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On Reflecting Brownian Motion with Drift

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Let $B = (B_t)_{t \ge 0}$ be a standard Brownian motion started at zero, and let $\mu \in \mathbb{R}$ be a given and fixed constant. Set $B_t^\mu = B_t + \mu t$ and $S_t^\mu = \max_{0 \le s \le t} B_s^\mu$ for $t \ge 0$. Then the process:

$$(x \vee S^\mu) - B^\mu = ((x \vee S_t^\mu) - B_t^\mu)_{t \ge 0}$$

realises an explicit construction of the reflecting Brownian motion with drift $-\mu$ started at x in $\mathbb{R}_+$. Moreover, if the latter process is denoted by $Z^x = (Z_t^x)_{t \ge 0}$, then the classic Lévy's theorem extends as follows:

$$\big((x \vee S^\mu) - B^\mu, (x \vee S^\mu) - x\big) \overset{\text{law}}{=} \big(Z^x, \ell^0(Z^x)\big)$$

where $\ell^0(Z^x)$ is the local time of Z^x at 0 . The Markovian argument for $(x \vee S^\mu) - B^\mu$ remains valid for any other process with stationary independent increments in place of B^μ . This naturally leads to a class of Markov processes which are referred to as reflecting Lévy processes. A point of view which both unifies and complements various approaches to these processes is provided by the extended Skorohod lemma.

1. Introduction

A successful treatment of optimal stopping problems (when solutions are not available in closed form) often requires that the underlying Markov process be expressed in terms of the *initial point* as explicitly as possible. The simplest example of this type is a standard Brownian motion $B = (B_t)_{t \ge 0}$ starting at 0 under P . Setting $B_t^x = x + B_t$ for $t \ge 0$ and $x \in \mathbb{R}$ one obtains a stochastic process $B^x = (B_t^x)_{t \ge 0}$ starting at x under P . Moreover, letting P_x denote $\text{Law}(B^x \,|\, \mathsf{P})$ on the canonical space $(C, \mathcal{B}(C))$ it follows that the coordinate process $c = (c_t)_{t \ge 0}$ defined by $c_t(\omega) = \omega(t)$ for $\omega \in C$ and $t \ge 0$ is a standard Brownian motion starting at x under P_x . A similar argument applies to a geometric Brownian motion $S = (S_t)_{t \ge 0}$ solving $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$ where one sets $S_t^x = x \exp(\sigma B_t + (\mu - \sigma^2/2)\, t)$ to obtain an explicit construction of the process in terms of the initial point x from $(0, \infty)$. Many other similar examples can be given to illustrate the same argument. Very often, however, such an explicit construction is not possible. In this case, moreover, any deeper treatment of the optimal stopping problem is much harder (if at all available).

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The present note is motivated by one example of this type which appeared in our treatment of the optimal prediction problem [1]. In this context one automatically arrives at the underlying Markov process $X = S^\mu - B^\mu$ where $B^\mu_t = B_t + \mu t$ and $S^\mu_t = \max_{0 \le s \le t} B^\mu_s$ for $t \ge 0$ and $\mu \in \mathbb{R}$. Recalling the result of [9] (see also [19, Exc. 2.19, p. 388]) one knows that:

$$X \overset{\text{law}}{=} |Y^0| \tag{1.1}$$

where $Y^x = (Y^x_t)_{t \ge 0}$ is the unique strong solution of the stochastic differential equation:

$$dY^x_t = -\mu \, \mathrm{sign}(Y^x_t) \, dt + dB_t \tag{1.2}$$

with $Y^x_0 = x$. Moreover, it was shown in [9] that $|Y^x|$ realises an explicit construction of the reflecting Brownian motion with drift $-\mu$ started at x in $\mathbb{R}_+$. A natural way to proceed in the context of our example was therefore to identify X^x with $|Y^x|$. The difficulty we faced at once, however, was that it is impossible to solve the equation (1.2) explicitly so to determine the actual dependence on x. The problem consequently appeared to be intractable.

It turns out, however, that there is another way to express X in terms of x so to preserve the Markov property. The process X^x so defined is given by:

$$X^x_t = (x \vee S^\mu_t) - B^\mu_t \tag{1.3}$$

for $t \ge 0$. To see that the Markovian structure is preserved note that:

$$\begin{aligned} X^x_{t+h} &= (x \vee S^\mu_{t+h}) - B^\mu_{t+h} \\ &= \Big(x \vee S^\mu_t \vee \max_{t \le s \le t+h} B^\mu_s \Big) - (B^\mu_{t+h} - B^\mu_t) - B^\mu_t \\ &= \Big((x \vee S^\mu_t) - B^\mu_t \Big) \vee \Big(\max_{t \le s \le t+h} (B^\mu_s - B^\mu_t) \Big) - (B^\mu_{t+h} - B^\mu_t) \\ &= (X^x_t \vee \widetilde{S}^\mu_h) - \widetilde{B}^\mu_h \end{aligned} \tag{1.4}$$

where $\widetilde{S}^\mu_h$ and $\widetilde{B}^\mu_h$ are independent of $\mathcal{F}^X_t$ and equally distributed as S^μ_h and B^μ_h respectively (upon using that B^μ has stationary independent increments). From this representation it is evident that X^x is a Markov process under P making $P_x = \mathrm{Law}(X^x \,|\, \mathsf{P})$ for $x \ge 0$ a family of probability measures on the canonical space $(C_+, \mathcal{B}(C_+))$ under which the coordinate process $c = (c_t)_{t \ge 0}$ is Markov with $P_x(X_0 = x) = 1$.

This simple fact being unveiled it comes with no surprise that the process X^x realises an explicit construction of the reflecting Brownian motion with drift $-\mu$ started at x in $\mathbb{R}_+$. This is formally verified in the proof of Theorem 2.1 below. The Itô-Tanaka calculus and Skorohod's lemma then naturally lead to an extension of classic Lévy's theorem (given in Theorem 3.1 below) where the process can start at arbitrary points in the state space $\mathbb{R}_+$. In the case when the initial point is zero this extension was derived in [9] using the Girsanov theorem and invoking Lévy's original theorem for Brownian motion with no drift (see also Section 4 in [9] for connections with [15] and [8]). Since the only two properties used in (1.4) to verify the Markov property are embodied in stationary and independent increments, we are naturally led (in Section 4) to discuss a class of Markov processes which are referred to as *reflecting Lévy processes*. A point of view which both unifies and complements various approaches to these processes is provided by the extended Skorohod lemma (cf. [7, Lemma 1]).

2. Reflecting Brownian motion with drift

1. Recall that the reflecting Brownian motion with drift $\nu \in \mathbb{R}$ started at x in $\mathbb{R}_+$ is a diffusion (strong Markov) process (with continuous sample paths) associated with the infinitesimal operator $\mathbb{L}^\nu$ acting on:

$$\mathcal{D}(\mathbb{L}^\nu) = \{\, f \in C_b^2(\mathbb{R}_+) \mid f'(0+) = 0 \,\} \tag{2.1}$$

according to the following formula:

$$\mathbb{L}^\nu f = \nu f' + \tfrac{1}{2} f'' \tag{2.2}$$

for $f \in \mathcal{D}(\mathbb{L}^\nu)$.

It is well known (cf. [12, Chap. 4, Sect. 5–7]) that the operator $(\mathbb{L}^\nu, \mathcal{D}(\mathbb{L}^\nu))$ generates a unique family of diffusion (strongly Markovian) measures $\{\, P_x \mid x \in \mathbb{R}_+ \,\}$ on the canonical space $(C_+, \mathcal{B}(C_+))$ with $P_x(c_0 = x) = 1$ such that:

$$f(c_t) - f(c_0) - \int_0^t (\mathbb{L}^\nu f)(c_s)\, ds \tag{2.3}$$

is a martingale under P_x for every $f \in \mathcal{D}(\mathbb{L}^\nu)$ and every $x \in \mathbb{R}_+$. (Throughout $C_+ = C([0,\infty), \mathbb{R}_+)$ denotes the family of continuous functions from the time set $[0,\infty)$ into the state space $\mathbb{R}_+$, the symbol $\mathcal{B}(C_+)$ denotes the Borel σ-algebra on C_+ , i.e. the smallest σ-algebra containing all Borel cylinder subsets of C_+ , and $c = (c_t)_{t \ge 0}$ denotes the coordinate process on $(C_+, \mathcal{B}(C_+))$ given by $c_t(\omega) = \omega(t)$ for $\omega \in C_+$ and $t \ge 0$.)

We let $\mathrm{RBM}^x(\nu)$ denote the unique law of the coordinate process $c = (c_t)_{t \ge 0}$ on the canonical space $(C_+, \mathcal{B}(C_+))$ under P_x for $x \in \mathbb{R}_+$ and $\nu \in \mathbb{R}$. Any other (continuous) process $Z^x(\nu)$ defined on (some) probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathrm{Law}(Z^x(\nu) \mid \mathsf{P}) = \mathrm{RBM}^x(\nu)$ for all $x \in \mathbb{R}_+$ with $\nu \in \mathbb{R}$ given and fixed is thus one realisation of the reflecting Brownian motion with drift ν started at x in $\mathbb{R}_+$. We will now see how one such realisation can be constructed explicitly.

2. Let $B = (B_t)_{t \ge 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that $B_0 = 0$ under P . Set $B_t^\mu = B_t + \mu t$ and $S_t^\mu = \max_{0 \le s \le t} B_s^\mu$ for $t \ge 0$ with $\mu \in \mathbb{R}$ given and fixed. Consider the process $X^x = (x \vee S^\mu) - B^\mu$ defined by:

$$X_t^x = (x \vee S_t^\mu) - B_t^\mu \tag{2.4}$$

for $t \ge 0$ and $x \in \mathbb{R}_+$. To indicate the dependence of X^x on μ we will write $X^x(\mu)$ instead of X^x when needed. The following theorem states that $X^x(\mu)$ is a reflecting Brownian motion with drift $-\mu$ started at x in $\mathbb{R}_+$.

Theorem 2.1. *The following identity in law holds:*

$$X^x(\mu) \overset{law}{=} RBM^x(-\mu) \tag{2.5}$$

for every $x \in \mathbb{R}_+$ and every $\mu \in \mathbb{R}$.

Proof. We apply the well-known method of proof. According to Theorem 5.2 in [12, p. 207] (upon invoking the basic transformation theorem for integrals with respect to image measures) it is sufficient to show that:

$$f(X_t^x) - f(X_0^x) - \int_0^t (\mathbb{L}^{-\mu} f)(X_s)\, ds \tag{2.6}$$

is a martingale under P for every $f \in \mathcal{D}(\mathbb{L}^{-\mu})$ and every $x \in \mathbb{R}_+$.

To this end, let $f \in \mathcal{D}(\mathbb{L}^{-\mu})$ and $x \in \mathbb{R}_+$ be given and fixed. Noting that X^x is a continuous semimartingale by Itô's formula we get:

$$\begin{aligned} f(X_t^x) &= f(X_0^x) + \int_0^t f'(X_s^x)\, dX_s^x + \frac{1}{2}\int_0^t f''(X_s^x)\, d\langle X^x, X^x\rangle_s \\ &= f(X_0^x) + \int_0^t f'(X_s^x)\, d(x \vee S_s^\mu) - \int_0^t f'(X_s^x)\, dB_s^\mu + \frac{1}{2}\int_0^t f''(X_s^x)\, d\langle X^x, X^x\rangle_s \\ &= f(X_0^x) + \int_0^t (-\mu f' + \tfrac{1}{2} f'')(X_s^x)\, ds - \int_0^t f'(X_s^x)\, dB_s \end{aligned} \tag{2.7}$$

since $d(x \vee S_s^\mu)$ is zero off the set of all s at which $X_s^x \neq 0$, while $f'(X_s^x) = 0$ for $X_s^x = 0$ so that $\int_0^t f'(X_s^x)\, d(x \vee S_s^\mu) \equiv 0$ for $t \geq 0$. Note also that $d\langle X^x, X^x\rangle_s = ds$ since $s \mapsto (x \vee S_s^\mu)$ is increasing and thus of bounded variation.

Since $M_t = \int_0^t f'(X_s^x)\, dB_s$ is a martingale under P for $t \geq 0$ (due to the fact that f' is bounded) we see that (2.6) holds as claimed and the proof is complete. □

3. In exactly the same way (using the Itô-Tanaka formula) it can be verified (cf. [9, Theorem 2]) that the process $|Y^x|$ stated following (1.2) above is a reflecting Brownian motion with drift $-\mu$ started at x in $\mathbb{R}_+$.

3. Extended Lévy's theorem

The classic Lévy's theorem (cf. [19, p. 240]) extends as follows. In the sequel we adopt the setting and notation introduced in Sections 1 and 2 above. Recall that $Z^x = Z^x(-\mu)$ denotes a reflecting Brownian motion with drift $-\mu$ started at x in $\mathbb{R}_+$.

Theorem 3.1. *The following identity in law holds:*

$$\big((x \vee S^\mu) - B^\mu, (x \vee S^\mu) - x\big) \overset{law}{=} \big(Z^x, \ell^0(Z^x)\big) \tag{3.1}$$

for every $x \in \mathbb{R}_+$ *and every* $\mu \in \mathbb{R}$.

Proof. We apply the well-known method of proof. Let $x \in \mathbb{R}_+$ and $\mu \in \mathbb{R}$ be given and fixed. As indicated following (1.2) above there is no loss of generality to identify Z^x with $|Y^x|$ where Y^x solves (1.2) with $Y_0^x = x$.

By the Itô-Tanaka formula we then have:

$$|Y_t^x| = x + \int_0^t \operatorname{sign}(Y_s^x)\, dY_s^x + \ell_t^0(Y^x) \tag{3.2}$$

$$= x - \mu t + \int_0^t \text{sign}(Y_s^x)\, dB_s + \ell_t^0(Y^x) = \beta_t^{x,-\mu} + \ell_t^0(Y^x)$$

where the final identity constitutes a definition of $\beta_t^{x,-\mu}$ for $t \ge 0$.

From the properties of functions $t \mapsto |Y_t^x|$, $t \mapsto \beta_t^{x,-\mu}$ and $t \mapsto \ell_t^0(Y^x)$ we see that Skorohod's lemma (cf. [19, p. 239] or (4.3)-(4.7) below) can be applied. This yields:

$$\begin{aligned}(3.3)\qquad \ell_t^0(Y^x) &= \sup_{0\le s\le t}\big(-\beta_s^{x,-\mu} \vee 0\big) = \sup_{0\le s\le t}\Big(\big(-x+\mu s+\widetilde{B}_t(x)\big)\vee 0\Big)\\ &= \Big(\sup_{0\le s\le t}\big(\widetilde{B}_s(x)+\mu s\big)\vee x\Big) - x\end{aligned}$$

where $(\widetilde{B}_s(x))_{s\ge 0} = (-\int_0^s \text{sign}(Y_r^x)\, dB_r)_{s\ge 0}$ is a standard Brownian motion (started at zero) by Lévy's characterisation theorem (cf. [19, p. 150]).

Inserting (3.3) into (3.2) and using that $Z_t^x = |Y_t^x|$ and $\ell_t^0(Z^x) = \ell_t^0(Y^x)$ one gets:

$$\begin{aligned}(3.4)\qquad \big(Z_t^x, \ell_t^0(Z^x)\big) = \Bigg(&\Big(\sup_{0\le s\le t}\big(\widetilde{B}_s(x)+\mu s\big)\vee x\Big) - \big(\widetilde{B}_t(x)+\mu t\big),\\ &\Big(\sup_{0\le s\le t}\big(\widetilde{B}_s(x)+\mu s\big)\vee x\Big) - x\Bigg)\end{aligned}$$

proving (3.1) as claimed. □

4. Reflecting Lévy processes

The Markovian argument for $(x \vee S^\mu) - B^\mu$ given in (1.4) above remains valid for any other process with stationary independent increments in place of the process B^μ. This naturally leads to a class of Markov processes described as follows.

1. Let $L = (L_t)_{t\ge 0}$ be a Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that $L_0 = 0$ under P. Set $\widetilde{L}_t = -L_t$ and $\widetilde{S}_t = \sup_{0\le s\le t} \widetilde{L}_s$ for $t \ge 0$. Consider the process $X^x = (x \vee \widetilde{S}) - \widetilde{L}$ defined by:

$$(4.1)\qquad X_t^x = (x \vee \widetilde{S}_t) - \widetilde{L}_t = (x + L_t) \vee \max_{0\le s\le t}(L_t - L_s)$$

for $t \ge 0$ and $x \in \mathbb{R}_+$.

In exactly the same way as in (1.4) (upon invoking the basic transformation theorem for integrals with respect to image measures) it follows that $P_x = \text{Law}(X^x\,|\,\mathsf{P})$ for $x \ge 0$ is a family of probability measures on the canonical space $(D_+, \mathcal{B}(D_+))$ under which the coordinate process $c = (c_t)_{t\ge 0}$ is Markov with $P_x(c_0 = x) = 1$. We let RLP^x denote the law of $c = (c_t)_{t\ge 0}$ on $(D_+, \mathcal{B}(D_+))$ obtained in this way for $x \ge 0$.

Definition 4.1. A Markov process $Z^x = (Z_t^x)_{t\ge 0}$ satisfying:

$$(4.2)\qquad \text{Law}(Z^x) = \text{RLP}^x$$

for all $x \in \mathbb{R}_+$ is referred to as a *reflecting Lévy process.*

Reflecting Lévy processes (4.1) have been intensively studied in the last fifty or so years by a number of authors (see [3], [2], [6], [13] and the reference therein). An early prominent paper on *quality control* where these processes (in discrete time) were considered is [16]. More recent applications, among others, include *optimal prediction* problems (cf. [10], [17], [1]).

2. Despite the fact that there are various ways which lead to these processes (either theoretically or practically) a simple unifying view can be obtained using the original Skorohod's path-by-path approach [18] to the reflection problem as follows.

Given a continuous process $L^x = (L^x_t)_{t \ge 0}$ with $L^x_0 = x \ge 0$ consider the Skorohod equation:

$$X_t = L^x_t + Y_t \tag{4.3}$$

for $t \ge 0$. Then Skorohod's lemma (cf. [19, p. 239]) states that there exist unique continuous processes X and Y satisfying (4.3) as well as:

$$X_t \ge 0 \tag{4.4}$$

$$Y_0 = 0 \;\text{ and }\; t \mapsto Y_t \;\text{ is increasing} \tag{4.5}$$

$$\int_0^t I(X_s > 0)\, dY_s = 0 \tag{4.6}$$

for $t \ge 0$. Moreover, the following explicit formula is valid:

$$Y_t = \sup_{0 \le s \le t} (-L^x_s) \vee 0 \tag{4.7}$$

for all $t \ge 0$. It may be noted that the whole result (4.3)-(4.7) is often referred to as Skorohod's lemma although Skorohod [18] only proved the existence and uniqueness of X and Y when L^x is an integral functional arising from diffusion processes, while the explicit expression (4.7) is due to Watanabe [21, p. 190] in the case of stable processes (giving a credit to Itô) and McKean [14, p. 86] in the case of Brownian motion and other diffusions as in [18].

It was recently observed (cf. [7, Lemma 1]; see also [20, Lemma 20], [11, p. 307], [5, Theorem 2.3]) that the Skorohod lemma (4.3)-(4.7) extends verbatim to the case when L^x is only assumed to be RCLL (right continuous with left limits). Moreover, if L^x is a Lévy process, then we can write $L^x_t = x + L_t$ with $L_0 = 0$ under P, and from (4.3)-(4.7) one finds:

$$\begin{aligned} X_t &= L^x_t + \max_{0 \le s \le t} (-L^x_s) \vee 0 = x + L_t + \max_{0 \le s \le t} (-x - L_s) \vee 0 \\ &= (x + L_t) \vee \max_{0 \le s \le t} (L_t - L_s) \end{aligned} \tag{4.8}$$

which is exactly (4.1). Thus, the reflecting Lévy process of Definition 4.1 can be seen as the unique solution of the Skorohod reflection problem extended verbatim from continuous to RCLL sample paths.

It should be noted that this process is different from the reflecting process introduced in [4] as the unique solution of a modified Skorohod reflection problem. The main difference is that when L^x at some t jumps from L^x_{t-} above zero to L^x_t below zero, then the reflecting process value is set to be 0 in the former case, while it is set to be the mirror image $-L^x_t$ in the latter case. Thus, in the case of the reflecting Lévy process (4.8), the jumps of L^x from $\mathbb{R}_+$ to $\mathbb{R}_-$ over 0 are not mirror-imaged but *absorbed* at 0. In the case when L^x is continuous the two definitions coincide.

3. Note that the previous two sections deal with reflecting Brownian motion with drift, which is the only reflecting Lévy process (in the sense of Definition 4.1) having continuous sample paths (up to B being replaced by σB for some $\sigma \geq 0$ constant). Moreover, this is also the only example of a continuous process L^x in the Skorohod equation (4.3) that makes the resulting process X Markovian (cf. [8]). One could expect that a similar uniqueness conclusion can be drawn for Leṽy processes within the class of RCLL processes.

A more detailed comparison and study of reflecting Lévy processes with jumps, including the boundary classification at zero in terms of the infinitesimal operator, appears to be worthy of further consideration.

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