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Limit at Zero of the Brownian First-Passage Density

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Let $(B_t)_{t \geq 0}$ be a standard Brownian motion started at zero, let $g : I\!R_+ \to I\!R$ be an upper function for B satisfying $g(0) = 0$, and let

$$\tau = \inf \{ \, t > 0 \mid B_t \geq g(t) \, \}$$

be the first-passage time of B over g. Assume that g is C^1 on $\langle 0, \infty \rangle$, increasing (locally at zero), and concave (locally at zero). Then the following identities hold for the density function f of τ:

$$f(0+) = \lim_{t \downarrow 0} \frac{1}{2} \frac{g(t)}{t^{3/2}} \varphi\left(\frac{g(t)}{\sqrt{t}}\right) = \lim_{t \downarrow 0} \frac{g'(t)}{\sqrt{t}} \varphi\left(\frac{g(t)}{\sqrt{t}}\right)$$

in the sense that if the second and third limit exist so does the first one and the equalities are valid (here $\varphi(x) = (1/\sqrt{2\pi})\, e^{-x^2/2}$ is the standard normal density). These limits can take any value in $[0, \infty]$. The method of proof relies upon the strong Markov property of B and makes use of real analysis.

1. Introduction

The result presented below was motivated by the question of A. Shiryaev** (see [2] and [1]) if one can find a continuous function $g : I\!R_+ \to I\!R$ such that the first-passage time of a standard Brownian motion $(B_t)_{t \geq 0}$ over g given by

$$\tau = \inf \{ \, t > 0 \mid B_t \geq g(t) \, \} \tag{1.1}$$

is exponentially distributed (i.e. has a density function $f(t) = \lambda e^{-\lambda t}$ with $t > 0$ and $\lambda > 0$). This problem can be viewed as an *inverse* to the problem of finding a density function f of τ when g is given.

Explicit solutions to the latter problem are known only in a limited number of special cases including linear or quadratic g. The law of τ is also known for a square-root boundary g but only in the form of a Laplace transform (which appears intractable to inversion).

The inverse problem seems even harder. While it is relatively easy to use a comparison argument and rule out those g satisfying $g(0+) > 0$ (since in this case $f(0+) = 0$), it is much less obvious to see if there is a continuous function g at all for which $f(0+)$ is strictly positive and finite. The knowledge of $f(0+)$ in terms of g, on the other hand, is of interest in various numerical methods found in the literature for computing f when g is known.

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**This question was formulated during a Banach centre meeting in 1976.

Motivated by these facts we state simple conditions on g under which the existence of the limit $f(0+)$ can be proved and the limit itself can be identified in terms of g (Theorem 2.1). A useful consequence of this identification is expressed through the boundaries:

$$g_0(t) = \sqrt{2\,t \log 1/t} \quad \text{and} \quad g_\varepsilon(t) = \sqrt{(2+\varepsilon)\,t \log 1/t} \quad (\varepsilon > 0) \tag{1.2}$$

which are found to separate those g implying $f(0+) = 0$ from those g implying $f(0+) = +\infty$ (for more details see Corollary 2.2 below). Another consequence is obtained by disclosing those g for which $f(0+)$ is strictly positive and finite (Example 2.3).

We base our proof on the following *master equation*:

$$\Psi\Big(\frac{z}{\sqrt{t}}\Big) = \int_0^t \Psi\Big(\frac{z-g(s)}{\sqrt{t-s}}\Big) f(s)\, ds \quad \big(z \ge g(t)\big) \tag{1.3}$$

which expresses the strong Markov property of the process $((t, B_t))_{t\ge 0}$ at time τ (and Ψ is given in (2.1) below). The initial idea in the derivation of this equation goes back to Schrödinger [10], and many other authors have studied equations of this type in connection with the first-passage problem (see e.g. [12, 5, 11, 7, 4, 9, 3]). A unifying approach to the integral equations arising in the first-passage problem is given in [8].

2. The result and proof

In the notation of the previous section recall that Blumenthal's 0-1 law implies that $P(\tau > 0)$ is either 0 or 1. A continuous function $g : \mathbb{R}_+ \to \mathbb{R}$ is said to be *an upper function* for B if $P(\tau > 0) = 1$ (otherwise g is said to be a lower function for B). Observe that each upper function g must satisfy $g(0) \ge 0$. If $g(0) > 0$ then $f(0+)$ can only be 0 (see Corollary 2.2 and Proposition 2.4 below). In the following theorem we treat the more difficult case when $g(0) = 0$.

The following notation will be used throughout:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \;,\quad \Phi(x) = \int_{-\infty}^{x} \varphi(z)\, dz \;,\quad \Psi(x) = 1 - \Phi(x) \tag{2.1}$$

for $x \in \mathbb{R}$. The main result of the paper may now be stated as follows.

Theorem 2.1

Let $(B_t)_{t\ge 0}$ be a standard Brownian motion started at zero, let $g : \mathbb{R}_+ \to \mathbb{R}$ be an upper function for B satisfying $g(0) = 0$, and let τ be the first-passage time of B over g given by (1.1).

Assume that g is C^1 on $\langle 0, \infty \rangle$, increasing (locally at zero), and concave (locally at zero). Then the following identities hold for the density function f of τ:

$$f(0+) = \lim_{t\downarrow 0} \frac{1}{2}\, \frac{g(t)}{t^{3/2}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \lim_{t\downarrow 0} \frac{g'(t)}{\sqrt{t}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) \tag{2.2}$$

in the sense that if the second and third limit exist so does the first one and the equalities are valid. These limits can take any value in $[0, \infty]$ (see Corollary 2.2 and Example 2.3 below).

Proof. 1. Fix $t > 0$ and $z \geq g(t)$. By the strong Markov property of the process $((t, B_t))_{t \geq 0}$ at τ , and the fact that g is continuous, we find that

$$P_0(B_t \geq z) = \int_0^t P_{g(s)}(B_{t-s} \geq z)\, F(ds) \tag{2.3}$$

where $B_0 = x$ under P_x and F is the distribution function of τ . By the scaling property of B we can rewrite (2.3) as follows:

$$\Psi\Big(\frac{z}{\sqrt{t}}\Big) = \int_0^t \Psi\Big(\frac{z-g(s)}{\sqrt{t-s}}\Big)\, F(ds)\ . \tag{2.4}$$

Differentiating (2.4) with respect to $z > g(t)$, and letting $z \downarrow g(t)$, we obtain:

$$\frac{1}{\sqrt{t}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \int_0^t \frac{1}{\sqrt{t-s}}\, \varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\, F(ds) \tag{2.5}$$

by standard means. Inserting $z = g(t)$ in (2.4) we get:

$$\Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \int_0^t \Psi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\, F(ds)\ . \tag{2.6}$$

Differentiating (2.6) with respect to t yields:

$$\frac{\partial}{\partial t}\, \Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \frac{1}{2}\, f(t) + \int_0^t \frac{\partial}{\partial t}\, \Psi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\, f(s)\; ds \tag{2.7}$$

where $f = F'$ is a continuous density of τ . This step requires some more precision, but since the existence of a continuous density f of τ for g which is C^1 on $\big\langle 0, \infty \big\rangle$ has been established in [13], we shall omit the details and refer instead to [8] for a simple proof.

More explicitly (2.7) can be rewritten as follows:

$$\Big(\frac{1}{2}\frac{g(t)}{t^{3/2}} - \frac{g'(t)}{\sqrt{t}}\Big)\varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \frac{1}{2}\, f(t) + \int_0^t \Big(\frac{1}{2}\frac{g(t)-g(s)}{(t-s)^{3/2}} - \frac{g'(t)}{\sqrt{t-s}}\Big)\varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) f(s)\; ds\ . \tag{2.8}$$

Recognizing now the identity (2.5) multiplied by $g'(t)$ within (2.8), and multiplying the remaining part of the identity (2.8) by 2 , we get:

$$\frac{g(t)}{t^{3/2}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = f(t) + \int_0^t \frac{g(t)-g(s)}{(t-s)^{3/2}}\, \varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) f(s)\; ds\ . \tag{2.9}$$

The same argument shows that the factor $1/2$ can be removed from (2.8) yielding:

$$\Big(\frac{g(t)}{t^{3/2}} - \frac{g'(t)}{\sqrt{t}}\Big)\varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = f(t) + \int_0^t \Big(\frac{g(t)-g(s)}{(t-s)^{3/2}} - \frac{g'(t)}{\sqrt{t-s}}\Big)\varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) f(s)\; ds\ . \tag{2.10}$$

This ends the first part of the proof.

