{A} = 2^X$, then $B(X) = B(X, \mathcal{A}) = B_i(X, \mathcal{A})$, and $B_s(X, \mathcal{A})$ will be shortly denoted by $B_s(X)$.

Suppose now, that $(X, \mathcal{A}, \mu)$ is a finite measure space. Define:

$$p^*(b) = \int^* b \; d\mu \quad , \quad \forall b \in B(X) \; .$$

Then by (2.16), (2.17) and (2.19) we see that $p^* \in \check{B}(X)$, and its *dual functional* $(p^*)_* = p_* \in \hat{B}(X)$ is given by:

$$p_*(b) = \int_* b \; d\mu \quad , \quad \forall b \in B(X) \; .$$

By (2.21) and (2.22) we find that p^* is $\| \cdot \|$-dominated, i.e. we have:

$$-M \| b \| \le p_*(b) \le p^*(b) \le M \| b \| \quad , \quad \forall b \in B(X)$$

where $M = v(\mu, X) = \mu(X)$. Define a partial ordering relation on $B(X)$ by requiring $b_1 \le b_2$ if and only if $b_1(x) \le b_2(x)$, $\forall x \in X$. Then evidently p^* and $\le$ are harmonious. Let us note that the map $q : B(X) \to B(X)$ defined by $q(b) = |\, b \,|$ is then subadditive with respect to $\le$, and that $p^*\big(q(b)\big) \ge 0$ for all $b \in B(X)$, i.e. p^* is q-positive and $q(0) = 0$. Thus by (4.14) we see that by:

$$d^*(b, c) = p^*\big(q(b - c)\big) = \int^* |\, b - c \,| \; d\mu$$

a pseudometric on $B(X)$ is defined. Moreover, by (4.16) we see that by:

$$\| b \|^* = p^*\big(q(b)\big) = \int^* |\, b \,| \; d\mu$$

a pseudonorm on $B(X)$ is defined. Let τ^* be the topology generated by the pseudometric d^* , and let τ be the topology generated by $\| \cdot \|$. Since p^* is $\| \cdot \|$-dominated and q is $\| \cdot \|$-continuous on $B(X)$, then by (4.15) it follows:

(4.31) $\qquad \tau^*$ is weaker than τ .

To conclude, we may apply Corollary 4.2 to this particular case with $x_1 = 1_{C_1}, \dots, x_n = 1_{C_n}$, where C_1 ,..., C_n are given subsets of X .

Corollary 4.10

If $\int^*(\sum_{i=1}^n \alpha_i 1_{C_i})\ d\mu = \sum_{i=1}^n \int^* \alpha_i 1_{C_i}\ d\mu$ *for all* $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *, then there exists* $F \in B^*(X)$ *such that* $F(\sum_{i=1}^n \alpha_i 1_{C_i}) = \sum_{i=1}^n \alpha_i \mu^*(C_i)$ *for all* $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *, and* $F(b) \le \int^* b\ d\mu$ *for all* $b \in B(X)$ *. In particular, we have* $\int_* b\ d\mu \le F(b) \le \int^* b\ d\mu$ *for all* $b \in B(X)$ *, and hence* $F(b) = \int_X b\ d\mu$ *if* b *is* μ*-measurable.*

Proof. Since p^* is $\|\cdot\|$-dominated, the proof follows from Corollary 4.2, (4.12) and (4.13). □

Remark 4.11 Note that in Corollary 4.10 we have $F(1_{C_i}) = \mu^*(C_i)$ for all $i = 1, \ldots, n$, and when $n = 1$ then the hypothesis above is trivially satisfied. However, in the case when $n \ge 2$, some further conditions on the sets $C_1, \ldots, C_n$ seem to be necessary in order that they satisfy this condition. For more information in this direction see [18].

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