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Levý-Khintchine Inequalities, Multiple Reflection, and Brownian Motion

GORAN PESKIR

Let $(\varepsilon_i)_{i\geq 1}$ be a sequence of independent Rademacher variables, and let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then the conjecture made in [5] that the best constant in the maximal Khintchine inequality:

$$\left(E\left(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k a_i\varepsilon_i\Big|^p\right)\right)^{1/p} \leq \mathbf{B}_p^*\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}$$

equals $\mathbf{B}_p^* = (E(\max_{0\leq t\leq 1}|B_t|^p))^{1/p}$ fails for p in a neighborhood of 1. This can be established by using a variant of the following general formula:

$$E\left(g\Big(z\vee\max_{0\leq t\leq T}|x+B_t|\Big)\right) = g(z) + 2\sum_{k=1}^\infty(-1)^{k-1}E\left(g\left(\frac{|x+B_T|}{2k-1}\right)-g(z)\right)^+$$

being valid for all $z\geq x\geq 0$ and all increasing functions $g : I\!R_+\rightarrow I\!R_+$ which are continuous at zero and satisfy $E\big(g\big(|x+B_T|\big)\big)<\infty$. The method of proof relies upon a multiple reflection argument. In addition, it is shown that

$$\left(E\left(\max_{1\leq k\leq n}\Big(\sum_{i=1}^k a_i\varepsilon_i\Big)_+^p\right)\right)^{1/p} \leq \left(E\Big(\max_{0\leq t\leq 1}B_t\Big)_+^p\right)^{1/p}\left(\sum_{i=1}^n|a_i|^2\right)^{1/2}$$

for all $p\geq 2$, and the constant appearing on the right-hand side is best possible. The main aim of the paper is to present some basic facts and formulas arising in the study of these questions with a special emphasis on unresolved issues.

1. Introduction

1. Let $\varepsilon_1,\ldots,\varepsilon_n$ be independent (Rademacher) random variables taking values ± 1 with probability $1/2$. The *Khintchine inequalities* (dating back to [8]) state that

$$\mathbf{A}_p\left(\sum_{i=1}^n|a_i|^2\right)^{1/2} \leq \left(E\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^p\right)^{1/p} \leq \mathbf{B}_p\left(\sum_{i=1}^n|a_i|^2\right)^{1/2} \tag{1.1}$$

for all $a_1,\ldots,a_n\in\mathbf{R}$, where $\mathbf{A}_p$ and $\mathbf{B}_p$ are universal constants and $0<p<\infty$. These inequalities are known to play a fundamental role in probability theory.

The best constants $\mathbf{A}_p$ and $\mathbf{B}_p$ in (1.1) are known (see [6]). In this paper we only focus on the right-hand side inequality and for further reference recall that

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$$\mathbf{B}_p = \Big(E|B_1|^p \Big)^{1/p} \quad (2 \le p < \infty) \tag{1.2}$$

where $B_1 \sim N(0,1)$. The case $0<p<2$ is less interesting as then $\mathbf{B}_p = 1$.

2. If $\xi_1, \ldots, \xi_n$ are independent random variables that are symmetrically distributed around zero (i.e. $-\xi_i \sim \xi_i$ for all i), then *Lévy's inequality* (cf. Lemma 3.1) states that

$$P\Big\{ \max_{1\le k\le n} |S_k| \ge t \Big\} \le 2\, P\big\{ |S_n| \ge t \big\} \tag{1.3}$$

for all $t \ge 0$, where we denote $S_k = \sum_{i=1}^k \xi_i$ for all $1 \le k \le n$. This inequality is a simple consequence of *the reflection principle* (see [9]).

Integration by parts, upon using (1.3) with $\xi_i = a_i \varepsilon_i$ and applying (1.1), then yields:

$$\mathbf{A}_p^* \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \le \Big(E\Big(\max_{1\le k\le n} \big| \sum_{i=1}^k a_i \varepsilon_i \big|^p \Big) \Big)^{1/p} \le \mathbf{B}_p^* \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \tag{1.4}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$, where $\mathbf{A}_p^*$ and $\mathbf{B}_p^*$ are universal constants and $0 < p < \infty$. These inequalities are usually referred to as the *maximal Khintchine inequalities*. In this paper we call them the *Lévy-Khintchine inequalities*.

3. The best constants $\mathbf{A}_p^*$ and $\mathbf{B}_p^*$ in (1.4) are not known. The only known but trivial fact is that $A_p^* = 1$ when $p \ge 2$. We may note from (1.3) that the following bound holds:

$$\mathbf{B}_p^* \le 2^{1/p}\, \mathbf{B}_p \tag{1.5}$$

for all $0 < p < \infty$. In view of a functional central limit theorem (Donsker's invariance principle) it was conjectured in [5] that the best value for $\mathbf{B}_p^*$ equals

$$\mathbf{B}_p^* = \Big(E\Big(\max_{0\le t\le 1} |B_t|^p \Big) \Big)^{1/p} \tag{1.6}$$

where $(B_t)_{t\ge 0}$ is standard Brownian motion.

4. It turns out however that this conjecture fails for p in a neighborhood of 1 . The method of disproval relies upon a multiple reflection argument which also enables one to establish a more informative connection between (1.3) and (1.4). The main aim of the paper is to present some basic facts and formulas arising in this study with a special emphasis on unresolved issues.

We begin our exposition with an analysis of the method of proof which would follow the line of arguments leading to the best constant (1.2) in the single-partial-sum inequality (1.1).

2. Successive integration arguments

Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent (Rademacher) random variables taking values ± 1 with probability $1/2$, and let $a_1, \ldots, a_n$ be real numbers.

1. We begin by recalling some known facts about the right-hand side inequality in (1.1). This inequality with $\mathbf{B}_p$ from (1.2) is equivalently rewritten as follows:

$$E\big| \sum_{i=1}^n a_i \varepsilon_i \big|^p \le E\big| \sum_{i=1}^n a_i g_i \big|^p \tag{2.1}$$

where $g_1, \dots, g_n$ are independent random variables identically distributed as $N(0,1)$.

To establish (2.1) we see by independence (Fubini's theorem) that it is sufficient to prove:

$$E|x+a\varepsilon|^p \le E|x+ag|^p \tag{2.2}$$

for all $x \in I\!R$ where $\varepsilon \sim \varepsilon_1$ and $g \sim g_1$. Clearly, we may assume in (2.2) that $a=1$, and therefore the problem reduces to verify the following inequality:

$$|x+1|^p + |x-1|^p \le \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x+t|^p \, e^{-t^2/2} \, dt \tag{2.3}$$

for all $x \ge 0$.

If $p \ge 3$ (or $p=2$) then the map

$$s \mapsto E|x+\varepsilon\sqrt{s}\,|^p \tag{2.4}$$

is convex on $I\!R_+$, and thus (2.3) follows by Jensen's inequality:

$$E\big|x+\varepsilon\sqrt{E(g^2)}\big|^p \le E\big|x+\varepsilon|g|\big|^p \tag{2.5}$$

where we use that $E(g^2)=1$ and $\varepsilon|g| \sim g$ upon independence (cf. [10] and [3]).

