J. Math. Anal. Appl. Vol. 222, No. 1, 1998, (244-254)
Research Report No. 355, 1996, *Dept. Theoret. Statist. Aarhus*

Optimal Stopping Inequalities for the Integral of Brownian Paths

GORAN PESKIR

Let $B = (B_t)_{t \geq 0}$ be a Brownian motion started at $x \in \mathbf{R}$. Given a stopping time τ for B and a real valued map F , we show how one can optimally bound:

$$E\bigg(\int_0^\tau F\big(|B_t|\big)\; dt \bigg)$$

in terms of $E(\tau)$. The method of proof relies upon solving the optimal stopping problem where one minimizes or maximizes:

$$E\bigg(\int_0^\tau F\big(|B_t|\big)\; dt - c\tau \bigg)$$

over all stopping times τ for B with $c > 0$. By Itô's formula and the optional sampling theorem this problem simplifies to the form where explicit computations are possible. The method is quantitatively demonstrated through the example of $F(x) = x^r$ with $r > -1$. This yields some new sharp inequalities.

1. Description of the problem and method of proof

Let $(B_t)_{t \geq 0}$ be standard Brownian motion. For simplicity in this section we assume that B starts at zero, but most of the considerations can be extended to the case when B starts at any $x \in \mathbf{R}$. Our starting point is Theorem 2.1 below where we prove the following inequalities:

$$A_p\, E\big(\tau^{1+p/2}\big) \leq E\bigg(\int_0^\tau |B_t|^p\; dt \bigg) \leq B_p\, E\big(\tau^{1+p/2}\big) \tag{1.1}$$

for all stopping times τ for B , and all $p > 0$, where A_p and B_p are numerical constants. Although the best values for the constants A_p and B_p in (1.1) are found below too, in most of the cases it is much easier to evaluate $E(\tau)$ rather than $E(\tau^{1+p/2})$.

In this paper we shall answer the question on how the inequality (1.1) can be optimally modified if the quantity $E(\tau^{1+p/2})$ is replaced by a function of $E(\tau)$. It turns out that the left-hand inequality in (1.1) admits such a modification. In Theorem 2.2 below we prove that:

$$E\bigg(\int_0^\tau |B_t|^p\; dt \bigg) \geq \frac{2}{(2{+}p)\,(1{+}p)}\, \big(E\tau\big)^{1+p/2} \tag{1.2}$$

for all stopping times τ for B and all $p > 0$, and moreover we show that $2/(2{+}p)(1{+}p)$ is the best possible constant. The analogue of such an inequality does not extend to the right-hand side in (1.1). In Example 2.4 below we show that the hitting times of the square root boundaries violate its validity. For this reason we are forced to change the functional of the Brownian path in

AMS 1980 *subject classifications*. Primary 60G40, 60J65, 60E15. Secondary 60G44, 60J25, 60J60.
Key words and phrases: Brownian motion, integral of Brownian path, optimal stopping, Burkholder-Davis' inequality, Itô-Tanaka formula, occupation times formula, Doob's optional sampling theorem, hitting time, square root boundary. goran@imf.au.dk

(1.1) from $t \mapsto |B_t|$ to $t \mapsto |B_t|^{-1}$. It turns out that such a modified path admits an extension of (1.2) to the right-hand side in (1.1). In Theorem 2.5 below we prove that:

$$E\bigg(\int_0^\tau \frac{1}{|B_t|^q}\; dt \bigg) \le \frac{2}{(2-q)\,(1-q)} \, \big(E\tau\big)^{1-q/2} \tag{1.3}$$

for all stopping times τ for B and all $0 \le q < 1$, and moreover we again show that $2/(2–q)(1–q)$ is the best possible constant. The optimality of the constant both in (1.2) and (1.3) is achieved through the hitting time of any positive number by the reflected Brownian motion $|B| = (|B_t|)_{t \ge 0}$. In fact if $\tau_a = \inf\{\, t>0 : |B_t| = a \,\}$ for some $a > 0$, then the equality is attained both in (1.2) and (1.3). Consequently, by letting $q \uparrow 1$ in (1.3) we see that:

$$E\bigg(\int_0^{\tau_a} \frac{1}{|B_t|}\; dt \bigg) = \infty \; . \tag{1.4}$$

This explains why q must be strictly less than 1 when dealing with $t \mapsto |B_t|^{-q}$ in (1.3).