2. Introduce the following notation:

$$\lambda = f(0+) = \lim_{t \downarrow 0} f(t) \tag{2.11}$$

$$R = \lim_{t \downarrow 0} \frac{g(t)}{t^{3/2}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) \tag{2.12}$$

$$L = \lim_{t \downarrow 0} \frac{g'(t)}{\sqrt{t}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big)\ . \tag{2.13}$$

Let λ^* denote the limsup of $f(t)$ as $t \downarrow 0$, and let λ_* denote the liminf of $f(t)$ as $t \downarrow 0$. We first show that the existence of the limits R and L implies that

$$\frac{R}{2} = L\ . \tag{2.14}$$

For this, first consider the case when $R < \infty$, and recall the classic estimate:

$$\Psi(x) \leq \frac{1}{2} e^{-x^2/2} \tag{2.15}$$

for all $x > 0$. This implies:

$$\frac{1}{t}\, \Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) \leq \frac{\sqrt{2\pi}}{2}\, \frac{g(t)}{t^{3/2}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big)\, \frac{\sqrt{t}}{g(t)} \leq \frac{\sqrt{2\pi}}{2}\, (R+\varepsilon)\, \frac{\sqrt{t}}{g(t)} \tag{2.16}$$

for all $0 < t < t_\varepsilon$ with $\varepsilon > 0$ given and fixed. Since g is an upper function for B , and $\Psi(x) \leq 1/2$ for all $x \in I\!R_+$, it follows from (2.6) that $\Psi(g(t)/\sqrt{t}) \to 0$ as $t \downarrow 0$, that is:

$$\frac{g(t)}{\sqrt{t}} \to +\infty \tag{2.17}$$

as $t \downarrow 0$. Passing to the limit in (2.16) using (2.17) we see that

$$\frac{1}{t}\, \Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) \to 0 \tag{2.18}$$

as $t \downarrow 0$. On the other hand, noting that

$$\frac{\partial}{\partial t}\, \Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \Big(\frac{1}{2}\, \frac{g(t)}{t^{3/2}} - \frac{g'(t)}{\sqrt{t}}\Big)\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) \tag{2.19}$$

contains the expressions appearing in R and L above, we see that the limit of the expression in (2.19) exists for $t \downarrow 0$. But then, in view of (2.18), a simple application of the L'Hospital's rule shows that this limit must be zero, proving that (2.14) holds in this case.

Now consider the case when $R = \infty$, and assume that $L < \infty$. Then as in (2.16) we find:

$$\frac{1}{t}\, \Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) \leq \frac{\sqrt{2\pi}}{2}\, \frac{g'(t)}{\sqrt{t}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big)\, \frac{1}{\sqrt{t}\, g'(t)} \leq \frac{\sqrt{2\pi}}{2}\, (L+\varepsilon)\, \frac{1}{\sqrt{t}\, g'(t)} \tag{2.20}$$

for all $0 < t < t_\varepsilon$ with $\varepsilon > 0$ given and fixed. Since g is an upper function for B , and $t \mapsto c\sqrt{t}$ is a lower function for B whenever $c > 0$, it follows easily that

$$\limsup_{t\downarrow 0} \sqrt{t}\, g'(t) = +\infty \ . \tag{2.21}$$

Passing to the limit in (2.20) using (2.21) we see that

$$\liminf_{t\downarrow 0} \frac{1}{t}\, \Psi\left(\frac{g(t)}{\sqrt{t}}\right) = 0 \ . \tag{2.22}$$

On the other hand, we see by the hypotheses that the limit of the expression in (2.19) equals $+\infty$ when $t \downarrow 0$, and thus by the L'Hospital's rule we find that

$$\lim_{t\downarrow 0} \frac{1}{t}\, \Psi\left(\frac{g(t)}{\sqrt{t}}\right) = +\infty \ . \tag{2.23}$$

This, however, is a contradiction with (2.22). Thus, we must have $L = \infty$, and the identity (2.14) holds in this case as well.

3. To prove (2.2) first consider the case when R and L are finite. Since g is concave, we note for further reference that

$$g'(t) \le \frac{g(t)-g(s)}{(t-s)} \le g'(s) \tag{2.24}$$

for all $0 < s < t$.

Using (2.24) in (2.10) we see that the integral appearing there is non-negative, so that by passing to the limit in (2.10) for $t \downarrow 0$, we obtain:

$$\lambda^* \le R-L \ . \tag{2.25}$$

Using (2.24) in (2.5), and substituting $u = g'(t)\sqrt{t-s}$, we find:

$$\begin{aligned} \frac{1}{\sqrt{t}}\, \varphi\left(\frac{g(t)}{\sqrt{t}}\right) &\le \int_0^t \frac{1}{\sqrt{t-s}}\, \varphi\Big(g'(t)\sqrt{t-s}\Big)\, f(s)\; ds \\ &\le (\lambda^* + \varepsilon) \int_0^t \frac{1}{\sqrt{t-s}}\, \varphi\Big(g'(t)\sqrt{t-s}\Big)\; ds \\ &\le \frac{2\,(\lambda^* + \varepsilon)}{g'(t)} \int_0^{\sqrt{t}\, g'(t)} \varphi(u)\, du \le \frac{(\lambda^* + \varepsilon)}{g'(t)} \end{aligned} \tag{2.26}$$

for all $0 < t < t_\varepsilon$ with $\varepsilon > 0$ given and fixed. Multiplying (2.26) by $g'(t)$, passing to the limit for $t \downarrow 0$, and then letting $\varepsilon \downarrow 0$, we obtain:

$$L \le \lambda^* \ . \tag{2.27}$$

From (2.27), (2.25) and (2.14) we see that

$$\lambda^* = \frac{R}{2} = L \ . \tag{2.28}$$

Thus, to complete the proof it is enough to show that the limit $f(0+)$ exists, i.e. that $\lambda_* = \lambda^*$.

For this, we first derive the following inequality (as a refinement of (2.26) above):

$$\int_0^t \frac{g(t)-g(s)}{(t-s)^{3/2}}\, \varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\, ds \le 1 \tag{2.29}$$

for all $t > 0$ (locally at zero). This follows by substituting $u = (g(t)-g(s))/\sqrt{t-s}$ so that:

$$\begin{aligned}\int_0^t \frac{g(t)-g(s)}{(t-s)^{3/2}}\, \varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\, ds &= \int_0^{g(t)/\sqrt{t}} \frac{\frac{g(t)-g(s)}{(t-s)^{3/2}}}{\frac{g'(s)}{\sqrt{t-s}} - \frac{1}{2}\frac{g(t)-g(s)}{(t-s)^{3/2}}}\, \varphi(u)\, du \\ &= \int_0^{g(t)/\sqrt{t}} \frac{\frac{g(t)-g(s)}{(t-s)}}{g'(s) - \frac{1}{2}\frac{g(t)-g(s)}{(t-s)}}\, \varphi(u)\, du \\ &\le 2\int_0^{g(t)/\sqrt{t}} \varphi(u)\, du \le 1\end{aligned} \tag{2.30}$$

where in the second last inequality we use (2.24). Passing to the limit in (2.9) for $t \downarrow 0$, upon recalling that $\liminf\,(x_n + y_n) \le \liminf\,(x_n) + \limsup\,(y_n)$ and using (2.29), we easily find:

$$R \le \lambda_* + \lambda^* \; . \tag{2.31}$$

This together with (2.28) implies that

$$\frac{R}{2} \le \lambda_* \tag{2.32}$$

and also shows that $\lambda_* = \lambda^*$. Thus, the limit $f(0+)$ exists and (2.2) holds when $R < \infty$.

When R and L are not finite, then the arguments leading to (2.27) imply that λ^* must be infinite. Moreover, a comparison argument can be used to show that λ_* is infinite too, however, these details will be omitted. The proof of the theorem is complete. □

Before we state a corollary to the theorem, we first make a few comments on the hypotheses used in the theorem.

1. Under the hypotheses of Theorem 2.1 stated prior to (2.2), suppose that $R = 0$. Then (2.9) implies directly that $f(0+) = 0$ no matter if g is concave or not. Thus, if $R = 0$ then g need not be concave (nor the second limit in (2.2) has to be considered). This shows that the main focus of Theorem 2.1 is on the limits $f(0+)$ that are strictly positive and finite. In Proposition 2.4 below we will see that the hypotheses on g can be further relaxed when $g(0+)$ is strictly positive.

2. The equation (2.6) holds for any continuous function $g : I\!R_+ \rightarrow I\!R$ satisfying $g(0) \ge 0$, so that whenever the limit $f(0+)$ exists (and is finite), the following explicit formula holds:

$$f(0+) = \lim_{t\downarrow 0} \frac{\Psi\Big(\frac{g(t)}{\sqrt{t}}\Big)}{\int_0^t \Psi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\, ds} \tag{2.33}$$

with no additional assumptions on g. While this fact may appear attractive since the right-hand side of (2.33) is expressed solely in terms of g, it is difficult to see how this limit is computed (or just found to exist). A similar remark can be made about the equation (2.5) yielding yet another explicit formula which we omit.

3. Under the hypotheses of Theorem 2.1 stated prior to (2.2), let R^*, R_* and L^*, L_* be defined analogously to λ^*, λ_* following (2.11)-(2.13) above, and suppose that the limit $f(0+)$ exists (and is finite). Then the arguments used in the proof above show:

$$L^* \le \frac{R^*}{2} \le f(0+) \le (R^* - L^*) \wedge (R_* - L_*) \ . \tag{2.34}$$

Moreover, whenever the mapping:

$$t \mapsto \Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) \quad \text{is } C^1 \text{ at } 0 \tag{2.35}$$

or in other words, the limit of the expression in (2.19) exists as $t \downarrow 0$, we see from (2.18) by means of the L'Hospital's rule that this limit must be zero. Hence we easily find:

$$\frac{R_*}{2} \le L_* \ . \tag{2.36}$$

This fact combined with (2.34) above shows that $R_* = R^*$, $L_* = L^*$ and $R/2 = L = f(0+)$. Hence we see that the equality $R/2 = L$ is not artificially imposed through the assumptions of the theorem but appears naturally.

4. Under the hypotheses of Theorem 2.1 stated prior to (2.2), assume that either the second limit $R/2$ or the third limit L exists in $\langle 0, \infty \rangle$. Then a necessary and sufficient condition for the existence of the other limit is that

$$\lim_{t \downarrow 0} \frac{g(t)}{2\, t\, g'(t)} = 1 \ . \tag{2.37}$$

This condition is useful in applications (cf. (2.42) below).

Corollary 2.2

Let $(B_t)_{t \ge 0}$ be a standard Brownian motion started at zero, let $g : I\!R_+ \rightarrow I\!R$ be a continuous function satisfying $g(0) \ge 0$, and let us assume that the first-passage time τ from (1.1) has a continuous density function f having a limit $f(0+)$ in $[0, \infty]$.

If there are $\varepsilon > 0$ and $\delta > 0$ such that

$$g(t) \ge \sqrt{(2+\varepsilon)\, t \log(1/t)} \tag{2.38}$$

for all $t \in \langle 0, \delta \rangle$, then $f(0+) = 0$. If there is $\delta > 0$ such that

$$g(t) \le \sqrt{2t \log(1/t)} \tag{2.39}$$

for all $t \in \langle 0, \delta \rangle$, then $f(0+) = +\infty$.

Proof. For the boundaries $t \mapsto g_\varepsilon(t)$ from (1.2) it is easily verified that

(2.40)
$$\lim_{t\downarrow 0} \frac{1}{2}\, \frac{g_\varepsilon(t)}{t^{3/2}}\, \varphi\Big(\frac{g_\varepsilon(t)}{\sqrt{t}}\Big) = \lim_{t\downarrow 0} \frac{g'_\varepsilon(t)}{\sqrt{t}}\, \varphi\Big(\frac{g_\varepsilon(t)}{\sqrt{t}}\Big) = 0 \;\text{ if }\; \varepsilon > 0$$
$$= +\infty \;\text{ if }\; \varepsilon = 0 \;.$$

Thus by Theorem 2.1 the density function f_ε of $\tau_\varepsilon = \inf\{\, t>0 \mid B_t \geq g_\varepsilon(t)\,\}$ satisfies:

(2.41)
$$f_\varepsilon(0+) = 0 \;\text{ if }\; \varepsilon > 0$$
$$= +\infty \;\text{ if }\; \varepsilon = 0 \;.$$

The two claims therefore follow from the following *comparison principle for first-passage densities*.

Given a continuous function $h_i : I\!R_+ \rightarrow I\!R$ satisfying $h_i(0) \geq 0$, assume that the first-passage time $\sigma_i = \inf\{\, t>0 \mid B_t \geq h_i(t)\,\}$ has a density function d_i having a limit $d_i(0+)$ in $[0,\infty]$ for $i = 1,2$. If $h_1(t) \leq h_2(t)$ for all $t \in \langle 0, \delta\rangle$ with some $\delta > 0$, then $d_1(0+) \geq d_2(0+)$.

To see this, note that h_1 being smaller than h_2 implies that $P(\sigma_2 \leq t) \leq P(\sigma_1 \leq t)$ and thus $P(\sigma_2 \leq t)/t \leq P(\sigma_1 \leq t)/t$ for all $t \in \langle 0, \delta\rangle$. Passing to the limit as $t \downarrow 0$ and using that the limits $d_1(0+)$ and $d_2(0+)$ exist, it follows that $d_2(0+) = D'_2(0+) \leq D'_1(0+) = d_1(0+)$, where D_i is the distribution function of σ_i for $i = 1,2$, and the proof is complete. □

Example 2.3

The limit $f(0+)$ in Theorem 2.1 can be strictly positive and finite. For this, define:

(2.42)
$$g(t) = \sqrt{2\,t \log 1/t + t \log\log 1/t + c\, t}$$

for $t \in \langle 0, \delta_c\rangle$ with $c \in I\!R$ given and fixed. Then a direct computation shows:

(2.43)
$$\lim_{t\downarrow 0} \frac{1}{2}\, \frac{g(t)}{t^{3/2}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \lim_{t\downarrow 0} \frac{g'(t)}{\sqrt{t}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \frac{e^{-c/2}}{\sqrt{4\pi}}$$

where the existence of the second limit can be easier proved by checking the condition (2.37). It thus follows by Theorem 2.1 that the limit $f(0+)$ exists and is given by

(2.44)
$$f(0+) = \frac{e^{-c/2}}{\sqrt{4\pi}} \;.$$

Observe that g locally at zero lies between g_0 and g_ε from (1.2) for any $\varepsilon > 0$.

Sufficient conditions for the existence of $f(0+)$ given in Theorem 2.1 (and the first remark following its proof) can be further relaxed when $g(0+)$ is strictly positive. The arguments used in the proof below can be extended to treat more complicated g as well.

Proposition 2.4

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g : \langle 0,\infty\rangle \rightarrow I\!R$ be a continuous function satisfying $g(0+) \in \langle 0,\infty]$, and let us assume that the first-passage time τ from (1.1) has a continuous density function f . If g is either increasing (locally at zero) or decreasing (locally at zero), then the limit $f(0+)$ exists and equals zero.

Proof. The case when g is increasing follows from (2.9) as stated in the first remark following the proof of Theorem 2.1. To prove the claim when g is decreasing, we will make use of the following *comparison principle for first-passage densities* (that can be viewed as an extension of the comparison principle used in the proof of Corollary 2.2 above).

Given a continuous function $h_i : I\!R_+ \rightarrow I\!R$ satisfying $h_i(0) \geq 0$, assume that the first-passage time $\sigma_i = \inf\{\, t > 0 \mid B_t \geq h_i(t)\,\}$ has a continuous density function d_i for $i = 1, 2$. If $h_1(s) \geq h_2(s)$ for all $s \in \langle 0, t \rangle$ and $h_1(s) \leq h_2(s)$ for all $s \in \langle t, t+\delta \rangle$ with some $t > 0$ and $\delta > 0$ given and fixed, then $d_1(t) \geq d_2(t)$.

To see this, note that the hypotheses imply that $P(t < \sigma_2 \leq t+h) \leq P(t < \sigma_1 \leq t+h)$ and thus $(D_2(t+h) - D_2(t))/h \leq (D_1(t+h) - D_1(t))/h$ for all $h \in \langle 0, \delta \rangle$ where D_i is the distribution function of σ_i for $i = 1, 2$. Passing to the limit when $h \downarrow 0$ we obtain $d_2(t) = D_2'(t) \leq D_1'(t) = d_1(t)$ as claimed.

Consider first the case when $0 < g(0+) < \infty$. Fix any $\alpha > 0$ and consider the first-passage time σ_{α,β_n} defined in (3.4) below, where for a given sequence $t_n \downarrow 0$ we choose $\beta_n \uparrow g(0)$ so that $g(t_n) = \alpha t_n + \beta_n$ and $g(t) > \alpha t + \beta_n$ for all $t \in \langle 0, t_n \rangle$. Note that such β_n exist since g is decreasing.

From the definition of β_n, the principle just stated above, and (3.5) below, we find:

$$f(t_n) \leq \frac{\beta_n}{t_n^{3/2}}\, \varphi\Big(\frac{\alpha t_n + \beta_n}{\sqrt{t_n}} \Big) \tag{2.45}$$

for all $n \geq 1$. Letting $n \rightarrow \infty$ and using that $\beta_n \rightarrow g(0) \in \langle 0, \infty \rangle$, it follows that $f(t_n) \rightarrow 0$. Since the sequence $(t_n)_{n \geq 1}$ was arbitrary, we may conclude that $f(0+) = 0$.

Consider now the case when $g(0+) = +\infty$. Fix any $\beta > 0$ and consider the first-passage time $\sigma_{\alpha_n,\beta}$ defined in (3.4) below, where for a given sequence $t_n \downarrow 0$ we choose $\alpha_n = (g(t_n) - \beta)/t_n$ so that $g(t_n) = \alpha_n t_n + \beta$ and $g(t) > \alpha_n t + \beta$ for all $t \in \langle 0, t_n \rangle$. Note that such α_n exist since g is decreasing.

From the definition of α_n, the principle just stated above, and (3.5) below, we find:

$$f(t_n) \leq \frac{\beta}{t_n^{3/2}}\, \varphi\Big(\frac{\alpha_n t_n + \beta}{\sqrt{t_n}} \Big) = \frac{\beta}{t_n^{3/2}}\, \varphi\Big(\frac{g(t_n)}{\sqrt{t_n}} \Big) \leq \frac{\beta}{t_n^{3/2}}\, \varphi\Big(\frac{g(t_1)}{\sqrt{t_n}} \Big) \tag{2.46}$$

for all $n \geq 1$. Letting $n \rightarrow \infty$ it follows that $f(t_n) \rightarrow 0$. Since the sequence $(t_n)_{n \geq 1}$ was arbitrary, we may conclude that $f(0+) = 0$. The proof is complete. □

3. Remarks on lower and upper bounds

We conclude this paper with a few remarks on lower and upper bounds for $P(\tau \leq t)$ where τ is the first-passage time of B over g given in (1.1) and $t > 0$ is given and fixed. It will be assumed throughout that $g : [0, t] \rightarrow I\!R$ is a continuous function satisfying $g(0) \geq 0$, but further hypotheses on g will also be introduced.

1. If g is C^1 on $\langle 0, t \rangle$ and increasing, then (2.6) and (2.9) in the proof above show that the following inequalities hold (using the second identity in (3.6) below):

$$\int_0^t \frac{g(t)}{s^{3/2}}\,\varphi\Big(\frac{g(t)}{\sqrt{s}}\Big)\,ds \le P\big(\tau \le t\big) \le \int_0^t \frac{g(s)}{s^{3/2}}\,\varphi\Big(\frac{g(s)}{\sqrt{s}}\Big)\,ds\ . \tag{3.1}$$

We now examine these inequalities in some more detail beginning with the second one.

2. Recall that *Kolmogorov's test* states that if g is continuous, increasing, and

$$s \mapsto \frac{g(s)}{\sqrt{s}} \text{ is decreasing} \tag{3.2}$$

then g is an upper function for B if and only if

$$\int_{0+}^t \frac{g(s)}{s^{3/2}}\,\varphi\Big(\frac{g(s)}{\sqrt{s}}\Big)\,ds < \infty\ . \tag{3.3}$$

A more careful inspection of the proof of the if-part of Kolmogorov's test given in [6, page 34] shows that the second inequality in (3.1) extends to the case when g is continuous, increasing and satisfies (2.17) above. Thus, for example, this inequality holds for each increasing upper function g for B. [Recall from the proof above that each upper function g for B must satisfy (2.17).] It shows, in particular, that the condition (3.2) in the if-part of Kolmogorov's test can be replaced by the condition (2.17), so that the if-part reads as follows: *If g is a continuous increasing function satisfying $g(s)/\sqrt{s} \to \infty$ as $s \downarrow 0$, then g is an upper function for B whenever (3.3) holds.* Note that this statement holds independently from the fact that (3.2) is verified or not.

3. For the first inequality in (3.1) consider the first-passage time of B over a linear boundary:

$$\sigma_{\alpha,\beta} = \inf\{\, s>0 \mid B_s \ge \alpha s+\beta \,\} \tag{3.4}$$

where $\alpha \in I\!R$ and $\beta > 0$. The density function $d_{\alpha,\beta}$ of $\sigma_{\alpha,\beta}$ is known to be given by

$$d_{\alpha,\beta}(s) = \frac{\beta}{s^{3/2}}\,\varphi\Big(\frac{\alpha s+\beta}{\sqrt{s}}\Big) \tag{3.5}$$

for $s > 0$. Choosing $\alpha = 0$ and $\beta = g(t)$, we note that

$$P\big(\sigma_{0,g(t)} \le t\big) = \int_0^t \frac{g(t)}{s^{3/2}}\,\varphi\Big(\frac{g(t)}{\sqrt{s}}\Big)\,ds = 2\,\Psi\Big(\frac{g(t)}{\sqrt{t}}\Big) \tag{3.6}$$

so that the first inequality in (3.1) can be rewritten as follows:

$$P\big(\sigma_{0,g(t)} \le t\big) \le P\big(\tau \le t\big)\ . \tag{3.7}$$

This fact is probabilistically obvious since g is increasing. It is moreover clear that if such a function g is concave, then the lower bound in (3.7) can be improved as follows:

$$P\big(\sigma_{\alpha(t),\beta(t)} \le t\big) \le P\big(\tau \le t\big) \tag{3.8}$$

where $\alpha(t) = g'(t)$ and $\beta(t) = g(t) - t\,g'(t)$ are chosen so that $s \mapsto \alpha(t)\,s + \beta(t)$ is a tangent to the curve $s \mapsto g(s)$ at the point $(t, g(t))$. The inequality (3.8) more explicitly reads as follows:

$$(3.9) \qquad \int_0^t \frac{g(t)-tg'(t)}{s^{3/2}}\,\varphi\Big(\frac{g(t)-(t-s)g'(t)}{\sqrt{s}}\Big)\,ds \le P\big(\tau \le t\big)$$

while (2.10) yields the following upper bound:

$$(3.10) \qquad P\big(\tau \le t\big) \le \int_0^t \left(\frac{g(s)}{s^{3/2}} - \frac{g'(s)}{\sqrt{s}}\right)\varphi\Big(\frac{g(s)}{\sqrt{s}}\Big)\,ds$$

when g is concave. The two inequalities (3.9) and (3.10) are better than the two inequalities in (3.1) when g is increasing and concave.

4. It is interesting to note that (2.6) holds for continuous g which are not necessarily increasing. Thus it follows from (2.6) (using the second identity in (3.6) above) that

$$(3.11) \qquad \frac{1}{2}\int_0^t \frac{g(t)}{s^{3/2}}\,\varphi\Big(\frac{g(t)}{\sqrt{s}}\Big)\,ds \le P\big(\tau \le t\big)$$

no matter if g is increasing or not. In probabilistic terms of the first-passage time (3.4), upon recalling (3.6), this inequality reads:

$$(3.12) \qquad P\big(\sigma_{0,g(t)} \le t\big) \le 2\,P\big(\tau \le t\big)$$

and may be seen as a simple consequence of *the reflection principle* (cf. [6, page 26]).

REFERENCES

[1] ANULOVA, S. V. (1980). On Markov stopping times with a given distribution for a Wiener process. *Theory Probab. Appl.* 25 (362-366).

[2] DUDLEY, R. M. *and* GUTMANN, S. (1977). Stopping times with given laws. *Sém. de Probab.* XI (Strasbourg 1975/76), *Lecture Notes in Math.* 581 (51-58).

[3] DURBIN, J. (1985). The first-passage density of a continuous Gaussian process to a general boundary. *J. Appl. Probab.* 22 (99-122).

[4] FEREBEE, B. (1982). The tangent approximation to one-sided Brownian exit densities. *Z. Wahrsch. Verw. Gebiete* 61 (309-326).

[5] FORTET, R. (1943). Les fonctions aléatoires du type Markoff associées à certaines équations linéaires aux dérivées partielles du type parabolique. *J. Math. Pures Appl. (9)* 22 (177-243).

[6] ITÔ, K. *and* MCKEAN, H. P. Jr. (1965). *Diffusion Processes and Their Sample Paths.* Reprint by Springer-Verlag 1996.

[7] PARK, C. *and* SCHUURMANN, F. J. (1976). Evaluations of barrier-crossing probabilities of Wiener paths. *J. Appl. Probab.* 13 (267-275).

[8] PESKIR, G. (2001). On integral equations arising in the first-passage problem for Brownian motion. *Research Report* No. 421, *Dept. Theoret. Statist. Aarhus* (20 pp). *Proc. Funct. Anal.* VII (Dubrovnik 2001), *Various Publ. Ser. Aarhus* 46, 2002 (159-175).

[9] RICCIARDI, L. M., SACERDOTE, L. *and* SATO, S. (1984). On an integral equation for first-

passage-time probability densities. *J. Appl. Probab.* 21 (302-314).

[10] SCHRÖDINGER, E. (1915). Zur Theorie der Fall- und Steigversuche an Teilchen mit Brownscher Bewegung. *Physik. Zeitschr.* 16 (289-295).

[11] SIEGERT, A. J. F. (1951). On the first passage time probability problem. *Phys. Rev. (2)* 81 (617-623).

[12] SMOLUCHOWSKI, M. v. (1915). Notiz über die Berechnung der Brownschen Molekularbewegung bei der Ehrenhaft-Millikanschen Versuchsanordnung. *Physik. Zeitschr.* 16 (318-321).

[13] STRASSEN, V. (1967). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* (Berkeley 1965/66) Vol II, Part 1, *Univ. California Press* (315-343).

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