If $2<p<3$ then it can be verified that the inequality (2.3) fails for $x>0$ large enough, and thus cannot be established by analytic methods either. This indicates that the method of proof cannot rely upon the same successive integration argument. The only known proof of (2.1) in this case is much more technically involved (see [6]).

2. Motivated by these facts we now consider the right-hand side inequality in (1.4) with B_p^* from (1.6). This inequality reads as follows:

$$E\Big(\max_{1\le k\le n} \big|\sum_{i=1}^k a_i\varepsilon_i\big|^p\Big) \le E\Big(\max_{0\le t\le 1} |B_t|^p\Big)\Big(\sum_{i=1}^n |a_i|^2\Big)^{p/2} := R_n \tag{2.6}$$

where $B=(B_t)_{t\ge 0}$ is a standard Brownian motion.

Set $t_k = \sum_{i=1}^k |a_i|^2$ for $1\le k\le n$ and observe by Brownian scaling that the right-hand side in (2.6) can be written as follows:

$$\begin{aligned} R_n &= E\Big(\max_{0\le t\le t_n} |B_t|^p\Big) \\ &= E\Big(\max_{0\le t\le t_1} \big|B_t^{(1)}\big|^p \vee \max_{t_1\le t\le t_2} \big|B_{t_1}^{(1)}+B_t^{(2)}\big|^p \vee \ldots \vee \max_{t_{n-1}\le t\le t_n} \big|\textstyle\sum_{i=1}^{n-1} B_{t_i}^{(i)}+B_t^{(n)}\big|^p\Big) \end{aligned} \tag{2.7}$$

where $B_t^{(i)} = B_t - B_{t_{i-1}}$ with $t_0=0$, and $(B_t^{(i)})_{t\ge t_{i-1}}$ is a standard Brownian motion independent from $\mathcal{F}_{t_{i-1}}^B := \sigma(B_s \mid 0\le s\le t_{i-1})$ when $1\le i\le n$.

By stationary independent increments of B , we find that the random variables

$$B_{t_i}^{(i)} := a_i g_i \sim N(0,|a_i|^2) \tag{2.8}$$

are independent for $1\le i\le n$. Thus (2.7) can be further rewritten as follows:

(2.9) $$R_n = E\Big(\max_{0\le t\le t_1} \big|B_t^{(1)}\big|^p \vee \max_{t_1\le t\le t_2} \big|a_1 g_1 + B_t^{(2)}\big|^p \vee \ldots \vee \max_{t_{n-1}\le t\le t_n} \big|\textstyle\sum_{i=1}^{n-1} a_i g_i + B_t^{(n)}\big|^p \Big)$$

where the Brownian motion $(B_t^{(i)})_{t_{i-1}\le t\le t_i}$ is independent from the entire past $\mathcal{F}_{t_{i-1}}^B$ for $1\le i\le n$.

If we neglect Brownian motions in (2.9), then for (2.6) it is enough to prove:

(2.10) $$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} a_i \varepsilon_i\Big|^p \Big) \le E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^{k} a_i g_i\Big|^p \Big)$$

where $g_1, \ldots, g_n$ are independent random variables identically distributed as $N(0,1)$.

To establish (2.10) we see by independence (Fubini's theorem) that it is sufficient to derive:

(2.11) $$E\Big(z \vee \big|x_1 + a\varepsilon\big|^p \vee \ldots \vee \big|x_n + a\varepsilon\big|^p \Big) \le E\Big(z \vee \big|x_1 + ag\big|^p \vee \ldots \vee \big|x_n + ag\big|^p \Big)$$

for all $z, x_1, \ldots, x_n \in I\!\!R$ where $\varepsilon \sim \varepsilon_1$ and $g \sim g_1$. Clearly, we may assume that $a=1$ in (2.11). As ε can take only two values, the problem of (2.11) therefore reduces to verify:

(2.12) $$E\Big(z \vee \big|x + \varepsilon\big|^p \vee \big|y + \varepsilon\big|^p \Big) \le E\Big(z \vee \big|x + g\big|^p \vee \big|y + g\big|^p \Big)$$

for all $z, x, y \in I\!\!R$. This inequality should be compared with the inequality (2.2) in the context of the single-partial-sum inequality (2.1).

It can be shown that (2.12) fails to hold at least for all $p \le 5$ or larger. This indicates that the question of proving (2.6) goes beyond a successive integration argument (based upon conditioning). It makes the case of maximal inequalities (2.6) similar to the case $2<p<3$ of the single-partial-sum inequality (2.1) that is known to be difficult. A multiple reflection approach is presented in the next section as a possible alternative to this drawback.

If Brownian motions in (2.9) are not neglected, then the following weaker version of (2.11):

(2.11') $$E\Big(z \vee \big|x_1 + a\varepsilon\big|^p \vee \big|x_2 + a\varepsilon\big|^p \vee \ldots \vee \big|x_n + a\varepsilon\big|^p \Big)$$
$$\le E\Big(z \vee \max_{0\le t\le 1} \big|x_1 + aB_t\big|^p \vee \big|x_2 + aB_1\big|^p \vee \ldots \vee \big|x_n + aB_1\big|^p \Big)$$

would be sufficient for (2.6), if valid for all $z, x_1, \ldots, x_n \in I\!\!R$, where $\varepsilon \sim \varepsilon_1$ and $(B_t)_{t\ge 0}$ is a standard Brownian motion. However, it can be shown too that this inequality fails in some cases, but we present no formal argument and leave it worthy of further consideration.

3. A main reason that (2.12) fails is that $E|g|<1$. Motivated by this fact we now show how the preceding idea can be applied to random variables having the first moment equal to one.

Let $\xi_1, \ldots, \xi_n$ be independent random variables that are symmetrically distributed around zero, and let us assume that $E|\xi_i| = 1$ for all $1\le i\le n$. Then Jensen's inequality implies:

(2.13) $$E\Big(z \vee \big|x + \varepsilon\big|^p \vee \big|y + \varepsilon\big|^p \Big) = E\Big(z \vee \big|x + \varepsilon E|\xi_i|\big|^p \vee \big|y + \varepsilon E|\xi_i|\big|^p \Big)$$
$$\le E\Big(z \vee E\big|x + \varepsilon|\xi_i|\big|^p \vee E\big|y + \varepsilon|\xi_i|\big|^p \Big) \le \Big(z \vee E\big|x + \xi_i\big|^p \vee \big|y + \xi_i\big|^p \Big)$$

for all $z, x, y \in I\!\!R$ due to the fact that $\varepsilon|\xi_i| \sim \xi_i$ for all $1\le i\le n$. Thus, the analogue of (2.12) holds, and by the independence argument above this yields the following inequality:

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^k a_i\varepsilon_i\Big|^p\Big) \le E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^k a_i\xi_i\Big|^p\Big) \tag{2.14}$$

for all $a_1,\dots,a_n \in I\!R$ and $p\ge 1$. This inequality is sharp when $p=1$.

Applying this inequality to $\xi_i = \sqrt{\pi/2}\, g_i$ for $1\le i\le n$ and $p=1$, we obtain:

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^k a_i\varepsilon_i\Big|\Big) \le \sqrt{\frac{\pi}{2}}\; E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^k a_i g_i\Big|\Big) \tag{2.15}$$

for all $a_1,\dots,a_n \in I\!R$. The constant $\sqrt{\pi/2}$ is best possible in this inequality.