Our proof of (1.2) and (1.3) is based upon solving the optimal stopping problem where one minimizes resp. maximizes:

$$E\bigg(\int_0^\tau F\big(|B_t|\big)\; dt \,-\, c\tau \bigg) \tag{1.5}$$

over all stopping times τ for B . Here $F : \mathbf{R} \to \mathbf{R}$ is a suitable function, and $c > 0$ is given and fixed. By Itô formula and the optional sampling theorem we find that (1.5) gets the form:

$$E\Big(H\big(|B_\tau|\big) - c|B_\tau|^2 \Big) \tag{1.6}$$

where $F = H''/2$ (with $H(0) = H'(0) = 0$). This further equals:

$$\int_0^\infty \big(H(x) - cx^2 \big) \; dP_{|B_\tau|}(x) \tag{1.7}$$

where $P_{|B_\tau|}$ denotes the distribution law of $|B_\tau|$. Thus if $x_*(c)$ denotes a minimum resp. maximum point of the map $x \mapsto H(x) - cx^2$ on $\mathbf{R}_+$, then:

$$\tau_*(c) = \inf \,\big\{\, t>0 \,:\, |B_t| \,=\, x_*(c) \,\big\} \tag{1.8}$$

is the optimal stopping time for the problem (1.5). This enables us to evaluate the minimum $Z_*(c)$ resp. the maximum $Z^*(c)$ in (1.5) exactly. By definition of the minimum then it follows:

$$E\bigg(\int_0^\tau F\big(|B_t|\big)\; dt \bigg) \ge \sup_{c>0} \Big(cE(\tau) + Z_*(c) \Big) \tag{1.9}$$

whenever τ is a stopping time for B . Similarly, by definition of the maximum we get:

$$E\bigg(\int_0^\tau F\big(|B_t|\big)\; dt \bigg) \le \inf_{c>0} \Big(cE(\tau) + Z^*(c) \Big) \tag{1.10}$$

for all stopping times τ for B . Supposing that the supremum in (1.9) resp. the infimum in (1.10) is attained at some $c_* > 0$, the right-hand side in (1.9) resp. (1.10) defines a function of $E(\tau)$.

Since the stopping time $\tau_*(c)$ is optimal for the problem (1.5) regardless of $c > 0$, it is clear that the inequalities (1.9) and (1.10) are sharp (the equality may be attained in a non-trivial manner).

In this paper we apply this method and make explicit computations only for the function $F(x) = x^r$ with $r > -1$, and this yields the inequalities (1.2) and (1.3), respectively. We want to mention that the cases of other functions F, including the case when B may start at any point, could be treated similarly. We believe that the presentation of our method above enables one to recognize the constraints on the function F needed to derive the inequalities (1.9) and (1.10) in a precise manner. For instance, if F is continuous and the map $x \mapsto H(x) - cx^2$ is bounded from above resp. below on $\mathbf{R}_+$, then (1.9) resp. (1.10) holds. Below we will see that the continuity condition on F may be relaxed (by treating the case where $F(x) = x^r$ for $-1 < r < 0$). In such a case one should use Itô-Tanaka's formula (rather than Itô's formula). It should be noted too (particularly when applying the optional sampling theorem) that the stopping times appearing throughout may be assumed bounded.

The results in this paper are similar in nature and can be compared with the results in [4]. In some sense the optimal stopping appearing here could be called a *Wald's type optimal stopping for the integral of Brownian paths*. Despite their appealing simplicity we are unaware of similar results.

2. Optimal stopping inequalities

In this section we present the main results of the paper. Throughout $B = (B_t)_{t\geq 0}$ is standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We begin by giving a proof of the inequalities (1.1). To the best of our knowledge these inequalities have not been written down earlier, but we believe that they are known.

Theorem 2.1
Let $B = (B_t)_{t\geq 0}$ be standard Brownian motion. Then the following inequalities are satisfied:

$$A_p\, E\big(\tau^{1+p/2}\big) \leq E\bigg(\int_0^\tau |B_t|^p\, dt\bigg) \leq B_p\, E\big(\tau^{1+p/2}\big) \tag{2.1}$$

for all stopping times τ for B, and all $p > 0$, where A_p and B_p are numerical constants.
The best possible values for A_p and B_p are:

$$A_p^* = 2(z_{2+p})^{2+p}/(2+p)(1+p) \tag{2.2}$$

$$B_p^* = 2(z_{2+p}^*)^{2+p}/(2+p)(1+p) \tag{2.3}$$

where z_{2+p} denotes the smallest positive zero of the confluent hypergeometric function $x \mapsto M(-(2+p)/2, 1/2, x^2/2)$, and z_{2+p}^* denotes the largest positive zero of the parabolic cylinder function $x \mapsto D_{2+p}(x)$ (see [1]). (In particular z_{2n} resp. z_{2n}^* is the smallest resp. largest positive zero of the Hermite polynomial $x \mapsto He_{2n}(x)$ for $n \geq 1$.)

Proof. Let $p > 0$ be given and fixed. By Itô formula we have:

$$|B_t|^{2+p} = (2+p)\int_0^t |B_s|^{1+p} sign(B_s)\, dB_s + 2^{-1}(2+p)(1+p)\int_0^t |B_s|^p\, ds \tag{2.4}$$

$$= (2+p)\, M_t + C_p \int_0^t |B_s|^p\, ds$$

where $M = (M_t)_{t \ge 0}$ is a continuous local martingale started at zero under P, and $C_p = 2^{-1}(2+p)(1+p)$. Let $[M] = ([M]_t)_{t \ge 0}$ denote the quadratic variation of M, and let τ be any stopping time for B. In view of (2.1) it is no restriction to assume that τ is bounded by a constant $\lambda > 0$, so that we have:

(2.5) $$E\big([M]_\tau\big)^{1/2} = E\bigg(\int_0^\tau |B_t|^{2(p+1)}\, dt\bigg)^{1/2} \le \lambda E\bigg(\max_{0 \le t \le \lambda} |B_t|^{2(p+1)}\bigg) = K\lambda^{p+2} < \infty$$

for some $K > 0$. Therefore by Doob's optional sampling theorem (see [3] and [5]):

(2.6) $$E(M_\tau) = 0\ .$$

Thus from (2.4) we obtain the following identity:

(2.7) $$E\bigg(\int_0^\tau |B_t|^p\, dt\bigg) = C_p^{-1} E|B_\tau|^{2+p}\ .$$

Recall Burkholder-Davis' inequalities:

(2.8) $$d_q E\big(\tau^{q/2}\big) \le E|B_\tau|^q \quad \text{if} \quad 1 < q < \infty \quad \text{and} \quad E\big(\tau^{q/2}\big) < \infty$$

(2.9) $$E|B_\tau|^q \le D_q E\big(\tau^{q/2}\big) \quad \text{for all} \quad q > 0$$

with the best possible values for D_q and d_q given by (see [2]):

(2.10) $$d_q = \begin{cases} (z_q)^q & \text{if} \quad 2 \le q < \infty \\ (z_q^*)^q & \text{if} \quad 1 < q \le 2 \end{cases}$$

(2.11) $$D_q = \begin{cases} (z_q^*)^q & \text{if} \quad 2 \le q < \infty \\ (z_q)^q & \text{if} \quad 0 < q \le 2 \end{cases}$$

where z_q and z_q^* are such as in the statement of the theorem. Setting $q = p+2$ the claims of the theorem clearly follow from (2.7)-(2.11). The proof is complete. □

In the next theorem we show how the left-hand inequality in (2.1) extends if the quantity $E(\tau^{1+p/2})$ is replaced by a function of $E(\tau)$. Note that the proof offers additional information as well, but we do not write it down separately (see Remark 2.3 below).

Theorem 2.2

Let $B = (B_t)_{t \ge 0}$ be standard Brownian motion. Then the following inequality is satisfied:

(2.12) $$E\bigg(\int_0^\tau |B_t|^p\, dt\bigg) \ge \frac{2}{(2+p)\,(1+p)}\, \big(E\tau\big)^{1+p/2}$$

for all stopping times τ for B and all $p > 0$. The constant $2/(2+p)(1+p)$ is the best possible.

Proof. Given $p > 0$ consider the following optimal stopping problem:

$$V(c) = \inf_{\tau} \; E\biggl(\int_0^\tau |B_t|^p \, dt - c\tau \biggr) \tag{2.13}$$

where $c > 0$ is given and fixed, and the infimum is taken over all stopping times τ for B having finite expectation. Let τ be any such stopping time for B . Then in view of (2.12) it is no restriction to assume that τ is bounded, so that by (2.7) we have:

$$\begin{aligned} E\biggl(\int_0^\tau |B_t|^p \, dt - c\tau \biggr) &= E\Bigl(C_p^{-1} |B_\tau|^{2+p} - c\tau \Bigr) = E\Bigl(C_p^{-1} |B_\tau|^{2+p} - c|B_\tau|^2 \Bigr) \\ &= \int_0^\infty \bigl(C_p^{-1} x^{2+p} - cx^2 \bigr) \; dP_{|B_\tau|}(x) \end{aligned} \tag{2.14}$$

where $C_p = 2^{-1}(2+p)(1+p)$ and $P_{|B_\tau|}$ denotes the distribution law of $|B_\tau|$.

Denote $f(x) = C_p^{-1} x^{2+p} - cx^2$ for $x > 0$. Then it is easily verified that f attains its minimum on $\mathbf{R}_+$ at $x_*(c) = (c(1+p))^{1/p}$ and $f(x_*(c)) = -p(1+p)^{2/p} c^{1+2/p}/(2+p)$. Hence from (2.14) we find:

$$V(c) = \frac{-p\,(1+p)^{2/p}}{(2+p)} \, c^{1+2/p} \tag{2.15}$$

and the optimal stopping time in (2.13) (the one at which the infimum is attained) is:

$$\tau_* = \inf \; \{ \, t > 0 \, : \, |B_t| = x_*(c) \, \} \tag{2.16}$$

(Since $E(\tau_*) < \infty$ note that the expectation in (2.5) with $\tau = \tau_*$ is finite, so that (2.6) and hence (2.7) holds for τ_* as well.) From (2.13) we get:

$$E\biggl(\int_0^\tau |B_t|^p \, dt \biggr) \geq \sup_{c>0} \Bigl(cE(\tau) + V(c) \Bigr) \; . \tag{2.17}$$

Denote $g(c) = cE(\tau) + V(c)$ for $c > 0$. Then it is easily verified that g attains its maximum on $\mathbf{R}_+$ at $c_* = (E\tau)^{p/2}/(1+p)$ and $g(c_*) = (2/(2+p)(1+p))(E\tau)^{1+p/2}$. Inserting this into (2.17) we get (2.12). The proof is complete. □

Remark 2.3

1. Since for τ_* defined in (2.16) we have $E(\tau_*) = E|B_{\tau_*}|^2 = x_*^2(c) = (c(1+p))^{2/p}$ for all $c > 0$, by taking $c = c_* = (E\tau)^{p/2}/(1+p)$ we see that each $\tau_a = \inf \{ \, t > 0 : |B_t| = a \, \}$ with $a > 0$ is optimal in (2.17), and therefore in (2.12) as well (the equality is attained at τ_a for all $a > 0$). (This is also verified directly by Itô-Tanaka formula and the optional sampling theorem.)

2. The proof presented above can be applied in the case when Brownian motion B starts at any $x \in \mathbf{R}$. In this way we can extend (2.12) by proving the following inequality:

$$E\biggl(\int_0^\tau |B_t + x|^p \, dt \biggr) \geq \frac{2}{(2+p)\,(1+p)} \biggl(\bigl(E(\tau) + x^2 \bigr)^{1+p/2} - |x|^{2+p} \biggr) \tag{2.18}$$

which is valid for all stopping times τ for B , all $p > 0$, and all $x \in \mathbf{R}$. Moreover, the constant $2/(2+p)(1+p)$ is the best possible for all $x \in \mathbf{R}$. In fact, if $\tau_{a,x} = \inf \{ \, t > 0 : |B_t + x| = a \, \}$ with $a > 0$, then the equality in (2.18) is attained at $\tau_{a,x}$ for all $a > 0$ and all $x \in \mathbf{R}$.

3. Note that (2.12) can be obtained more directly by using Jensen's inequality (observe in (2.7) that $E|B_\tau|^{2+p} = E(|B_\tau|^2)^{1+p/2} \geq (E|B_\tau|^2)^{1+p/2} = (E\tau)^{1+p/2}$ whenever $E(\tau) < \infty$). The main value of the proof given above lies in its applicability to the other functionals F (different from $x \mapsto |x|^p$) for which the convexity of $x \mapsto H(\sqrt{x})$ with $H'' = F$ (and thus Jensen's inequality as well) fails. The same facts hold for (2.18). See also Remark 2.4 in [4].

Example 2.4

From (2.1) and (2.12) it is clear that the quantity $E(\tau^{1+p/2})$ on the right-hand side in (2.1) cannot generally be replaced by $(E\tau)^{1+p/2}$. Here we exhibit a class of stopping times which violate such an inequality. For simplicity we only consider the case $p = 1$, but other cases could be treated similarly. So we shall show that the following inequality:

$$E\Big(\int_0^\tau |B_t|\ dt\Big) \leq C\big(E\tau\big)^{3/2} \tag{2.19}$$

cannot be satisfied for all stopping times τ for B with a fixed numerical constant $C > 0$.

For this consider the hitting times of the square root boundaries:

$$T_c = \inf\ \big\{\ t > 0 : |B_t| = c\,(t+1)^{1/2}\ \big\} \tag{2.20}$$

for $c > 0$. Then it is well-known that $E(T_c^q) < \infty$ if and only if $c < c_0(q)$ where $c_0(q)$ is the smallest positive zero of the confluent hypergeometric function $c \mapsto M(-q, 1/2, c^2/2)$ (see [6]). In particular, it is easily verified that $c_0(1) = 1 > c_0(3/2) = 0.84\ldots$.

Chose now $0 < c_n \uparrow c_0(3/2)$. Denote $T_n = T_{c_n}$ and $T_0 = T_{c_0(3/2)}$. Then $E(T_n^{3/2}) < \infty$ for any $n \geq 1$ given and fixed, so that:

$$\begin{aligned} E\Big(\int_0^{T_n} |B_t|^4\, dt\Big)^{1/2} &\leq E\Big(c_n^4\big(T_n+1\big)^2 T_n\Big)^{1/2} \leq c_n^2 E\big(T_n^{3/2} + \sqrt{2}T_n + T_n^{1/2}\big) \\ &\leq c_n^2\,(2+\sqrt{2})\big(1+E(T_n^{3/2})\big) < \infty\ . \end{aligned} \tag{2.21}$$

Therefore by Itô formula (applied to $|B_t|^3$) and Doob's optional sampling theorem we find:

$$E\Big(\int_0^{T_n} |B_t|\ dt\Big) = \frac{1}{3}\, E|B_{T_n}|^3 = \frac{c_n^3}{3}\, E(T_n+1)^{3/2}\ . \tag{2.22}$$

So, if (2.19) would be satisfied, then:

$$\frac{c_n^3}{3}\, E(T_n^{3/2}) \leq \frac{c_n^3}{3}\, E(T_n+1)^{3/2} \leq C(ET_n)^{3/2} \tag{2.23}$$

for all $n \geq 1$. Letting $n \to \infty$ we would get:

$$\frac{c_0^3(3/2)}{3}\, E(T_0^{3/2}) \leq C(ET_0)^{3/2} < \infty \tag{2.24}$$

since $c_0(3/2) < 1$. However, this contradicts the fact that $E(T_0^{3/2}) = \infty$. Thus (2.19) cannot be satisfied for all T_n with $n \geq 1$. The proof of the claim is complete. □

Motivated by the preceding example we change the functional of the Brownian path in (2.12) so that we obtain the following extension of this inequality. The proof is similar to the proof of Theorem 2.2 and we only sketch the most important parts for the convenience of the reader.

Theorem 2.5
Let $B = (B_t)_{t \geq 0}$ *be standard Brownian motion. Then the following inequality is satisfied:*

$$E\Big(\int_0^\tau \frac{1}{|B_t|^q}\, dt \Big) \leq \frac{2}{(2-q)(1-q)} \big(E\tau\big)^{1-q/2} \tag{2.25}$$

for all stopping times τ *for* B *and all* $0 \leq q < 1$ *. The constant* $2/(2-q)(1-q)$ *is the best possible.*

Proof. Given $0 \leq q < 1$ consider the following optimal stopping problem:

$$W(c) = \sup_\tau E\Big(\int_0^\tau \frac{1}{|B_t|^q}\, dt - c\tau \Big) \tag{2.26}$$

where $c > 0$ is given and fixed, and the supremum is taken over all stopping times τ for B having finite expectation. By Itô-Tanaka (and occupation times) formula (see [5] p.208-209) we find:

$$\begin{aligned} |B_t|^{2-q} &= (2-q) \int_0^t |B_s|^{1-q} sign(B_s)\, dB_s + 2^{-1}(2-q)(1-q) \int_0^t |B_s|^{-q}\, ds \\ &= (2-q)\, M_t + C_q \int_0^\tau \frac{1}{|B_s|^q}\, ds \end{aligned} \tag{2.27}$$

where $M = (M_t)_{t \geq 0}$ is a continuous local martingale started at zero under P , and $C_q = 2^{-1}(2-q)(1-q)$. Let τ be any stopping time for B having finite expectation. Then it is no restriction to assume that τ is bounded, so that in exactly the same way as in the proof of Theorem 2.1 (Doob's optional sampling theorem) we find $E(M_\tau) = 0$. Thus from (2.27) we obtain:

$$E\Big(\int_0^\tau \frac{1}{|B_t|^q}\, dt \Big) = C_q^{-1} E|B_\tau|^{2-q} \ . \tag{2.28}$$

Hence we find that:

$$\begin{aligned} E\Big(\int_0^\tau \frac{1}{|B_t|^q}\, dt - c\tau \Big) &= E\Big(C_q^{-1}|B_\tau|^{2-q} - c\tau \Big) = E\Big(C_q^{-1}|B_\tau|^{2-q} - c|B_\tau|^2 \Big) \\ &= \int_0^\infty \big(C_q^{-1} x^{2-q} - cx^2 \big)\, dP_{|B_\tau|}(x) \end{aligned} \tag{2.29}$$

where $P_{|B_\tau|}$ denotes the distribution law of $|B_\tau|$.

Denote $f(x) = C_q^{-1} x^{2-q} - cx^2$ for $x > 0$. Then it is easily verified that f attains its maximum on $\mathbf{R}_+$ at $x^*(c) = (c(1-q))^{-1/q}$ and $f(x^*(c)) = qc^{1-2/q}/(2-q)(1-q)^{2/q}$. Hence from (2.23) we find:

$$W(c) = \frac{q}{(2-q)(1-q)^{2/q}}\, c^{1-2/q} \tag{2.30}$$

and the optimal stopping time in (2.26) (the one at which the infimum is attained) is:

$$\tau^* = \inf \big\{ t > 0 : |B_t| = x^*(c) \big\} \tag{2.31}$$

(Note that (2.28) holds for τ^* as well.) From (2.26) we get:

$$E\Big(\int_0^\tau \frac{1}{|B_t|^q}\,dt\Big) \le \inf_{c>0}\Big(cE(\tau) + W(c)\Big)\ . \tag{2.32}$$

Denote $g(c) = cE(\tau) + W(c)$ for $c > 0$. Then it is easily verified that g attains its minimum on $\mathbf{R}_+$ at $c_* = 1/(1-q)(E\tau)^{q/2}$ and $g(c_*) = (2/(2-q)(1-q))(E\tau)^{1-q/2}$. Inserting this into (2.32) we get (2.25). The proof is complete. □

Remark 2.6

1. In exactly the same way as in part 1 of Remark 2.3 above we can verify that each $\tau_a = \inf\{\, t>0 : |B_t| = a \,\}$ with $a > 0$ is optimal in (2.32), and therefore in (2.25) as well (the equality is attained at τ_a for all $a > 0$). (This is also obtained directly by Itô-Tanaka formula and the optional sampling theorem.) Finally, letting $q \uparrow 1$ in (2.25) with $\tau = \tau_a$, we see that (1.4) holds for all $a > 0$. This explains why q in Theorem 2.5 must be strictly less than 1.

2. The proof above can be applied in the case when Brownian motion B starts at any $x \in \mathbf{R}$. In this way we can extend (2.25) by proving the following inequality:

$$E\Big(\int_0^\tau \frac{1}{|B_t+x|^q}\,dt\Big) \le \frac{2}{(2-q)\,(1-q)}\Big(\big(E(\tau)+x^2\big)^{1-q/2} - |x|^{2-q}\Big) \tag{2.33}$$

which is valid for all stopping times τ for B, all $0 \le q < 1$, and all $x \in \mathbf{R}$. Moreover, the constant $2/(2-q)(1-q)$ is the best possible for all $x \in \mathbf{R}$ (the equality is attained at $\tau_{a,x} = \inf\{\, t>0 : |B_t+x| = a \,\}$ for all $a>0$ and all $x \in \mathbf{R}$).

3. Note that (2.25) can be obtained more directly by using Jensen's inequality (observe in (2.28) that $E|B_\tau|^{2-q} = E(|B_\tau|^2)^{1-q/2} \le (E|B_\tau|^2)^{1-q/2} = (E\tau)^{1-q/2}$ whenever $E(\tau) < \infty$). The main value of the proof given above lies in its applicability to the other functionals F (different from $x \mapsto |x|^{-q}$) for which the concavity of $x \mapsto H(\sqrt{x})$ with $H'' = F$ (and thus Jensen's inequality as well) fails. The same facts hold for (2.33). See also Remark 2.4 in [4].

From (2.28) in the proof of Theorem 2.5 and (2.8)-(2.11) in the proof of Theorem 2.1 we obtain the following result.

Corollary 2.7

Let $B = (B_t)_{t\ge 0}$ be standard Brownian motion. Then the following inequalities are satisfied:

$$A_q\, E\big(\tau^{1-q/2}\big) \le E\Big(\int_0^\tau \frac{1}{|B_t|^q}\,dt\Big) \le B_q\, E\big(\tau^{1-q/2}\big) \tag{2.34}$$

for all stopping times τ for B and all $0 \le q < 1$, where A_q and B_q are numerical constants. The best possible values for A_q and B_q are:

$$A_q^* = 2(z_{2-q}^*)^{2-q}/(2-q)(1-q) \tag{2.35}$$

$$B_q^* = 2(z_{2-q})^{2-q}/(2-q)(1-q) \tag{2.36}$$

where z_{2-q}^* denotes the largest positive zero of the parabolic cylinder function $x \mapsto D_{2-q}(x)$, and z_{2-q} denotes the smallest positive zero of the confluent hypergeometric function $x \mapsto M(-(2-q)/2, 1/2, x^2/2)$ (see [1]). □

REFERENCES

[1] ABRAMOWICZ, M. *and* STEGUN, I. A. (1970). *Handbook of Mathematical Functions*. Dover Publications, New York.

[2] DAVIS, B. (1976). On the L^p norms of stochastic integrals and other martingales. *Duke Math. J.* 43 (697-704).

[3] DOOB, J. L. (1953). *Stochastic Processes*. John Wiley & Sons.

[4] GRAVERSEN, S. E. *and* PESKIR, G. (1994). On Wald-type optimal stopping for Brownian motion. *Math. Inst. Aarhus, Preprint Ser.* No. 10, (15 pp). *J. Appl. Probab.* 34, 1997 (66-73).

[5] REVUZ, D. *and* YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer-Verlag.

[6] SHEPP, L. A. (1967). A first passage problem for the Wiener process. *Ann. Math. Statist.* 38 (1912-1914).

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk