The Azéma-Yor Embedding in Non-Singular Diffusions

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Let $(X_t)_{t\geq 0}$ be a non-singular (not necessarily recurrent) diffusion on $\mathbb{R}$ starting at zero, and let ν be a probability measure on $\mathbb{R}$. Necessary and sufficient conditions are established for ν to admit the existence of a stopping time τ_* of (X_t) solving the Skorokhod embedding problem, i.e. X_{τ_*} has the law ν. Furthermore, an explicit construction of τ_* is carried out which reduces to the Azéma-Yor construction [1] when the process is a recurrent diffusion. In addition, this τ_* is characterized uniquely to be a pointwise smallest possible embedding that stochastically maximizes (minimizes) the maximum (minimum) process of (X_t) up to the time of stopping.

1. Introduction

Let $(X_t)_{t\geq 0}$ be a non-singular (not necessarily recurrent) diffusion on $\mathbb{R}$ starting at zero, and let ν be a probability measure on $\mathbb{R}$. In this paper, we consider the problem of embedding the given law ν in the process (X_t), i.e. the problem of constructing a stopping time τ_* of (X_t) satisfying $X_{\tau_*} \sim \nu$ and determining conditions on ν which make this possible. This problem is known as the *Skorokhod embedding problem*.

The proof (see below) leads naturally to explicit construction of an extremal embedding of ν in the following sense. The embedding is an extension of the Azéma-Yor construction [1] that is pointwise the smallest possible embedding that stochastically maximizes $\max_{0\leq t\leq\tau_*} X_t$ (or stochastically minimizes $\min_{0\leq t\leq\tau_*} X_t$) over all embeddings τ_*.

The Skorokhod embedding problem has been investigated by many authors and was initiated in Skorokhod [17] when the process is a Brownian motion. In this case Azéma and Yor [1] (see Rogers [14] for an excursion argument) and Perkins [10] yield two different explicit extremal solutions of the Skorokhod embedding problem in the natural filtration. An extension of the Azéma-Yor embedding, when the Brownian motion has an initial law, was given in Hobson [7]. The existence of an embedding in a general Markov process was characterized by Rost [16], but no explicit construction of the stopping time was given. Bertoin and Le Jan [3] constructed a new class of embeddings when the process is a Hunt process starting at a regular recurrent point. Furthermore, Azéma and Yor [1] give an explicit solution when the process is a recurrent diffusion. The case when the process is a Brownian motion with drift (non-recurrent diffusion) was recently studied in Grandits [5] and Peskir [12], and then again in Grandits and Falkner [6]. A necessary and sufficient condition on ν that makes an explicit Azéma-Yor construction possible is given in [12]. The same necessary and sufficient condition is also given in [6] with the embedding that is a randomized stopping time obtained by the general result of Rost [16].

More general embedding problems for martingales are considered in Rogers [15] and Brown, Hobson and Rogers [4].

Applications of Skorokhod embedding problems have gained some interest to option pricing theory. How to design an option given the law of a risk is studied in [11], and bounds on the prices of Lookback options obtained by robust hedging are studied in [8].

This paper was motivated by the works of Grandits [5], Peskir [12] and Grandits and Falkner [6] where they consider the embedding problem for the non-recurrent diffusion of Brownian motion with drift. In this paper, we extend the condition given there and the Azéma-Yor construction to the case of a general non-recurrent non-singular diffusions. The approach of finding a solution to the Skorokhod problem is the following. First, the initial problem is transformed by composing (X_t) with its scale function into an analogous embedding problem for a continuous local martingale. Secondly, by the time-change given in the construction of the Dambis-Dubins-Schwarz Brownian motion (see [13]) the martingale embedding is shown to be equivalent to embedding in Brownian motion. Finally, when (X_t) is Brownian motion we have the embedding given in [1]. This methodology is well-known to the specialists in the field (see e.g. [1]), although we could not find the result in the literature on Skorokhod embedding problems. The embedding problem for a continuous local martingale introduces some novelty since the martingale is convergent when the initial diffusion is non-recurrent. Also some properties of the constructed embedding are given so as to characterize the embedding uniquely (Section 3).

2. **The main result**

Let $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ be two Borel functions such that $1/\sigma^2(\cdot)$ and $|\mu(\cdot)|/\sigma^2(\cdot)$ are locally integrable at every point in $\mathbb{R}$. Let $(X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ be the unique weak solution up to an explosion time e of the one-dimensional time-homogeneous stochastic differential equation

$$(2.1) \qquad dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \quad, \quad X_0 = 0$$

where (B_t) is a standard Brownian motion and $e = \inf\{\,t > 0 \,:\, X_t \notin \mathbb{R}\,\}$. See Karatzas and Shreve [9, Chapter 5.5] for a survey on existence, uniqueness and basic facts of the solutions to the stochastic differential equation (2.1). For simplicity, the state space of (X_t) is taken to be $\mathbb{R}$, but it will be clear that the considerations are generally valid for any state space which is an interval.

The scale function of (X_t) is given by

$$S(x) = \int_0^x \exp\left(-2 \int_0^u \frac{\mu(r)}{\sigma^2(r)}\,dr\right) du$$

for $x \in \mathbb{R}$. The scale function $S(\cdot)$ has a strictly positive continuous derivative and the second derivative exists almost everywhere. Thus $S(\cdot)$ is strictly increasing with $S(0) = 0$. Define the open interval $I = (S(-\infty), S(\infty))$. If $I = \mathbb{R}$ then (X_t) is recurrent and if I is bounded from below or above then (X_t) is non-recurrent (see [9, Proposition 5.22]).

Let ν be a probability measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |S(u)|\,\nu(du) < \infty$$

and denote

$$m = \int_{\mathbb{R}} S(u)\,\nu(du)\ .$$

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Let $\alpha = \inf\{\,x \in \mathbb{R} \mid \nu((-\infty, S^{-1}(x)]) > 0\,\}$ and $\beta = \sup\{\,x \in \mathbb{R} \mid \nu([S^{-1}(x), \infty)) > 0\,\}$. If $m \geq 0$, define the stopping time

$$(2.2) \qquad \tau_{h_+} = \inf\{\,t > 0 \,:\, X_t \leq h_+(\max_{0 \leq r \leq t} X_r)\,\}$$

where the increasing function $s \mapsto h_+(s)$ for $S^{-1}(m) < s < S^{-1}(\beta)$ is expressed through its right inverse by

$$h_+^{-1}(x) = S^{-1}\left(\frac{1}{\nu([x, \infty))} \int_{[x, \infty)} S(u)\, \nu(du)\right) \quad (x < S^{-1}(\beta))$$

and set $h_+(s) = -\infty$ for $s \leq S^{-1}(m)$ and $h_+(s) = s$ for $s \geq S^{-1}(\beta)$. If $m \leq 0$, define the stopping time

$$(2.3) \qquad \tau_{h_-} = \inf\{\,t > 0 \,:\, X_t \geq h_-(\min_{0 \leq r \leq t} X_r)\,\}$$

where the increasing function $s \mapsto h_-(s)$ for $S^{-1}(\alpha) < s < S^{-1}(m)$ is expressed through its right inverse by

$$h_-^{-1}(x) = S^{-1}\left(\frac{1}{\nu((-\infty, x])} \int_{(-\infty, x]} S(u)\, \nu(du)\right) \quad (x > S^{-1}(\alpha))$$

and set $h_-(s) = \infty$ for $s \geq S^{-1}(m)$ and $h_-(s) = s$ for $s \leq S^{-1}(\alpha)$.

The main problem under consideration in this paper is the following. Given the probability measure ν , find a stopping time τ_* of (X_t) satisfying

$$(2.4) \qquad X_{\tau_*} \sim \nu$$

and determine the necessary and sufficient conditions on ν which make such a construction possible.

The following theorem states that the above stopping times are solutions to the Skorokhod embedding problem (2.4).

Theorem 2.1. *Let* (X_t) *be a non-singular diffusion on* $\mathbb{R}$ *starting at zero, let* $S(\cdot)$ *denote its scale function satisfying* $S(0) = 0$ *, and let* ν *be a probability measure on* $\mathbb{R}$ *satisfying* $\int_{\mathbb{R}} |S(x)|\, \nu(dx) < \infty$ *. Set* $m = \int_{\mathbb{R}} S(x)\, \nu(dx)$ *.*

Then there exists a stopping time τ_* *for* (X_t) *such that* $X_{\tau_*} \sim \nu$ *if and only if one of the following four cases holds:*

(i) $S(-\infty) = -\infty$ *and* $S(\infty) = \infty$ *;*
(ii) $S(-\infty) = -\infty$ *,* $S(\infty) < \infty$ *and* $m \geq 0$ *;*
(iii) $S(-\infty) > -\infty$ *,* $S(\infty) = \infty$ *and* $m \leq 0$ *;*
(iv) $S(-\infty) > -\infty$ *,* $S(\infty) < \infty$ *and* $m = 0$ *.*

Moreover, if $m \geq 0$ *then* τ_* *can be defined by (2.2), and if* $m \leq 0$ *then* τ_* *can be defined by (2.3).*

Proof. First, we verify that the conditions in cases (i)-(iv) are sufficient.

1. The first step in finding a solution to the problem (2.4) is to introduce the continuous local martingale $(M_t)_{t \geq 0}$ which shall be used in transforming the original problem into an analogous Skorokhod problem. Let (M_t) be the continuous local martingale given by composing (X_t) with the scale function $S(\cdot)$, i.e.

$$(2.5) \qquad M_t = S(X_t) .$$

Then $S(-\infty) < M_t < S(\infty)$ for $t < e$ and if I is bounded from below or above, M_t converges to the boundary of I for $t \uparrow e$ and $M_t = M_e$ on $\{e < \infty\}$ for $t \geq e$. By Itô-Tanaka formula it follows that (M_t) is a solution to the stochastic differential equation

$$dM_t = \tilde{\sigma}(M_t)\, dB_t$$

where

$$\tilde{\sigma}(x) = \begin{cases} S'\big(S^{-1}(x)\big)\, \sigma\big(S^{-1}(x)\big) & \text{for } x \in I \\ 0 & \text{else .} \end{cases}$$

The quadratic variation process is therefore given by

$$\langle M, M \rangle_t = \int_0^t \tilde{\sigma}^2(M_u)\, du = \int_0^{t \wedge e} \big(S'(X_u)\, \sigma(X_u)\big)^2 du$$

and it is immediately seen that $t \mapsto \langle M, M \rangle_t$ is strictly increasing for $t < e$. If I is bounded from below or above then $\langle M, M \rangle_e < \infty$, and if $I = \mathbb{R}$ the local martingale (M_t) is recurrent, or equivalently $\langle M, M \rangle_e = \infty$ and $e = \infty$. The process (M_t) does not explode, but the explosion time e for (X_t) can be expressed as $e = \inf\{t > 0 : M_t \notin I\}$.

Let U be a random variable satisfying $U \sim \nu$ and let μ be the probability measure satisfying $S(U) \sim \mu$. For a stopping time τ_* of (X_t) it is not difficult to see that $X_{\tau_*} \sim \nu$ if and only if $M_{\tau_*} \sim \mu$. Therefore, the initial problem (2.4) is analogous to the problem of finding a stopping time τ_* of (M_t) satisfying

$$(2.6) \qquad\qquad\qquad M_{\tau_*} \sim \mu .$$

Moreover, if τ_* is an embedding for (M_t) by the above observations, it follows that $S(-\infty) < M_{\tau_*} < S(\infty)$ and hence $\tau_* < e$.

2. The second step is to apply time-change and verify that the embedding problem of continuous local martingale (2.6) is equivalent to the embedding problem of Brownian motion. Let (T_t) be the time-change given by

$$(2.7) \qquad\qquad T_t = \inf\{s > 0 : \langle M, M \rangle_s > t\} = \langle M, M \rangle_t^{-1}$$

for $t < \langle M, M \rangle_e$. Define the process $(W_t)_{t \geq 0}$ by

$$(2.8) \qquad\qquad W_t = \begin{cases} M_{T_t} & \text{if } t < \langle M, M \rangle_e \\ M_e & \text{if } t \geq \langle M, M \rangle_e . \end{cases}$$

Since $t \mapsto T_t$ is strictly increasing for $t < \langle M, M \rangle_e$, we have that $\big(\mathcal{F}_{T_t}^M\big) = \big(\mathcal{F}_t^W\big)$. This implies that, if $\tau < \langle M, M \rangle_e$ is a stopping time for (W_t) then T_τ is a stopping time for (M_t), and vice versa if $\tau < e$ is a stopping time for (M_t) then $\langle M, M \rangle_\tau$ is a stopping time for (W_t). The process (W_t) is a Brownian motion stopped at $\langle M, M \rangle_e$ according to Dambis-Dubins-Schwarz theorem (see [13, (1.7) Theorem, Chapter V]). By the definition of (W_t) it is clear that $\langle M, M \rangle_e = \inf\{t > 0 : W_t \notin I\}$ and hence the two processes $(W_t)_{t \geq 0}$ and $(B_{\tau_{S(-\infty),S(\infty)} \wedge t})_{t \geq 0}$ have the same law where $\tau_{S(-\infty),S(\infty)} = \inf\{t > 0 : B_t \notin I\}$.

From the above observation we deduce that the embedding problem for the continuous local martingale is equivalent to embedding in the stopped Brownian motion, i.e the martingale case (2.6) is equivalent to finding a stopping time $\tilde{\tau}_*$ of (W_t) satisfying

$$(2.9) \qquad\qquad\qquad W_{\tilde{\tau}_*} \sim \mu .$$

3. For constructing a stopping time $\tilde{\tau}_*$ of (W_t) that satisfies the embedding problem (2.9) we shall make use of the Azéma-Yor construction. Assume that $m \geq 0$ and define the stopping time

(2.10)
$$\tilde{\tau}_* = \inf \left\{ t > 0 \, : \, W_t \leq b_+\big(\max_{0 \leq r \leq t} W_r \big) \right\}$$

where the increasing function $s \mapsto b_+(s)$ for $m < s < \beta$ is expressed through its right inverse by

(2.11)
$$b_+^{-1}(x) = \frac{1}{\mu([x,\infty))} \int_{[x,\infty)} u\,\mu(du) \quad (x < \beta)$$

and for $s \leq m$ set $b_+(s) = -\infty$ and for $s \geq \beta$ set $b_+(s) = s$. The stopping time $\tilde{\tau}_*$ can then be described by $\tilde{\tau}_* = \tilde{\tau}_m + \tilde{\tau}_* \circ \theta_{\tilde{\tau}_m}$, where $\tilde{\tau}_m = \inf\{t > 0 \, : \, W_t = m\}$. Note that $s \mapsto b_+^{-1}(s)$ is the barycentre function of the probability measure μ. Moreover, the following connection between $h_+^{-1}(\cdot)$ and $b_+^{-1}(\cdot)$ is valid

(2.12)
$$h_+^{-1}(\cdot) = (S^{-1} \circ b_+^{-1} \circ S)(\cdot) \,.$$

Due to $\langle M, M \rangle_e = \inf\{t > 0 \, : \, W_t \notin (S(-\infty), S(\infty))\}$ and the construction of $b_+(\cdot)$ it follows that $\tilde{\tau}_* < \langle M, M \rangle_e$ if either $S(-\infty) = -\infty$, or $m = 0$ with $S(-\infty) > -\infty$ and $S(\infty) < \infty$. Therefore in the cases (i), (ii) and (iv) we have that $\tilde{\tau}_* < \langle M, M \rangle_e$. (Note that $\tilde{\tau}_* < \langle M, M \rangle_e$ fails in the other cases.) The process (W_t) is a Brownian motion stopped at $\langle M, M \rangle_e$. Note if $\tilde{\tau}_*$ is an embedding of the centered distribution of $S(U) - m$ then the strong Markov property ensures that the stopping time $\tilde{\tau}_m + \tilde{\tau}_* \circ \theta_{\tilde{\tau}_m}$ is an embedding of $S(U) \sim \mu$. By this observation we then have from Azéma and Yor [1] that $W_{\tilde{\tau}_*} \sim \mu$. Then the stopping time τ_* for (M_t) given by

$$\tau_* = T_{\tilde{\tau}_*} = \inf\left\{ t > 0 \, : \, M_t \leq b_+\big(\max_{0 \leq r \leq t} M_r \big) \right\}$$

satisfies $M_{\tau_*} = W_{\tilde{\tau}_*} \sim \mu$ where (T_t) is the time change given in (2.7). From (2.12) and the definition of (M_t) we see that τ_* is given in (2.2) and it clearly fulfills $X_{\tau_*} \sim \nu$. The same arguments hold for $m \leq 0$.

4. Finally the conditions in the cases (i)-(iv) are necessary as well. Indeed, case (i) is trivial because there is no restriction on the class of probability measures we are considering. In case (ii) let τ_* be a stopping time for (X_t) satisfying $X_{\tau_*} \sim \nu$ or equivalently $M_{\tau_*} \sim \mu$. Then the process $(M_{\tau_* \wedge t})$ is a continuous local martingale which is bounded from above by $S(\infty) < \infty$. Letting $\{\gamma_n\}_{n \geq 1}$ be a localization for the local martingale, and applying Fatou's lemma and the optional sampling theorem, we see that $m = \mathbf{E}(M_{\tau_*}) \geq \liminf_n \mathbf{E}(M_{\tau_* \wedge \gamma_n}) = 0$. Cases (iii) and (iv) are proved in exactly in the same way. Note that (M_t) is a bounded martingale in case (iv). $\qquad \square$

3. Characterization of the embedding stopping time

In this section, we examine some extremal properties of the embedding from Theorem 2.1 that are given in [11]-[12] when the process is a Brownian motion with drift. Loosely speaking, the embedding τ_* is pointwise the smallest embedding that stochastically maximizes $\max_{0 \leq t \leq \tau_*} X_t$. This characterizes τ_* uniquely. In the sequel we assume that $m \geq 0$. The results for $m \leq 0$ can easily be translated from the $m \geq 0$ case.

Proposition 3.1. *Let* $m \geq 0$ *and under the assumptions of Theorem 2.1, let* τ *be any stopping time of* (X_t) *satisfying* $X_\tau \sim \nu$. *If* $\mathbf{E}\big(\max_{0 \leq t \leq \tau} S(X_t)\big) < \infty$ *then*

$$(3.1) \qquad \mathbf{P}\big(\max_{0 \leq t \leq \tau} X_t \geq s\big) \leq \mathbf{P}\big(\max_{0 \leq t \leq \tau_*} X_t \geq s\big)$$

for all $s \geq 0$. *If furthermore* ν *satisfies*

$$(3.2) \qquad \int_0^\infty S(u) \log(S(u))\, \nu(du) < \infty$$

and the stopping time τ *satisfies* $\max_{0 \leq t \leq \tau} X_t \sim \max_{0 \leq t \leq \tau_*} X_t$ *(i.e. there is equality in (3.1) for all* $s > 0$*) then* $\tau = \tau_*$ $\mathbf{P}$-*a.s.*

Proof. Let τ be the stopping time given in the proposition. Then we have that $M_\tau \sim \mu$ and $\mathbf{E}\big(\max_{0 \leq t \leq \tau} M_t\big) < \infty$. Since τ and τ_* are two embeddings we have from Section 2 that the two stopping times $\tilde{\tau}$ and $\tilde{\tau}_*$ for (W_t) given by $\tilde{\tau} = \langle M, M \rangle_\tau$ and $\tilde{\tau}_* = \langle M, M \rangle_{\tau_*}$ satisfy $W_{\tilde{\tau}} \sim W_{\tilde{\tau}_*} \sim \mu$. Note that $\tilde{\tau}_*$ is given in (2.10) and that $\mathbf{E}\big(\max_{0 \leq t \leq \tilde{\tau}} W_t\big) = \mathbf{E}\big(\max_{0 \leq t \leq \tau} M_t\big) < \infty$. Thus it is enough to verify

$$(3.3) \qquad \mathbf{P}\big(\max_{0 \leq t \leq \tilde{\tau}} W_t \geq s\big) \leq \mathbf{P}\big(\max_{0 \leq t \leq \tilde{\tau}_*} W_t \geq s\big)$$

for all $s \geq 0$. Given the following fact (see [4])

$$(3.4) \qquad \mathbf{P}\big(\max_{0 \leq t \leq \tilde{\tau}_*} W_t \geq s\big) = \inf_{y < s} \frac{\mathbf{E}\big(W_{\tilde{\tau}_*} - y\big)^+}{s - y}$$

the proof of (3.3) in essence is the same as the proof of [4, Lemma 2.1] and we include it merely for completeness. First note that $\max_{0 \leq t \leq \tilde{\tau}_*} W_t \geq m$ $\mathbf{P}$-a.s. and (3.3) is trivial for $0 \leq s \leq m$. Let $s > m$ be given and fix $y < s$. We have the inequality

$$(3.5) \qquad \frac{\big(W_{\tilde{\tau} \wedge t} - y\big)^+}{s - y} + \frac{s - W_{\tilde{\tau} \wedge t}}{s - y} \mathbf{1}_{[s,\infty)}\big(\max_{0 \leq r \leq \tilde{\tau} \wedge t} W_r\big) \geq \mathbf{1}_{[s,\infty)}\big(\max_{0 \leq r \leq \tilde{\tau} \wedge t} W_r\big)$$

which can be verified on a case by case basis. Taking expectation in (3.5) we have by Doob's submartingale inequality that

$$\mathbf{P}\big(\max_{0 \leq r \leq \tilde{\tau} \wedge t} W_r \geq s\big) \leq \frac{\mathbf{E}\big(W_{\tilde{\tau} \wedge t} - y\big)^+}{s - y}\ .$$

Since $\mathbf{E}\big(\max_{0 \leq t \leq \tilde{\tau}} W_t\big) < \infty$, we can apply Fatou's lemma and letting $t \to \infty$ we obtain that

$$\mathbf{P}\big(\max_{0 \leq r \leq \tilde{\tau}} W_r \geq s\big) \leq \frac{\mathbf{E}\big(W_{\tilde{\tau}} - y\big)^+}{s - y}$$

for all $y < s$. Taking infimum over all $y < s$ and since $W_{\tilde{\tau}_*} \sim W_{\tilde{\tau}}$ together with (3.4) we have the inequality (3.3).

In order to prove the second part, we have by the foregoing that it is clearly sufficient to show that

$$(3.6) \qquad \tilde{\tau} = \tilde{\tau}_* \quad \mathbf{P}\text{-a.s.}$$

We shall use a modified proof of [18, Theorem 1] to prove (3.6). First note that from [2] (see also [12]) that condition (3.2) is satisfied if and only if $\mathbf{E}\big(\max_{0 \leq t \leq \tilde{\tau}_*} W_t\big) < \infty$. Therefore $\big((W_{\tilde{\tau} \wedge t} - s)^+\big)_{t \geq 0}$ is uniform integrable for any s. Fix s which is not an atom for the

probability measure μ and set $x = b_+^{-1}(s)$ where $b_+^{-1}(\cdot)$ is the barycentre function in (2.11). Thus by the fact $W_{\tilde{\tau}} \sim W_{\tilde{\tau}_*}$ and the optional sampling theorem, we get that

$$\mathbf{E}\big(W_{\tilde{\tau}} - s\big)^+ \geq \mathbf{E}\big(W_{\tilde{\tau} \wedge \tilde{\tau}_x} - s\big)^+ = \big(b_+^{-1}(s) - s\big)\,\mathbf{P}(\tilde{\tau}_x \leq \tilde{\tau}) + \mathbf{E}\big((W_{\tilde{\tau}} - s)^+ \,;\, \tilde{\tau} > \tilde{\tau}_x\big)$$

$$= \mathbf{E}\big((W_{\tilde{\tau}} - s)^+ \,;\, W_{\tilde{\tau}} \geq s\big) + \mathbf{E}\big((W_{\tilde{\tau}} - s)^+ \,;\, \tilde{\tau} > \tilde{\tau}_x\big)$$

$$= \mathbf{E}\big(W_{\tilde{\tau}} - s\big)^+ + \mathbf{E}\big((W_{\tilde{\tau}} - s)^+ \,;\, W_{\tilde{\tau}} \geq s \,,\, \tilde{\tau} < \tilde{\tau}_x\big)$$

where we have used (see [2]) that

(3.7) $$\mathbf{P}(W_{\tilde{\tau}} \geq s) = \mathbf{P}\big(\max_{0 \leq t \leq \tilde{\tau}} W_t \geq b_+^{-1}(s)\big)$$

and the definition of the barycentre function. Hence $\mathbf{E}\big((W_{\tilde{\tau}} - s)^+ \,;\, W_{\tilde{\tau}} \geq s \,,\, \tilde{\tau} < \tilde{\tau}_x\big) \leq 0$ and therefore $\{W_{\tilde{\tau}} \geq s \,,\, \tilde{\tau} < \tilde{\tau}_x\}$ is a $\mathbf{P}$-nullset due to the fact that $\{W_{\tilde{\tau}} = s\}$ is also a $\mathbf{P}$-nullset. Because of (3.7), we conclude that $\{W_{\tilde{\tau}} \geq s\} = \{\max_{0 \leq t \leq \tilde{\tau}} W_t \geq b_+^{-1}(s)\}$ $\mathbf{P}$-a.s. for all s which is not an atom for μ. Since $s \mapsto b_+^{-1}(s)$ is left continuous, we have that $\max_{0 \leq t \leq \tilde{\tau}} W_t \geq b_+^{-1}(W_{\tilde{\tau}})$ $\mathbf{P}$-a.s. and we deduce that $\tilde{\tau}_* \leq \tilde{\tau}$ $\mathbf{P}$-a.s. Finally, let $\tilde{\sigma}$ be an any stopping time for (W_t) satisfying $\tilde{\tau}_* \leq \tilde{\sigma} \leq \tilde{\tau}$ $\mathbf{P}$-a.s. Then, optional sampling theorem implies that $\mathbf{E}\big(W_{\tilde{\sigma}} - s\big)^+ = \mathbf{E}\big(W_{\tilde{\tau}_*} - s\big)^+$ for all s and therefore $W_{\tilde{\sigma}} \sim \mu$. Clearly, this is only possible if $\tilde{\tau} = \tilde{\tau}_*$ $\mathbf{P}$-a.s. The proof is complete. $\square$

Remark 3.2. Observe that no uniform integrability condition is needed for the second part of the result, which normally is assumed in similar statements (see e.g. [2] and [18]), and it is only necessary to control the size of the maximum process (i.e. condition (3.2)). Furthermore, note that $\mathbf{E}\big(\max_{0 \leq t \leq \tau} S(X_t)\big) < \infty$ and (3.2) are trivial when $S(\cdot)$ is bounded from above (i.e. when the process (X_t) is non-recurrent).

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