On the other hand, from (2.9) with (2.6) upon using that $E(\max_{0\le t\le 1}|B_t|) = \sqrt{\pi/2}$, we get:

$$E\Big(\max_{1\le k\le n} \Big|\sum_{i=1}^k a_i g_i\Big|\Big) \le \sqrt{\frac{\pi}{2}}\; \Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \tag{2.16}$$

for all $a_1,\dots,a_n \in I\!R$. The constant $\sqrt{\pi/2}$ is best possible in this inequality.

4. The only property of the map $x\mapsto |x|^p$ that matters in the proof above is its convexity. Thus the same argument implies that the following inequality holds:

$$E\Big(\max_{1\le k\le n} G\Big(\sum_{i=1}^k a_i\varepsilon_i\Big)\Big) \le E\Big(\max_{1\le k\le n} G\Big(\sum_{i=1}^k a_i\xi_i\Big)\Big) \tag{2.17}$$

for all $a_1,\dots,a_n \in I\!R$ whenever $G: I\!R\to I\!R$ is convex. Extracting the crucial properties even more, the following general result of Hunt [7] can be established with a simpler proof.

Lemma 2.1 (Hunt)

Let $\rho_1,\dots,\rho_n$ be independent random variables satisfying $|\rho_i|\le 1$ and $E(\rho_i)=0$ for all $1\le i\le n$, and let $\xi_1,\dots,\xi_n$ be independent random variables that are symmetrically distributed around zero and satisfy $E|\xi_i|\ge 1$ for all $1\le i\le n$.

If $F: I\!R^n\to I\!R$ is convex (in each argument) then:

$$EF(\rho_1,\dots,\rho_n) \le EF(\xi_1,\dots,\xi_n)\ . \tag{2.18}$$

Proof. Since the random variables are independent, a successive integration argument based upon Fubini's theorem reduces the proof to the case $n=1$. Set $\rho=\rho_1$ and $\xi=\xi_1$.

By convexity of F and the fact that $|\rho|\le 1$, we find:

$$F(\rho) \le \frac{1-\rho}{2}\, F(-1) + \frac{1+\rho}{2}\, F(1)\ . \tag{2.19}$$

Taking the expectation on both sides in (2.19) and using that $E(\rho)=0$, we get:

$$EF(\rho) \le EF(\varepsilon) \tag{2.20}$$

where ε is a Bernoulli random variable taking values ± 1 with probability $1/2$.

Set $\mu = E|\xi|$ and note that $\mu\ge 1$. Thus $F(-1)+F(1)\le F(-\mu)+F(\mu)$ since F is convex. Hence by Jensen's inequality we may conclude:

(2.21) $$EF(\varepsilon) \le EF(\varepsilon\mu) = EF(\varepsilon E|\xi|) \le EF(\varepsilon|\xi|) = EF(\xi)$$

due to $\varepsilon|\xi| \sim \xi$. This completes the proof. □

Since the supremum of a family of convex functions defines a convex function, it follows that for any given $a_1, \ldots, a_n \in I\!R$ the function

(2.22) $$F(x_1, \ldots, x_n) = G(a_1x_1) \vee G(a_1x_1 + a_2x_2) \vee \ldots \vee G(a_1x_1 + a_2x_2 + \ldots + a_nx_n)$$

is convex whenever G is so. Thus (2.17) is a direct consequence of (2.18).

It should be noted that under the hypotheses of Lemma 2.1 we can deduce in exactly the same way that the following "mixing" inequality is satisfied:

(2.23) $$EF(\rho_1, \ldots, \rho_n) \le EF(\eta_1, \ldots, \eta_n)$$

where each η_i is either ρ_i , ε_i or ξ_i for all $1 \le i \le n$. To a large extent this mixing property characterises the successive integration approach presented. For example, such mixing inequalities will not be valid when $2 < p < 3$ in the case of a single partial sum (2.1).

Corollary 2.2

Under the same hypotheses as in Lemma 2.1 assume that $F : I\!R^n \to I\!R$ *is convex (in each argument) and that satisfies the following identity:*

(2.24) $$F(\nu x_1, \ldots, \nu x_n) = \nu^p F(x_1, \ldots, x_n)$$

for all $\nu > 0$ *and all* $x_1, \ldots, x_n \in I\!R$ *with some* $p > 0$.

If $E|\xi_i| = \mu > 0$ *for all* $1 \le i \le n$ *, then:*

(2.25) $$EF(\rho_1, \ldots, \rho_n) \le \frac{1}{\mu^p} EF(\xi_1, \ldots, \xi_n) .$$

A typical example of such a function F is given by (2.22) above with $G(x) = |x|^p$ or $G(x) = x^p$ for $p \ge 1$, but there are also plenty other possibilities.

It should be observed that (2.25) improves upon (2.18) when $\mu > 1$, as well as that it extends it to the case when $\mu < 1$.

5. We shall now describe a simple connection between inequalities derived and some known inequalities for Brownian motion. The central role in this description is played by Skorokhod's embedding. Below we assume that $a_1, \ldots, a_n \in I\!R$ are given and fixed.

Applying (2.17) with $\xi_i = g_i/E|g_i| = \sqrt{\pi/2}\, g_i$ and $G(x) = x$, we get:

(2.26) $$E\Big(\max_{1\le k\le n}\Big(\sum_{i=1}^k a_i\varepsilon_i\Big)\Big) \le \sqrt{\frac{\pi}{2}}\, E\Big(\max_{1\le k\le n}\Big(\sum_{i=1}^k a_i g_i\Big)\Big) \le \sqrt{\frac{\pi}{2}}\, E\Big(\max_{0\le t\le t_n} B_t\Big)$$
$$= \sqrt{\frac{\pi}{2}}\, E\Big(\max_{0\le t\le 1} B_t\Big)\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}$$

by means of Brownian scaling. Using further that $E(\max_{0\le t\le 1} B_t) = \sqrt{2/\pi}$, we obtain:

$$E\Big(\max_{1\le k\le n} \Big(\sum_{i=1}^k a_i\varepsilon_i \Big) \Big) \;\le\; \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \; . \tag{2.27}$$

This inequality can be related to the following inequality (see [2]):

$$E\Big(\max_{0\le t\le \tau} B_t \Big) \;\le\; \sqrt{E(\tau)} \tag{2.28}$$

which is valid for all stopping times τ of the standard Brownian motion $B=(B_t)_{t\ge 0}$. It can be established by Skorokhod's embedding. Consider the hitting times:

$$\begin{aligned} \tau_1 &= \inf\,\{\, t>0 \;:\; |B_t| = |a_1| \,\} \\ \tau_2 &= \inf\,\{\, t>0 \;:\; |B_{\tau_1+t} - B_{\tau_1}| = |a_2| \,\} \\ &\vdots \\ \tau_n &= \inf\,\{\, t>0 \;:\; |B_{\tau_1+\ldots+\tau_{n-1}+t} - B_{\tau_1+\ldots+\tau_{n-1}}| = |a_n| \,\} \; . \end{aligned} \tag{2.29}$$

Then $\tau_1,\ldots,\tau_n$ are independent random variables satisfying $E(\tau_k)=|a_k|^2$, and $T_k=\sum_{i=1}^k \tau_i$ is a stopping time of B for all $1\le k\le n$. Moreover, the following identity in law holds:

$$(a_1\varepsilon_1, a_2\varepsilon_2, \ldots, a_n\varepsilon_n) \sim (B_{T_1}, B_{T_2}-B_{T_1} \ldots, B_{T_n}-B_{T_{n-1}}) \; . \tag{2.30}$$

In particular, from (2.30) we see that

$$\max_{1\le k\le n} S_k \;\sim\; \max_{1\le k\le n} B_{T_k} \tag{2.31}$$

where we set $S_k=\sum_{i=1}^k a_i\varepsilon_i$ for all $1\le k\le n$. It remains to apply (2.28) to the stopping time T_n upon noting that $\max_{1\le k\le n} B_{T_k}$ is dominated by $\max_{0\le t\le T_n} B_t$. This yields (2.27).

The inequality (2.28) is known to be sharp as the equality is attained at the stopping time

$$\tau_* \;=\; \inf\,\{\, t>0 \;:\; S_t - B_t = a \,\} \tag{2.32}$$

for all $a\ge 0$, where $S_t=\max_{0\le r\le t} B_r$. Recalling that $(S_t-B_t)_{t\ge 0} \sim (|B_t|)_{t\ge 0}$, we see that τ_* is equally distributed as the stopping time

$$\sigma_* \;=\; \inf\,\{\, t>0 \;:\; |B_t| = a \,\} \; . \tag{2.33}$$

It may be interesting to observe that each T_k appearing above is equally distributed as $(\sum_{i=1}^k |a_i|^2)\,\sigma_*$ for all $1\le k\le n$.

We shall see in the next section that (2.27) can also be obtained by a much less sophisticated argument that relies upon a simple reflection of $S_1,\ldots,S_n$.

Similarly, from (1.4) and (1.5) when $p=2$ we see by Jensen's inequality that the following "reflected" analogue of (2.27) is valid:

$$E\Big(\max_{1\le k\le n} \Big| \sum_{i=1}^k a_i\varepsilon_i \Big| \Big) \;\le\; \sqrt{2}\, \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \; . \tag{2.34}$$

This inequality can be deduced from the following inequality (see [2]):

$$E\Big(\max_{0\le t\le\tau}|B_t|\Big) \;\le\; \sqrt{2}\sqrt{E(\tau)} \tag{2.35}$$

which is valid for all stopping times τ of the standard Brownian motion $(B_t)_{t\ge0}$. It can also be established by Skorokhod's embedding in exactly the same way as above.

The inequality (2.35) is known to be sharp as the equality is attained at the stopping time

$$\widetilde{\tau}_* \;=\; \inf\,\{\, t>0 \;:\; \widetilde{S}_t - B_t = a \,\} \tag{2.36}$$

for all $a\ge0$, where $\widetilde{S}_t = \max_{0\le r\le t}|B_r|$. The stopping time $\widetilde{\tau}_*$ is equally distributed as the convolution of two independent copies of σ_* from (2.33) above.

6. We conclude this section by pointing out a few easy facts for comparison with (1.4). Throughout we assume that $a_1,\dots,a_n\in I\!R$ and $p>0$ are given and fixed.

From (2.9) we see by Brownian scaling that

$$E\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k a_ig_i\Big|^p\Big) \le E\Big(\max_{0\le t\le t_n}|B_t|^p\Big) = E\Big(\max_{0\le t\le 1}|B_t|^p\Big)\Big(\sum_{i=1}^n|a_i|^2\Big)^{p/2}. \tag{2.37}$$

On the other hand, by a Gaussian property we have:

$$E\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k a_ig_i\Big|^p\Big) \ge E\Big(\Big|\sum_{i=1}^n a_ig_i\Big|^p\Big) \;=\; E|g_1|^p\Big(\sum_{i=1}^n|a_i|^2\Big)^{p/2}. \tag{2.38}$$

Taking (2.37) and (2.38) together, we obtain:

$$\begin{aligned}\big(E|B_1|^p\big)^{1/p}\Big(\sum_{i=1}^n|a_i|^2\Big)^{1/2} &\le \Big(E\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k a_ig_i\Big|^p\Big)\Big)^{1/p} \le \\ &\le \Big(E\Big(\max_{0\le t\le 1}|B_t|^p\Big)\Big)^{1/p}\Big(\sum_{i=1}^n|a_i|^2\Big)^{1/2}.\end{aligned} \tag{2.39}$$

Both constants appearing in (2.39) are best possible.

3. Single and multiple reflection arguments

Our main aim in this section is to enter into a deeper analysis of the problem (1.4) with (1.6) that is motivated by induction arguments given below.

1. We shall begin by recalling *Lévy inequalities* which are obtained by a single reflection argument. These considerations will be complemented later by a multiple reflection argument.

Lemma 3.1 (Single reflection)
Let $\xi_1,\dots,\xi_n$ be independent random variables that are symmetrically distributed around zero, and let $S_k=\sum_{i=1}^k\xi_i$ for all $1\le k\le n$ with $S_0=0$. Then for all $t\ge0$ we have:

$$P\Big\{\max_{0\le k\le n}S_k \ge t\Big\} \;\le\; 2\,P\big\{S_n\ge t\big\} - P\big\{S_n=t\big\} \tag{3.1}$$

$$P\Big\{ \max_{0\le k\le n} |S_k| \ge t \Big\} \le 2\,P\big\{|S_n|\ge t\big\} - P\big\{|S_n| = t\big\}\ . \tag{3.2}$$

Moreover, if $\xi_i = a\,\varepsilon_i$ *for all* $1\le i\le n$ *with some* $a\in I\!\!R$ *where* $\varepsilon_1,\dots,\varepsilon_n$ *are independent random variables taking values* ± 1 *with probability* $1/2$ *, then equality holds in (3.1) when* $t = ja$ *for some non-negative integer* j *.*

Proof. We begin by proving (3.1). By a single reflection argument we find:

$$\begin{aligned} P\Big\{ \max_{0\le k\le n} S_k \ge t \Big\} &= P\Big\{ \max_{0\le k\le n} S_k \ge t\ ,\ S_n \ge t \Big\} + P\Big\{ \max_{0\le k\le n} S_k \ge t\ ,\ S_n < t \Big\} \\ &= P\big\{S_n \ge t\big\} + P\Big\{ \max_{0\le k\le n} S_k \ge t\ ,\ S_n < t \Big\} \\ &\le P\big\{S_n \ge t\big\} + P\Big\{ \max_{0\le k\le n} S_k \ge t\ ,\ S_n > t \Big\} \\ &= P\big\{S_n \ge t\big\} + P\big\{S_n > t\big\} = 2\,P\big\{S_n \ge t\big\} - P\big\{S_n = t\big\}\ . \end{aligned} \tag{3.3}$$

Observe that the inequality follows since for the stopping time

$$\tau = \inf\ \big\{\ 0\le k\le n\ \big|\ S_k \ge t\ \big\} \tag{3.4}$$

the following identity in law holds:

$$(\xi_1,\dots,\xi_\tau,\xi_{\tau+1},\dots,\xi_n) \sim (\xi_1,\dots,\xi_\tau,-\xi_{\tau+1},\dots,-\xi_n) \tag{3.5}$$

with obvious extensions if τ equals either 0 or n .

Moreover, if the final hypothesis is fulfilled and $t = ja$ for some $j\ge 0$, then clearly $S_\tau = t$ and thus the reverse inequality in (3.3) holds too. This establishes the final claim about (3.1).

To prove (3.2) we can use (3.1). In this way for $t > 0$ we get:

$$\begin{aligned} P\Big\{ \max_{0\le k\le n} |S_k| \ge t \Big\} &\le P\Big\{ \max_{0\le k\le n} S_k \ge t \Big\} + P\Big\{ \min_{0\le k\le n} S_k \le -t \Big\} \\ &= 2\,P\Big\{ \max_{0\le k\le n} S_k \ge t \Big\} \le 2\Big(2P\big\{S_n\ge t\big\} - P\big\{S_n = t\big\}\Big) \\ &= 2\,P\big\{|S_n|\ge t\big\} - P\big\{|S_n| = t\big\} \end{aligned} \tag{3.6}$$

due to the fact that $(-S_1,\dots,-S_n) \sim (S_1,\dots,S_n)$ and $-S_n \sim S_n$. The proof is complete.

□

It should be observed in the proof above that we cannot repair the inequality in (3.6) under the final hypothesis of the lemma. This is exactly the place where a multiple reflection argument should enter in order to obtain a higher precision.

In accordance with this general remark we shall now show how the simple inequality (3.1) still can be used to find the best constant in a version of (1.4) where the absolute value is replaced by a maximal positive value. Below for notational convenience we denote $x\vee 0$ by x_+ .

Theorem 3.2

Let $\varepsilon_1,\dots,\varepsilon_n$ *be independent random variables taking values* ± 1 *with probability* $1/2$ *, and* *let* $a_1, \dots, a_n$ *be real numbers. Then the following inequality is satisfied:*

$$\Big(E\Big(\max_{1\le k\le n} \Big(\sum_{i=1}^k a_i \varepsilon_i \Big)_+^p \Big) \Big)^{1/p} \le \Big(E\Big(\max_{0\le t\le 1} B_t \Big)_+^p \Big)^{1/p} \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \tag{3.7}$$

for all $p \ge 2$ *. The constant appearing on the right-hand side is best possible.*

Proof. Set $S_k = \sum_{i=1}^k a_i \varepsilon_i$ for $1 \le k \le n$. Since $-S_n \sim S_n$ we have $2P\{S_n \ge t\} = P\{|S_n| \ge t\}$ for all $t \ge 0$, and upon integrating by parts in (3.1) , we get:

$$E\Big(\max_{1\le k\le n} S_k \Big)_+^p \le E|S_n|^p \le (\mathbf{B}_p)^p \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \tag{3.8}$$

where the second inequality follows by (1.1) with $\mathbf{B}_p$ in (1.2). Since $\max_{0\le t\le 1} B_t \sim |B_1|$, we see that $\mathbf{B}_p$ equals the constant appearing in (3.7), and thus (3.7) is established.

To show that the constant $\mathbf{B}_p$ is best possible in (3.7), take $a_1 = \ldots = a_n = \sqrt{1/n}$. Then by a functional central limit theorem (see e.g. [1]) it follows that

$$E\Big(\frac{1}{\sqrt{n}} \max_{1\le k\le n} \Big(\sum_{i=1}^k \varepsilon_i \Big)_+ \Big)^p \longrightarrow E\Big(\max_{0\le t\le 1} B_t \Big)^p \quad (n \to \infty) \tag{3.9}$$

and as $\max_{0\le t\le 1} B_t \sim |B_1|$, the right-hand side in (3.9) equals $E|B_1|^p = (\mathbf{B}_p)^p$. This shows that equality in (3.8) is attained in the limit, and thus $\mathbf{B}_p$ is best possible. □

Remark 3.3

Observe that the argument used in the proof above shows that

$$E\Big(\max_{1\le k\le n} S_k \Big)_+^p \le E|S_n|^p \tag{3.10}$$

for all $p > 0$, and that the constant 1 is best possible in this inequality.

In the case $0 < p < 2$ this yields by Jensen's inequality:

$$\Big(E\Big(\max_{1\le k\le n} S_k \Big)_+^p \Big)^{1/p} \le \mathbf{B}_p \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \tag{3.11}$$

with $\mathbf{B}_p = 1$. It is however unclear what the best value for $\mathbf{B}_p$ is in this case.

In view of a result presented in Corollary 3.8 below, the following fact seems interesting in this context. The choice of $a_1 = x$ and $a_2 = \ldots = a_{n+1} = 1/\sqrt{n}$ in (3.11) with $p=1$ does not disprove (in the limit) that the best value for $\mathbf{B}_1$ is smaller than or equal to $E|B_1| = \sqrt{2/\pi}$.

For this set $\widetilde{S}_k = \sum_{i=2}^{k+1} a_i \varepsilon_i$ for $1 \le k \le n$ with $\widetilde{S}_0 = 0$ and note that by Fubini's theorem and a functional central limit theorem (see e.g. [1]) we obtain:

$$\begin{aligned} E\Big(\max_{1\le k\le n} (x\varepsilon_1 + S_k) \Big)^+ &= \frac{1}{2} E\Big(\max_{1\le k\le n} (x + S_k) \Big)^+ + \frac{1}{2} E\Big(\max_{1\le k\le n} (-x + S_k) \Big)^+ \longrightarrow \\ \frac{1}{2} E\Big(\max_{0\le t\le 1} (x + B_t) \Big)^+ + \frac{1}{2} E\Big(\max_{0\le t\le 1} (-x + B_t) \Big)^+ &= \frac{1}{2} E\Big(x + |B_1| \Big)^+ + \frac{1}{2} E\Big(-x + |B_1| \Big)^+ \end{aligned} \tag{3.12}$$

as $n \to \infty$ since $\max_{0\le t\le 1} B_t \sim |B_1|$. Thus the problem reduces to show that

$$E\Big(x+|B_1|\Big)^+ + E\Big(-x+|B_1|\Big)^+ \le 2E|B_1|\sqrt{1+x^2} \tag{3.13}$$

fails for some $x \in I\!R$. By standard means, however, it is possible to verify that (3.11) is valid for all $x \in I\!R$, and this proves the claim.

2. We turn to multiple reflection arguments. Consider the problem (1.4)+(1.6) where $\varepsilon_1, \ldots, \varepsilon_n$ are independent (Rademacher) random variables taking values ± 1 with probability $1/2$, and $a_1, \ldots, a_n$ are real numbers. By Brownian scaling the problem can be rewritten as:

$$E\Big(\max_{1\le k\le n} |S_k|^p\Big) \le E\Big(\max_{0\le t\le t_n} |B_t|^p\Big) \tag{3.14}$$

where $S_k = \sum_{i=1}^k a_i \varepsilon_i$ for $1 \le k \le n$ and $t_n = \sum_{i=1}^n |a_i|^2$.

A natural attempt (suggested by I. Pinelis) to establish (3.14) would be by induction in n. It is then easily seen that this approach requires (3.14) to be extended to the following stronger form:

$$E\Big(z \vee \max_{0\le k\le n} |x+S_k|^p\Big) \le E\Big(z \vee \max_{0\le t\le t_n} |x+B_t|^p\Big) \tag{3.15}$$

where $z \ge x^p$ with $x \ge 0$ and we put $S_0 = 0$. Setting

$$f_p(z,x,T) = E\Big(z \vee \max_{0\le t\le T} |x+B_t|^p\Big) \tag{3.16}$$

we see by Fubini's theorem that for (3.15) it suffices to show:

$$f_p(z,x+a,T) + f_p(z,x-a,T) \le 2\, f_p(z,x,T+a^2) \tag{3.17}$$

for all $z \ge x^p$, $x > 0$, $a > 0$ and $T > 0$. (It is no restriction to assume $T=1$.)

In this way the initial problem (3.14) is completely translated to a problem about the function f_p in (3.16). We thus proceed by establishing a formula for this function that is convenient for computation. The main ingredient in the verification below is contained in the following well-known lemma on *multiple reflection* (see e.g. [4] pp. 286-288).

Lemma 3.4 (Multiple reflection)

Let $(B_t)_{0\le t\le T}$ be a standard Brownian motion, let $I =]a_-, a_+[$ for some $a_- < 0 < a_+$, let $I_0 \subset I$ be a Borel set, and let I_{k+1} be the reflection of I_k over $a_+ + k(a_+ - a_-)$ for $k \in \mathbf{Z}$. Then the following identity is satisfied:

$$P\big\{\, B_t \in I \text{ for all } t \in [0,T] \text{ and } B_T \in I_0 \,\big\} = \sum_{k=-\infty}^{+\infty} (-1)^k P\{B_T \in I_k\}\,. \tag{3.18}$$

Proof. We sketch the well-known argument. Consider the following events:

$$A_0 = \big\{\, (B_t)_{0\le t\le T} \text{ hits neither } a_- \text{ nor } a_+ \text{ and } B_T \in I_0 \,\big\} \tag{3.19}$$

$$C_n^\pm = \big\{\, (B_t)_{0\le t\le T} \text{ first hits } a_\pm \text{, then crosses } I \text{ at least } n \text{ times, and } B_T \in I_0 \,\big\} \tag{3.20}$$

for all $n \in \{0, 1, \ldots\}$, and observe that the left-hand side in (3.18) equals $P(A_0)$. Then (3.18) follows by induction upon using the following identities:

$$P(A_0) = P\{B_T \in I_0\} - P(C_0^+ \cup C_0^-) \tag{3.21}$$

$$P(C_n^+ \cup C_n^-) = P(C_n^+) + P(C_n^-) - P(C_n^+ \cap C_n^-) \quad (n \in \mathbf{N}_0) \tag{3.22}$$

$$C_n^+ \cap C_n^- = C_{n+1}^+ \cup C_{n+1}^- \quad (n \in \mathbf{N}_0) \tag{3.23}$$

$$P(C_n^\pm) = P\{B_T \in I_{\pm(n+1)}\} \tag{3.24}$$

where (3.24) follows by the reflection property of Brownian motion. The proof is complete.

□

In the next proposition we shall make use of the following general fact on alternating series. If $(f_k)_{k \geq 1}$ is a decreasing sequence of non-negative μ-measurable functions converging to zero as $k \to \infty$ and satisfying $\int_X f_1 \, d\mu < \infty$, then $\int (\sum_{k=1}^\infty (-1)^{k-1} f_k) \, d\mu = \sum_{k=1}^\infty (-1)^{k-1} \int f_k \, d\mu$. This can be easily obtained by the Leibnitz theorem on alternating series which states that, if $(c_k)_{k \geq 1}$ is a decreasing sequence of non-negative real numbers converging to zero, then $\sum (-1)^{k-1} c_k$ is convergent. In particular, the preceding general fact shows that the sum and expectation sign in (3.25) below can be interchanged under the hypotheses of the proposition.

Proposition 3.5

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion, let $g : I\!R_+ \to I\!R_+$ be an increasing function that is continuous at zero, and let $T > 0$ be given and fixed. Then we have:

$$E\Big(g\Big(z \vee \max_{0 \leq t \leq T} |x + B_t| \Big) \Big) = g(z) + 2 \sum_{k=1}^\infty (-1)^{k-1} E\left(g\Big(\frac{|x + B_T|}{2k-1} \Big) - g(z) \right)^+ \tag{3.25}$$

for all $z \geq x$ and all $x \geq 0$ such that $E\big(g(|x + B_T|) \big) < \infty$.

Proof. Set $M_{x,T} = \max_{0 \leq t \leq T} |x + B_t|$. Then the left-hand side in (3.25) equals

$$E\Big(g(z) \vee g(M_{x,T}) \Big) = g(z) + \int_{g(z)}^\infty P\{ \, g(M_{x,T}) \geq u \, \} \; du \tag{3.26}$$

and the right-hand side reads

$$g(z) + 2 \sum_{k=1}^\infty (-1)^{k-1} \int_{g(z)}^\infty P\Big\{ \, g\Big(\frac{|x + B_T|}{2k-1} \Big) \geq u \, \Big\} \; du \tag{3.27}$$

where the series and integral sign can be interchanged (due to the remark stated above).

Moreover, since g is increasing we see that $g^{-1}([u, \infty[)$ equals either $[v, \infty[$ or $]v, \infty[$ for some v . Thus, it is sufficient to prove that

$$P\{ \, M_{x,T} \geq v \, \} = 2 \sum_{k=1}^\infty (-1)^{k-1} P\Big\{ \, \frac{|x + B_T|}{2k-1} \geq v \, \Big\} \tag{3.28}$$

for all $v > x$.

To prove (3.28) we shall use Lemma 3.4 with $I_0 = I = \,]-v-x, v-x[$. This yields:

$$(3.29) \qquad P\{\, M_{x,T} \geq v \,\} = 1 - \sum_{k\in\mathbf{Z}} (-1)^k P\{B_T \in I_k\} = 2 \sum_{k\in\mathbf{Z}} P\{B_T \in I_{2k-1}\}$$

due to the fact that $I_k = \,](2k-1)v-x\,,\,(2k+1)v-x[$ forms a partition of $I\!R$ (up to a countable set that B_T hits with probability zero) when $k \in \mathbf{Z}$.

Introduce sets $D_k = \,]-\infty\,,-(2k-1)v-x] \cup [(2k-1)v-x\,,+\infty[$ for $k \geq 1$, and note that the right-hand side in (3.28) equals

$$(3.30) \qquad 2\sum_{k=1}^{\infty} (-1)^{k-1} P\Big\{ \frac{|x+B_T|}{2k-1} \geq v \Big\} = 2\sum_{k=1}^{\infty} (-1)^{k-1} P\{B_T \in D_k\} \ .$$

As it is easily seen that

$$(3.31) \qquad \sum_{k\in\mathbf{Z}} 1_{I_{2k-1}} = \sum_{k\in\mathbf{N}} (-1)^{k-1}\, 1_{D_k}$$

(on the complement of a countable set that B_T hits with probability zero), the identity (3.28) follows by (3.29) and (3.30), and the proof is complete. □

Corollary 3.6

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, and let $T>0$ be given and fixed. Then:

$$(3.32) \qquad E\Big(z \vee \max_{0\leq t\leq T} |x+B_t|^p \Big) = z + 2\sum_{k=1}^{\infty} (-1)^{k-1} E\bigg(\Big(\frac{|x+B_T|}{2k-1} \Big)^p - z \bigg)^+$$

for all $z \geq x^p$, $x \geq 0$ and $p>0$. In particular, the following identities are valid:

$$(3.33) \qquad E\Big(\max_{0\leq t\leq T} |B_t|^p \Big) = 2\,\beta_p\, E|B_T|^p = 2\,\beta_p \sqrt{\frac{2^p}{\pi}}\; \Gamma\Big(\frac{p+1}{2} \Big)\; T^{p/2}$$

for all $p>0$, where the constant β_p is given by

$$(3.34) \qquad \beta_p = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^p} \ .$$

Proof. It follows directly from (3.25) upon Brownian scaling and a well-known formula for $E|B_1|^p$ when $p>0$. This completes the proof. □

It can be verified that

$$(3.35) \qquad \beta_1 = \frac{\pi}{4} \quad , \quad \beta_2 = \text{Catalan's constant} = 0.91\ldots \quad , \quad \beta_3 = \frac{\pi^3}{32} \quad \text{etc.}$$

$$(3.36) \qquad \lim_{p\downarrow 0}\, (2\beta_p) = 1 \quad \text{and} \quad \lim_{p\uparrow\infty}\, (2\beta_p) = 2$$

for all $p>0$. The function $p \mapsto \beta_p$ is known to be related to the Riemann zeta function.

Remark 3.7

It is possible to compute the expectation in (3.33) by a different method which uses known properties of the stopping time:

$$\tau_1 = \inf\{\ t>0 : |B_t| = 1\ \}\ . \tag{3.37}$$

For this first note that Brownian scaling easily implies:

$$\max_{0\le t\le 1} |B_t| \sim \frac{1}{\sqrt{\tau_1}}\ . \tag{3.38}$$

Using further that $(e^{\sigma B_t - \sigma^2 t/2})_{t\ge 0}$ is a martingale for each $\sigma>0$, by the optional sampling theorem we obtain the well-known identity:

$$E\Big(e^{-\lambda\tau_1}\Big) = \frac{1}{\cosh(\sqrt{2\lambda})} \tag{3.39}$$

for all $\lambda>0$. Recalling the well-known formula:

$$\int_0^\infty \lambda^{q-1} e^{-\lambda t}\, d\lambda = \frac{\Gamma(q)}{t^q} \quad (q>0\ ,\ t>0) \tag{3.40}$$

we find by (3.38) and (3.39) the following integral formula:

$$E\Big(\max_{0\le t\le 1} |B_t|^p\Big) = E\Big((\tau_1)^{-p/2}\Big) = \frac{1}{\Gamma(p/2)} \int_0^\infty \frac{\lambda^{(p/2)-1}}{\cosh(\sqrt{2\lambda})}\, d\lambda \tag{3.41}$$

for the right-hand side of (3.33) when $p>0$.

The preceding method cannot be applied to compute the expectation in (3.32) as the scaling property breaks down in this case.

3. The formula (3.32) offers an explicit expression for the function f_p in (3.16) and in this way provides a tool to attack the problem (3.17). It turns out however that the inequality (3.17) fails in many cases we considered, and we were not able to find $p>0$ for which it would be satisfied. Nonetheless, the formula (3.32) can be used to disprove the conjecture (1.6) in the case $p=1$. (One should observe that this fact is equivalent to the fact that (3.17) fails when $z=x=0$ and $p=T=1$.) The following fact (with proof) was observed by I. Pinelis.

Corollary 3.8
The best constant $\mathbf{B}_1^*$ *in the inequality (1.4) is strictly larger than* $E(\max_{0\le t\le 1}|B_t|) = \sqrt{\pi/2}$.

Proof. In the setting of (1.4) take $a_1 = x$ and $a_2 = \ldots = a_{n+1} = 1/\sqrt{n}$, apply Fubini's theorem, and then use a functional central limit theorem (see e.g. [1]) upon letting $n\to\infty$ just like in (3.9) above. In this way (1.4) with $\mathbf{B}_1^*$ from (1.6) implies:

$$E\Big(\max_{0\le t\le 1} |x+B_t|\Big) \le E\Big(\max_{0\le t\le 1} |B_t|\Big)\sqrt{1+x^2}\ . \tag{3.42}$$

Thus, if we can show that (3.42) fails for some x, the proof will be complete.

To do so recall that (3.32) states that for all $x\ge 0$ we have:

$$E\Big(\max_{0\le t\le T} |x+B_t|\Big) = x + 2\sum_{k=1}^\infty (-1)^{k-1} E\Big(\frac{|x+B_T|}{2k-1} - x\Big)^+ \tag{3.43}$$

and note that the expectation term in the series above is decreasing in k . Thus the partial sums fluctuate up and down from the limiting value. Therefore to disprove (3.42) it is enough to show that

$$x + 2\Big(E\Big(|x+B_1|-x\Big)^+ - E\Big(\frac{|x+B_1|}{3}-x\Big)^+\Big) > \sqrt{\frac{\pi}{2}}\,\sqrt{1+x^2} \tag{3.44}$$

for some $x>0$, where we recall from (3.33) that $E(\max_{0\le t\le 1}|B_t|) = \sqrt{\pi/2}$.

For this, we note that

$$E\Big(|x+B_1|-y\Big)^+ = \int_y^\infty \Big(\Phi(-t+x) + \Phi(-t-x)\Big)\,dt := G(x,y) \tag{3.45}$$

for all x , $y \in I\!R$, where $\Phi(z) = (1/\sqrt{2\pi})\int_{-\infty}^z e^{-t^2/2}dt$ is the distribution function of B_1 . Thus, the problem reduces to show that

$$L(x) := x + 2\,G(x,x) - \frac{2}{3}\,G(x,3x) > \sqrt{\frac{\pi}{2}}\,\sqrt{1+x^2} =: R(x) \tag{3.46}$$

for some $x>0$. Numerical calculations show that $G(1,1) = 0.407\ldots$ and $G(1,3) = 0.008\ldots$ Hence $L(1) = 1.80\ldots$ and $R(1) = 1.77\ldots$ showing that (3.46) holds when $x=1$. Thus (3.42) fails for $x=1$, and the proof is complete. □

The preceding considerations naturally raise the question as to determine those $p>0$ for which the following inequality is valid:

$$E\Big(\max_{0\le t\le 1}|x+B_t|^p\Big) \le \big(1+x^2\big)^{p/2}\,E\Big(\max_{0\le t\le 1}|B_t|^p\Big) \tag{3.47}$$

for all $x \in I\!R$. The proof above shows that this inequality fails for $p=1$, and similar calculations suggest that (3.47) (and possibly the conjecture (1.6) itself) fails for $0<p<2$ and holds for $p\ge 2$. We make no attempt here to prove this formally but leave it worthy of further consideration.

Observe by Brownian scaling that (3.47) is equivalent to

$$E\Big(\max_{0\le t\le T}|x+B_t|^p\Big) \le \big(T+x^2\big)^{p/2}\,E\Big(\max_{0\le t\le 1}|B_t|^p\Big) \tag{3.47'}$$

where $x \in I\!R$ and $T>0$. Note also that the inequality (3.47) is an immediate consequence of the conjecture (1.6) if taking $a_1 = x$ and $a_2 = \ldots = a_{n+1} = 1/\sqrt{n}$ and letting $n \to \infty$ as in the proof above.

4. The following consequence of Lemma 3.4 was communicated to me by I. Pinelis. Although it might appear somewhat unexpected, it should be noted that this identity in law does not extend to the stochastic processes. This fact can be verified e.g. by considering hitting times.

Proposition 3.9

Let $(B_t)_{0\le t\le T}$ be a standard Brownian motion, and let $x \in I\!R$ be given and fixed. Then the following identity in law holds:

$$\Big(\max_{0\le t\le T}|x+B_t|\Big) \sim \Big(\max_{0\le t\le T}|B_t|\Big) \vee \Big(|x|+|B_T|\Big)\,. \tag{3.48}$$

Proof. By Brownian scaling and the fact that $(-B_t)_{t\geq 0}$ is a standard Brownian motion, there is no restriction to assume that $T=1$ and $x>0$.

Let $u > x$ be given and fixed. Then by (3.18) we find:

$$P\Big\{ \max_{0\leq t\leq 1} |x+B_t| < u \Big\} = P\big\{ -u-x < B_t < u-x \,,\ \forall t \in [0,1] \big\} = \sum_{k\in\mathbf{Z}} (-1)^k P\{B_1 \in I_k\} \tag{3.49}$$

where $I_k = \,](2k-1)u-x \,,\, (2k+1)u-x[$ for $k \in \mathbf{Z}$.

On the other hand, by (3.18) we also find:

$$P\Big\{ \max_{0\leq t\leq 1} |B_t| \vee (x+|B_1|) < u \Big\} = P\big\{ -u < B_t < u \,,\ \forall t \in [0,1] \ \&\ -u+x < B_1 < u-x \big\} = \sum_{k\in\mathbf{Z}} (-1)^k P\{B_1 \in J_k\} \tag{3.50}$$

where $J_k = \,](2k-1)u+x \,,\, (2k+1)u-x[$ for $k \in \mathbf{Z}$.

Now note that $I_k = J_k \cup L_k$ where $L_k =](2k-1)u-x \,,\, (2k-1)u+x]$ for all $k \in \mathbf{Z}$. Thus, from (3.49) and (3.50) we see that it is enough to show that

$$\sum_{k\in\mathbf{Z}} (-1)^k P\{B_1 \in L_k\} = 0 \ . \tag{3.51}$$

For this, note that for all $k \geq 1$ we have:

$$P\{B_1 \in L_k\} = \Phi\Big((2k-1)u+x\Big) - \Phi\Big((2k-1)u-x\Big) \tag{3.52}$$

$$P\{B_1 \in L_{-(k-1)}\} = \Phi\Big((-2k+1)u+x\Big) - \Phi\Big((-2k+1)u-x\Big) \tag{3.53}$$

where Φ is the distribution function of a standard normal random variable. Since $\Phi(t)+\Phi(-t) = 1$ for all $t \in I\!\!R$, we see that

$$P\{B_1 \in L_k\} = P\{B_1 \in L_{-(k-1)}\} \tag{3.54}$$

for all $k \geq 1$. Hence we find that (3.51) holds, and the proof is complete.

$\square$

5. If instead of Brownian motion in Lemma 3.4 one deals with a random walk, then the analogue of the multiple reflection formula (3.18) becomes messy. This is not the case if the random walk has equidistant steps. We demonstrate this fact in the proof of the following result.

Proposition 3.10

The inequality (1.4) holds with B_p^* *from (1.6) for* $p \geq 2$ *if all* $a_1, \ldots, a_n$ *are equal.*

Proof. Denote $M_n = \max_{1\leq k\leq n} |S_k|$ where $S_k = \sum_{i=1}^k \varepsilon_i$ for $1 \leq k \leq n$. Then in exactly the same way as in the proof of (3.18) by the reflection property we find:

(3.55) $$P\{M_n < m\} = \sum_{k\in\mathbf{Z}} (-1)^k P\{S_n \in I_k\} = \sum_{k\in\mathbf{Z}} (-1)^k P\{\, |S_n - 2km| < m \,\}$$

for all $m \in \mathbf{N}$, where $I_k = \,](2k-1)m, (2k+1)m[$.

From the preceding formula we obtain:

(3.56) $$\begin{aligned} P\{M_n \geq m\} &= 2\sum_{k\in\mathbf{Z}} P\{S_n \in I_{2k-1}\} + \sum_{k\in\mathbf{Z}} P\{S_n = 2k-1\} \\ &= 2\sum_{k\in\mathbf{N}} P\{|S_n| \in I_{2k-1}\} + \sum_{k\in\mathbf{N}} P\{|S_n| = 2k-1\} := G(|S_n|) \end{aligned}$$

where the map G satisfies:

(3.57) $$\begin{aligned} G(s) &= 2\sum_{k\in\mathbf{N}} 1_{I_{2k-1}}(s) + \sum_{k\in\mathbf{N}} 1_{\{2k-1\}}(s) \\ &= \sum_{k\in\mathbf{N}} (-1)^{(k-1)} \Big(1_{[(2k-1)m,\infty[}(s) + 1_{[(2k-1)m+1,\infty[}(s) \Big) \ . \end{aligned}$$

From (3.56) and (3.57) we get:

(3.58) $$\begin{aligned} P\{M_n \geq m\} &= \sum_{k\in\mathbf{N}} (-1)^{(k-1)} \Big(P\big\{|S_n| \geq (2k-1)m\big\} + P\big\{|S_n| \geq (2k-1)m + 1\big\} \Big) \\ &= \sum_{k\in\mathbf{N}} (-1)^{(k-1)} \Big(P\big\{ \big[|S_n|/(2k-1)\big] \geq m \big\} + P\big\{ \big[(|S_n|-1)/(2k-1)\big] \geq m \big\} \Big) \end{aligned}$$

where $[x]$ denotes the integer part of x . By linearity we can conclude:

(3.59) $$Eg(M_n) = EH_g(|S_n|)$$

for all functions $g : I\!\!R_+ \mapsto I\!\!R_+$ such that $g(0) = 0$, where the map H_g is given by

(3.60) $$H_g(s) = \sum_{k\in\mathbf{N}} (-1)^{(k-1)} \bigg(g\Big(\Big[\frac{s}{2k-1} \Big] \Big) + g\Big(\Big[\frac{s-1}{2k-1} \Big] \Big) \bigg)$$

for all $s \geq 0$.

Taking $g(x) = x^p$ for $x \geq 0$, and letting H_p denote H_g , it is possible to verify that

(3.61) $$H_p(s) \leq 2\, \beta_p\, s^p$$

for all $p \geq 2$ where β_p is given by (3.34). From (3.59)-(3.61) we find:

(3.62) $$E\Big(\max_{1\leq k\leq n} \Big| \sum_{i=1}^{k} \varepsilon_i \Big|^p \Big) \leq 2\,\beta_p\, E|S_n|^p \leq 2\,\beta_p\, E|B_1|^p \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{p/2} = (\mathbf{B}_p^*)^p \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{p/2}$$

for all $p \geq 2$ by means of (1.1)+(1.2) and (3.33). The proof is complete.

□

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Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk