Proc. Funct. Anal. VII (Dubrovnik 2001), *Various Publ. Ser.* Vol. 46, 2002, (159-175)
Research Report No. 426, 2002, *Dept. Theoret. Statist. Aarhus*

On Nonlinear Integral Equations Arising in Problems of Optimal Stopping

J. L. PEDERSEN *and* G. PESKIR*

Let $B = (B_t)_{0 \le t \le 1}$ be a standard Brownian motion started at zero, let $\lambda \ge 0$ be given and fixed, and let $G : [0,1] \times I\!R \rightarrow I\!R$ be a measurable function. Consider the optimal stopping problem:

$$V = \sup_{0 \le \tau \le 1} E\Big(e^{-\lambda \tau} G(\tau, B_\tau) \Big)$$

where τ is a stopping time of B. Then, under certain regularity conditions on the map G, the optimal stopping time is given by

$$\tau_b = \inf \{ \, 0 \le t \le 1 \mid B_t \ge b(t) \, \}$$

where the optimal stopping boundary $t \mapsto b(t)$ is characterized as a unique solution of the nonlinear integral equation:

$$\int_0^{1-t} e^{-\lambda s} E\bigg(\Big(G_t + \tfrac{1}{2} G_{xx} - \lambda G \Big)(t+s, b(t)+B_s) \cdot I\Big(b(t)+B_s > b(t+s) \Big) \bigg) ds$$
$$= e^{-\lambda(1-t)} E\Big(G\big(1, b(t)+B_{1-t}\big) \Big) - G\big(t, b(t)\big)$$

being valid for all $t \in [0,1]$. The key argument in the proof is based upon an extension of the Itô-Tanaka formula yielding local times of B at curved boundaries.

1. Introduction

Finite-horizon problems of optimal stopping for Markov processes have been studied by a number of authors ever since the basic principles of optimal stopping have been established in the works of Snell [16] and Dynkin [1] (for general theory see [15]). These problems are inherently two-dimensional and therefore analytically more difficult.

One way to handle the problem is to formulate a free-boundary problem (see [3]) for the parabolic operator associated with the Markov process (cf. [8], [4], [17], [5], [6], [11]) making use of the principle of smooth fit (dating back to [9]). The question then reduces to prove the existence and uniqueness of a solution to the free-boundary problem which then leads to the optimal stopping boundary and the value function of the optimal stopping problem.

Solutions to free-boundary problems for parabolic equations are rarely known explicitly and one is therefore confined to apply numerical methods to get out numbers. For such methods in the context of the American put option, which is the best known example of a finite-horizon optimal stopping problem, we refer to pages 86-87 in [7]. Newer references dealing with similar numerical methods appear continuously in the literature and may be found elsewhere.

*Centre for Mathematical Physics and Stochastics (funded by the Danish National Research Foundation).
Mathematics Subject Classification 2000. Primary 60G40, 35R35, 45G10. Secondary 60J65, 60J60, 45G15.
Key words and phrases: Optimal stopping, finite horizon, Brownian motion, a free-boundary problem, nonlinear integral equation, the Itô-Tanaka formula, local time, curved boundary, the first-passage problem.

It was shown in the papers by Jacka [5] and Jacka & Lynn [6] that the optimal stopping boundary can be characterized as a unique solution of the system of (at least) countably many nonlinear integral equations (see Theorem 4.3 in [5] and Theorem 3.5 in [6]). In order to determine the boundary explicitly one is therefore faced with a complicated task of making sure that all these equations are satisfied by a candidate. The latter fact is typically ignored in the papers dealing with numerical approximations of the solution.

The main aim of the present paper is to show that only one equation from this system may be sufficient to characterize the optimal stopping boundary in the case when the process is standard Brownian motion. The key argument in the proof is based upon an extension of the Itô-Tanaka formula (cf. [14; Theorem 1.5, p. 223]) dealing with local times of Brownian motion at curved boundaries. In order to establish this extended formula, we make use of general results on the Itô formula due to Föllmer, Protter and Shiryaev [2], although a direct approach could also be given. A similar method of proof can be applied to more general continuous Markov processes (diffusions) and will be discussed elsewhere.

A similar system of nonlinear integral equations arises in the first-passage problem for Brownian motion (see Theorems 3.1 and 6.1 in [13]). The inverse first-passage problem seeks to determine the boundary when the distribution of the first-passage time is known and may thus be reformulated as the existence (and uniqueness) problem for the system. Since such existence results can be obtained by probabilistic methods in problems of optimal stopping discussed above, the present paper is partly aimed at highlighting some analytic aspects of this interesting connection.

2. The result and proof

1. Let $B = (B_t)_{0 \le t \le 1}$ be a standard Brownian motion started at zero under P, let $\lambda \ge 0$ be given and fixed, and let $G : [0,1] \times I\!R \to I\!R$ be a measurable function.

We consider the optimal stopping problem:

$$V = \sup_{0 \le \tau \le 1} E\Big(e^{-\lambda\tau} G(\tau, B_\tau)\Big) \tag{2.1}$$

where τ is a stopping time of B.

Recognizing (t, B_t) as the Markov process, and thinking of $e^{-\lambda\tau}$ as a term of killing, we are naturally led to extend the problem (2.1) as follows:

$$V(t,x) = \sup_{0 \le \tau \le 1-t} E_{t,x}\Big(e^{-\lambda\tau} G(t+\tau, B_{t+\tau})\Big) \tag{2.2}$$

where τ is as above and $P_{t,x}(B_t = x) = 1$ for $t \in [0,1]$ and $x \in I\!R$. Thus, under $P_{t,x}$ the process B is a standard Brownian motion which at time t starts at x.

2. We assume that the map G satisfies some regularity conditions implying the existence of a strictly decreasing continuous function $b : [0,1] \to I\!R$ such that the first-passage time:

$$\tau_b = \inf\{\, 0 \le s \le 1-t \mid B_{t+s} \ge b(t+s) \,\} \tag{2.3}$$

is optimal in (2.2) (the infimum of an empty set being equal $1-t$). These conditions can be studied by probabilistic methods (see [6] and [17]) but we will not study them in the present paper.

Among known regularity conditions of this type we mention the following:

(2.4) $$(t,x) \mapsto G(t,x) \text{ is } C^{1,2} \text{ on } [0,1]\times I\!R$$

(2.5) $$x \mapsto \big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\big)(t,x) \text{ is decreasing on } I\!R \text{ for each } t\in[0,1]$$

(2.6) $$t \mapsto \big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\big)(t,x) \text{ is decreasing on } [0,1] \text{ for each } x\in I\!R$$

(cf. Theorem 4.3, Propositions 4.4 and 4.5 in [6]). A non-trivial problem will also require that

(2.7) $$\big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\big)(t,x) > 0 \text{ for } x < \gamma(t)$$

(2.8) $$\big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\big)(t,x) = 0 \text{ for } x = \gamma(t)$$

(2.9) $$\big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\big)(t,x) < 0 \text{ for } x > \gamma(t)$$

for all $(t,x)\in[0,1]\times I\!R$ where $\gamma:[0,1]\to I\!R$ is a continuous function. It then easily follows by Itô's formula that $b(t)>\gamma(t)$ for all $0<t<1$. We will assume in the sequel that (2.4)-(2.9) are satisfied although these hypotheses are by no means necessary. Further regularity conditions on G will also be introduced in the sequel when needed.

Let us define the continuation region C and the stopping region D as follows:

(2.10) $$C = \{\,(t,x)\in[0,1]\times I\!R \mid x < b(t)\,\}$$

(2.11) $$D = \{\,(t,x)\in[0,1]\times I\!R \mid x > b(t)\,\}\ .$$

It will be assumed that the regularity conditions on G imply that V is continuous on $[0,1]\times I\!R$ and $C^{1,2}$ in $C\cup D$ as well as that $x\mapsto V(t,x)$ is C^1 at $b(t)$ and $s\mapsto V(s,b(t))$ is sufficiently regular at t with $0<t<1$ so that Itô's formula can be applied (for sufficient conditions for the latter see (2.37)-(2.43) and (2.46)-(2.48) as well as (2.49)-(2.52) below).

For clearer exposition of the method and main idea we will deal in the sequel only with the class of boundaries specified above. We thus denote by $\mathcal{C}$ the class of all strictly decreasing continuous functions $c:[0,1]\to I\!R$ satisfying $c(t)>\gamma(t)$ for all $0<t<1$. It should be noted that the method of proof presented below is not restricted to this class of boundaries. However, if the boundary c happens to oscillate wildly then the study also needs to focus harder on regularity properties of the map $t\mapsto V_c(t,x)$ from (2.29) below, which for a strictly decreasing $c\in\mathcal{C}$ is much simpler (see sufficient conditions (2.41)-(2.43) below).

3. As another set of regularity conditions imposed throughout we will assume (without further mentioning) that the optional sampling theorem (see e.g. [14]) is applicable to all local martingales (i.e. stochastic integrals with respect to B) appearing throughout. In view of the BDG inequality (see e.g. [14]) this fact will be satisfied if a growth condition on the function under consideration is imposed. These growth conditions will not be stated explicitly in the sequel, but each concrete example of the problem (2.2) will require such a verification.

Given that the two sets of regularity conditions specified above are satisfied, and consequently that τ_b from (2.3) is optimal in (2.2), our initial aim in the sequel will be to determine *two nonlinear integral equations* which characterize the optimal stopping boundary b .

4. Under the regularity conditions on G discussed above we are naturally led to formulate the following *free-boundary problem* for the unknown V and b :

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} = \lambda V \quad \text{in } C \tag{2.12}$$

$$V(t,x)\Big|_{x=b(t)-} = G(t,x)\Big|_{x=b(t)+} \quad \text{(instantaneous stopping)} \tag{2.13}$$

$$\frac{\partial V}{\partial x}(t,x)\Big|_{x=b(t)-} = \frac{\partial G}{\partial x}(t,x)\Big|_{x=b(t)+} \quad \text{(smooth fit)} \tag{2.14}$$

$$V > G \quad \text{in } C \tag{2.15}$$

$$V = G \quad \text{in } D \tag{2.16}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} < \lambda V \quad \text{in } D \tag{2.17}$$

where (2.13) and (2.14) hold for all $0 < t < 1$.

Dynkin's superharmonic characterization of the value function (see [1] and [15]) implies that V from (2.2) is the smallest function satisfying (2.12), (2.13), (2.15), (2.16) and (2.17).

5. Let us now show that V satisfying (2.12)-(2.17) coincides with the value function (2.2). For this, we may apply Itô's formula (using that $x \mapsto V(t,x)$ is C^1 at $b(t)$ by (2.14) above) and for a given and fixed $(t,x) \in [0,1] \times I\!R$ in this way obtain:

$$e^{-\lambda s} V(t+s, B_{t+s}) = V(t,x) + \int_0^s e^{-\lambda u}\Big(V_t + \tfrac{1}{2}V_{xx} - \lambda V\Big)(t+u, B_{t+u})\, du + M_s \tag{2.18}$$

where $M_s = \int_0^s e^{-\lambda u} V_x(t+u, B_{t+u})\, dB_u$ for $0 \le s \le 1-t$ is a continuous (local) martingale. Since $V_t + \frac{1}{2}V_{xx} - \lambda V \le 0$ by means of (2.12) and (2.17), using also that the value of V_{xx} can be set arbitrarily at $(t+u, b(t+u))$ for each u , from (2.18) we get:

$$e^{-\lambda\sigma} V(t+\sigma, B_{t+\sigma}) \le V(t,x) + M_\sigma \tag{2.19}$$

for any stopping time $0 \le \sigma \le 1-t$ of B given and fixed. On the other hand, by (2.13), (2.15) and (2.16) we know that $V \ge G$ so that (2.19) implies:

$$e^{-\lambda\sigma} G(t+\sigma, B_{t+\sigma}) \le V(t,x) + M_\sigma \ . \tag{2.20}$$

Taking the $P_{t,x}$ expectation we obtain:

$$\sup_{0\le\sigma\le 1-t} E_{t,x}\Big(e^{-\lambda\sigma} G(t+\sigma, B_{t+\sigma})\Big) \le V(t,x) \ . \tag{2.21}$$

To see that equality in (2.21) is attained at τ_b from (2.3), we may use (2.12) with (2.13) and (2.16) to see that (2.18) above with τ_b in place of s reads as follows:

$$e^{-\lambda\tau_b} G(t+\tau_b, B_{t+\tau_b}) = V(t,x) + M_{\tau_b} \ . \tag{2.22}$$

Then as in (2.21) above it follows that

$$E_{t,x}\Big(e^{-\lambda\tau_b}G(t+\tau_b, B_{t+\tau_b})\Big) = V(t,x) \tag{2.23}$$

proving the claim.

6. In the next step we shall try to determine the two integral equations for b mentioned prior to (2.12) above. We begin by noting that (2.18) can be written using (2.12) and (2.16) as follows:

$$\begin{aligned} e^{-\lambda s}V(t+s, B_{t+s}) &= V(t,x) \\ &+ \int_0^s e^{-\lambda u}\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u, B_{t+u}) \cdot I\Big(B_{t+u} > b(t+u)\Big)\, du + M_s\ . \end{aligned} \tag{2.24}$$

Setting $s = 1-t$, using that $V(1,x) = G(1,x)$, and taking the $P_{t,x}$ expectation, we get:

$$\begin{aligned} e^{-\lambda(1-t)}E_{t,x}\big(G(1,B_1)\big) &= V(t,x) \\ &+ \int_0^{1-t} e^{-\lambda u}E_{t,x}\Big(\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u, B_{t+u}) \cdot I\Big(B_{t+u} > b(t+u)\Big)\Big)\, du \end{aligned} \tag{2.25}$$

for all $(t,x) \in [0,1] \times I\!R$. If $x \geq b(t)$ then $V(t,x) = G(t,x)$ by (2.13) and (2.16), so that (2.25) becomes an integral equation for b .

Yet another integral equation can be naturally obtained by differentiating both sides in (2.25) with respect to x upon using (2.14). This gives the following equation:

$$\begin{aligned} e^{-\lambda(1-t)}\frac{\partial}{\partial x}E_{t,x}\big(G(1,B_1)\big) &= \frac{\partial V}{\partial x}(t,x) \\ &+ \int_0^{1-t} e^{-\lambda u}\frac{\partial}{\partial x}E_{t,x}\Big(\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u, B_{t+u}) \cdot I\Big(B_{t+u} > b(t+u)\Big)\Big)\, du \end{aligned} \tag{2.26}$$

for all $(t,x) \in [0,1] \times I\!R$. Similarly, if $x \geq b(t)$ then $V_x(t,x) = G_x(t,x)$ by (2.14) and (2.16), so that (2.26) becomes an integral equation for b .

In this way we have obtained two systems of (at least) countably many nonlinear integral equations for the unknown boundary b . The system (2.25) has been derived by Jacka [5] and Jacka & Lynn [6] in the case of more general Markov diffusions. In these papers it was shown that the system (2.25) characterizes the optimal boundary b uniquely.

7. It is of practical interest to reduce the number of these equations so to obtain a smaller system which still characterizes the optimal boundary b . We begin to tackle this question by noting that two natural equations are obtained by inserting $x = b(t)$ in (2.25) and (2.26) respectively. We shall term them *the instantaneous-stopping equation*:

$$\begin{aligned} e^{-\lambda(1-t)}E_{t,b(t)}\big(G(1,B_1)\big) &= G(t,b(t)) \\ &+ \int_0^{1-t} e^{-\lambda u}E_{t,b(t)}\Big(\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u, B_{t+u}) \cdot I\Big(B_{t+u} > b(t+u)\Big)\Big)\, du \end{aligned} \tag{2.27}$$

and *the smooth-fit equation*:

$$e^{-\lambda(1-t)} \frac{\partial}{\partial x} E_{t,x}\big(G(1,B_1)\big)\Big|_{x=b(t)} = \frac{\partial G}{\partial x}\big(t,b(t)\big) + \int_0^{1-t} e^{-\lambda u} \frac{\partial}{\partial x} E_{t,x}\Big(\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u,B_{t+u}) \cdot I\Big(B_{t+u} > b(t+u)\Big)\Big)\Big|_{x=b(t)} du \tag{2.28}$$

both being valid for all $t \in [0,1]$. Our first aim in the sequel is to show that the two equations (2.27) and (2.28) are sufficient to determine the optimal boundary b uniquely.

8. For this, let us assume that a function $c : [0,1] \to I\!R$ from the class $\mathcal{C}$ solves (2.27) and (2.28). Define a function V_c in accordance with (2.25) above:

$$V_c(t,x) = e^{-\lambda(1-t)} E_{t,x}\big(G(1,B_1)\big) - \int_0^{1-t} e^{-\lambda u} E_{t,x}\Big(\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u,B_{t+u}) \cdot I\Big(B_{t+u} > c(t+u)\Big)\Big) du \tag{2.29}$$

for all $x < c(t)$ with $t \in [0,1]$ given and fixed, and set:

$$V_c(t,x) = G(t,x) \tag{2.30}$$

for all $x \geq c(t)$. Let C and D be defined as in (2.10) and (2.11) with c instead of b when we speak of V_c . Then due to (2.27) and (2.28) we see that V_c satisfies (2.13) and (2.14) (the latter follows from the fact that the function of x defined by the right-hand side of (2.29) with $t \in [0,1]$ given and fixed is C^1 on $I\!R$ under a regularity condition on G easily specified), which in turn is a key to establish the following representation:

$$V_c(t,x) = E_{t,x}\Big(e^{-\lambda \tau_c} G(t+\tau_c, B_{t+\tau_c})\Big) \tag{2.31}$$

for all $(t,x) \in [0,1] \times I\!R$ where τ_c is defined as in (2.3).

To verify (2.31) one can check directly (using the Markov property) that V_c satisfies (2.12), then using that V_c satisfies (2.13) and (2.14) (the latter in particular) one verifies (2.24) for V_c and c by Itô's formula, and then finally inserting τ_c in place of s in (2.24) one gets (2.31) upon taking the $P_{t,x}$ expectation. Thus V_c satisfies (2.12)-(2.14), and clearly (2.16) by definition (2.30), while (2.17) must be satisfied as a regularity condition singled out earlier in (2.9) above. Therefore, upon recalling our discussion in Subsection 5 above, it remains to establish (2.15) i.e. that $V_c > G$ in C .

We shall verify the latter in two steps. For this, first recall that by our assumptions above we know that V from (2.2) and b from (2.3) solve the free-boundary problem (2.12)-(2.17). In particular, by definition of V in (2.2) and the representation (2.31), it follows that

$$V_c(t,x) \leq V(t,x) \tag{2.32}$$

for all $(t,x) \in [0,1] \times I\!R$.

To prove the reverse inequality, take any $0 < t < 1$ and let x be greater than both $b(t)$ and $c(t)$. Define a stopping time σ by setting:

$$\sigma = \inf\{\, 0 \leq s \leq 1-t \mid B_{t+s} \leq b(t+s) \,\} . \tag{2.33}$$

Then using (2.24) with σ in place of s, we find:

$$(2.34)\qquad e^{-\lambda\sigma}V(t+\sigma, B_{t+\sigma}) = V(t,x) + \int_0^\sigma e^{-\lambda u}\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u, B_{t+u})\,du + M_\sigma\ .$$

Likewise, using (2.24) for V_c and c with σ in place of s, we find:

$$(2.35)\qquad e^{-\lambda\sigma}V_c(t+\sigma, B_{t+\sigma}) \;=\; V_c(t,x) + \int_0^\sigma e^{-\lambda u}\Big(G_t + \tfrac{1}{2}G_{xx} - \lambda G\Big)(t+u, B_{t+u})\cdot I\Big(B_{t+u} > c(t+u)\Big)\,du + M_\sigma^c\ .$$

Taking the $P_{t,x}$ expectation in (2.34) and (2.35), using (2.32) and that $V(t,x) = V_c(t,x) = G(t,x)$ for the given and fixed t and x, as well as that $G_t + \frac{1}{2}G_{xx} - \lambda G < 0$ in D, we see that $b(s) \geq c(s)$ for all $t < s < 1$, and thus $b \geq c$ on $[0,1]$. It thus follows from (2.25) with V_c and c as well as V and b that

$$(2.36)\qquad V(t,x) \;\leq\; V_c(t,x)$$

implying also that $V_c = V$, and so $c = b$. This completes the proof of the claim that the two equations (2.27) and (2.28) are sufficient to determine the optimal boundary b uniquely.

9. Our final aim in the sequel is to show that the instantaneous-stopping equation (2.27) alone is sufficient to determine the optimal boundary b uniquely.

For this, we shall proceed as earlier by assuming that a function $c : [0,1] \rightarrow I\!R$ from the class $\mathcal{C}$ solves (2.27) and define a function V_c by (2.29) and (2.30) in accordance with (2.25) above. Then by (2.27) we see that V_c satisfies (2.13), while in order to derive the representation (2.31) just as above in the paragraphs following (2.31), the key difficulty arises from the fact that we no longer know that V_c satisfies the smooth-fit condition (2.14) (due to the absence of the smooth-fit equation (2.28)). Given however that (2.14) is verified the rest of the proof following (2.31) can be repeated word by word to show that $V_c = V$ and so $c = b$.

The proof of the claim will therefore be completed if we show that (2.27) has a power of implying (2.14), and this is what we do in the rest of the section.

10. For this, we shall establish the following *extension of Itô's formula* (which is tailored for situations arising from the problem (2.2)).

Let b, C and D be as above, and let $F : [0,1] \times I\!R \rightarrow I\!R$ be a measurable function satisfying the following conditions:

(2.37) F is $C^{1,2}$ in $C \cup D$

(2.38) $x \mapsto F(t,x)$ is continuous at $b(t)$

(2.39) limits $F_x(t, b(t)\pm)$ exist in $I\!R$ through bounded variation

(2.40) $t \mapsto F_x(t, b(t)\pm)$ are continuous

(2.41) $t \mapsto F(t,x)$ is absolutely continuous

(2.42) $x \mapsto F_t(t,x)$ belongs to L^1_{loc}

(2.43) $t \mapsto F_t(t,\cdot\,)$ is continuous from $[0,1]$ into L^1_{loc}

for all $(t,x) \in [0,1] \times I\!R$. Then we have:

$$F(t,B_t) = F(0,B_0) + \int_0^t F_t(s,B_s)\,ds + \int_0^t F_x(s,B_s)\,I\big(B_s \neq b(s)\big)\,dB_s + \frac{1}{2}\int_0^t F_{xx}(s,B_s)\,I\big(B_s \neq b(s)\big)\,ds + \frac{1}{2}\int_0^t \Delta_x F_x(s,b(s))\,d\ell_s^b \tag{2.44}$$

where $\Delta_x F_x(s,b(s)) = F_x(s,b(s)+) - F_x(s,b(s)-)$ and $(\ell_s^b)_{0 \le s \le t}$ is *the local time of* B *at* b :

$$\ell_t^b = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big(b(s)-\varepsilon < B_s < b(s)+\varepsilon\big)\,ds \tag{2.45}$$

uniformly over $t \in [0,1]$ as a limit in probability.

In parallel to (2.38)-(2.40) note that if (2.37) holds and we have:

(2.46) $\quad s \mapsto F(s,b(t))$ is continuous at t

(2.47) $\quad$ limits $F_t(t\pm,b(t))$ exist in $I\!R$

(2.48) $\quad t \mapsto F_t(t\pm,b(t))$ are continuous

for all $t \in [0,1]$, then the conditions (2.41)-(2.43) are satisfied.

To verify (2.44) we shall make use of general results on Itô's formula obtained in [2]. We will begin by recalling some facts from this paper.

11. Recall the extension of Itô's formula derived in Theorem 5.1 of [2]. Let $F : [0,1] \times I\!R \to I\!R$ be a measurable function satisfying the following conditions:

(2.49) $\quad t \mapsto F(t,x)$ and $x \mapsto F(t,x)$ are absolutely continuous

(2.50) $\quad x \mapsto F_t(t,x)$ belongs to L^1_{loc} and $x \mapsto F_x(t,x)$ belongs to L^2_{loc}

(2.51) $\quad t \mapsto F_t(t,\cdot\,)$ is continuous from $[0,1]$ into L^1_{loc}

(2.52) $\quad t \mapsto F_x(t,\cdot\,)$ is continuous from $[0,1]$ into L^2_{loc}

for all $(t,x) \in [0,1] \times I\!R$. Then we have:

$$F(t,B_t) = F(0,B_0) + \int_0^t F_t(s,B_s)\,ds + \int_0^t F_x(s,B_s)\,dB_s + \frac{1}{2}[F_x(\,\cdot\,,B),B]_t \tag{2.53}$$

where $[F_x(\,\cdot\,,B),B]_t$ is the *quadratic covariation* given by

$$[F_x(\,\cdot\,,B),B]_t = \lim_{n \to \infty} \sum_{t_i \in D_t^n} \Big(F_x(t_i,B_{t_i}) - F_x(t_{i-1},B_{t_{i-1}})\Big)\big(B_{t_i} - B_{t_{i-1}}\big) \tag{2.54}$$

and this limit exists uniformly over $t \in [0,1]$ in probability (the set D_t^n consists of points $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ satisfying $\max_{1 \le i \le n}(t_i - t_{i-1}) \to 0$ as $n \to \infty$).

We will proceed by summarizing some useful facts about the quadratic covariation (2.54) which are needed for the formula (2.44).

12. Let $f : [0,1] \times I\!R \to I\!R$ be a measurable function (which will be F_x above) satisfying the following two conditions:

(2.55) $$x \mapsto f(t,x) \;\text{ belongs to }\; L^2_{loc}$$

(2.56) $$t \mapsto f(t,\cdot\,) \;\text{ is continuous from }\; [0,1] \;\text{ into }\; L^2_{loc}$$

for all $t \in [0,1]$. Then the quadratic covariation:

(2.57) $$[f(\,\cdot\,,B),B]_t = \lim_{n\to\infty} \sum_{t_i \in D_t^n} \Big(f(t_i,B_{t_i}) - f(t_{i-1},B_{t_{i-1}})\Big)\big(B_{t_i} - B_{t_{i-1}}\big)$$

exists uniformly over $t \in [0,1]$ as a limit in probability (cf. Theorem 3.3 in [2]).

Moreover, if the following condition holds:

(2.58) $$x \mapsto f(t,x) \;\text{ is absolutely continuous}$$

for all $t \in [0,1]$, then we have:

(2.59) $$[f(\,\cdot\,,B),B]_t = \int_0^t f_x(s,B_s)\, ds$$

(cf. part (c) of Remark 3.2 in [2]).

13. Given a measurable function $f : [0,1] \times I\!R \to I\!R$ introduce the following norms:

(2.60) $$\|f\|_\alpha = \int_0^1 \Big(\int f^2(t,x)\, dx \Big)^{1/2} \frac{1}{t^{3/4}}\, dt$$

(2.61) $$\|f\|_\beta = \Big(\int_0^1 \Big(\int f^2(t,x)\, dx \Big) \frac{1}{\sqrt{t}}\, dt \Big)^{1/2} .$$

If for a sequence of measurable functions $f_n : [0,1] \times I\!R \to I\!R$ satisfying (2.55) and (2.56) we have $f_n \to f$ in the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ as $n \to \infty$, then:

(2.62) $$\lim_{n\to\infty} [f_n(\,\cdot\,,B),B]_t = [f(\,\cdot\,,B),B]_t$$

uniformly over $t \in [0,1]$ as a limit in probability. This follows from the proof of Theorem 3.3 in [2], see part (b) of Remark 3.2 in the same paper.

14. Given a continuous function $a : [0,1] \to I\!R$ set:

(2.63) $$f(t,x) = 1_{[a(t),\infty[}(x)$$

and follow Example 3.1 in [2] by defining the local time of B at a as follows:

(2.64) $$\ell_t^a = [f(\,\cdot\,,B),B]_t$$

where the quadratic variation exists uniformly over $t \in [0,1]$ as a limit in probability by (2.57).

Taking any sequence $\varepsilon_n \downarrow 0$ for $n \to \infty$ and setting:

(2.65) $$f_n(t,x) = \frac{1}{2\varepsilon_n} \int_{-\infty}^x 1_{]a(t)-\varepsilon_n, a(t)+\varepsilon_n[}(z)\, dz$$

we see that f_n satisfies (2.58) and therefore by (2.59) we have:

(2.66) $$[f_n(\,\cdot\,,B),B]_t = \frac{1}{2\varepsilon_n} \int_0^t I\big(a(s)-\varepsilon_n < B_s < a(s)+\varepsilon_n\big)\, ds\ .$$

Moreover $f_n \rightarrow f$ in the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ as $n \rightarrow \infty$ so that (2.66) implies:

(2.67) $$\ell_t^a = \lim_{n\rightarrow\infty} \frac{1}{2\varepsilon_n} \int_0^t I\big(a(s)-\varepsilon_n < B_s < a(s)+\varepsilon_n\big)\, ds$$

uniformly over $t \in [0,1]$ as a limit in probability.

The path $t \mapsto \ell_t^a$ is increasing and continuous, and the measure $d\ell_t^a$ is concentrated at times when the Brownian path $t \mapsto B_t$ intersects the curve $t \mapsto a(t)$ on $[0,1]$, i.e.

(2.68) $$\int_0^1 I\big(B_t \neq a(t)\big)\, d\ell_t^a = 0\ .$$

15. Example 3.1 in [2] has been followed by Example 5.1 where the following extension of the Itô-Tanaka formula has been derived upon applying (2.53) above:

(2.69) $$\big(B_t - a(t)\big)^+ = \big(B_0 - a(0)\big)^+ - \int_0^t I\big(a(s) < B_s\big) a'(s)\, ds + \int_0^t I\big(a(s) < B_s\big)\, dB_s + \frac{1}{2}\, \ell_t^a$$

when $a : [0,1] \rightarrow I\!R$ is C^1 . The formula (2.69) has been also written for general continuous a including also a of bounded variation (cf. (5.12) and (5.14) in [2]). With the aim of establishing (2.44) under (2.37)-(2.43), we will now continue these considerations.

16. Let us therefore assume that F satisfies (2.37)-(2.43), denote $f(t,x) = F_x(t,x)$, and similarly to (2.65) define the linear approximation:

(2.70) $$\begin{aligned} f_n(t,x) &= f(t,x) \quad \text{if} \quad x \notin\,]b(t)-\varepsilon_n, b(t)+\varepsilon_n[\\ &= \text{linear} \quad \text{if} \quad x \in [b(t)-\varepsilon_n, b(t)+\varepsilon_n] \end{aligned}$$

where a sequence $\varepsilon_n \downarrow 0$ for $n \rightarrow \infty$ is given and fixed. Then $x \mapsto f_n(t,x)$ is absolutely continuous so that from (2.57) using (2.59) it follows that

(2.71) $$\begin{aligned} [f_n(\,\cdot\,,B),B]_t = &\int_0^t f_x(s,B_s)\, I\big(B_s \notin\,]b(s)-\varepsilon_n, b(s)+\varepsilon_n[\big)\, ds \\ &+ \frac{1}{2\varepsilon_n} \int_0^t \Big(f(s,b(s)+\varepsilon_n) - f(s,b(s)-\varepsilon_n)\Big)\, I\big(b(s)-\varepsilon_n < B_s < b(s)+\varepsilon_n\big)\, ds\ . \end{aligned}$$

Moreover $f_n \rightarrow f$ in the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ as $n \rightarrow \infty$ so that (2.62) and (2.39) imply:

(2.72) $$\begin{aligned} [f(\,\cdot\,,B),B]_t = &\int_0^t f_x(s,B_s)\, I\big(B_s \neq b(s)\big)\, ds \\ &+ \lim_{n\rightarrow\infty} \frac{1}{2\varepsilon_n} \int_0^t \Big(f(s,b(s)+\varepsilon_n) - f(s,b(s)-\varepsilon_n)\Big)\, I\big(b(s)-\varepsilon_n < B_s < b(s)+\varepsilon_n\big)\, ds \end{aligned}$$

uniformly over $t \in [0,1]$ as a limit in probability.

17. By (2.40) we know that

(2.73) $$t \mapsto f(t, b(t)+) - f(t, b(t)-) \text{ is continuous on } [0,1] .$$

Adding and subtracting $f(s, b(s)+)$ as well as $f(s, b(s)-)$ under the second integral sign in (2.72) and using Lebesgue's theorem (to control the first and the third term), we are therefore led to establish the following representation:

(2.74) $$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t g(s)\, I\big(b(s) - \varepsilon < B_s < b(s) + \varepsilon\big)\, ds = \int_0^t g(s)\; d\ell_s^b$$

for a continuous function $g : [0,1] \to I\!R$.

For this, denote the left-hand side in (2.74) by L and note that

(2.75) $$\begin{aligned} L &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g(s)\, I\big(b(s) - \varepsilon < B_s < b(s) + \varepsilon\big)\, ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \sum_{i=1}^n g(s_i^\varepsilon) \int_{t_{i-1}}^{t_i} I\big(b(s) - \varepsilon < B_s < b(s) + \varepsilon\big)\, ds \\ &\le \sum_{i=1}^n \Big(\limsup_{\varepsilon \downarrow 0} g(s_i^\varepsilon) \Big) \Big(\ell_{t_i}^b - \ell_{t_{i-1}}^b \Big) \le \sum_{i=1}^n g(s_i^*) \Big(\ell_{t_i}^b - \ell_{t_{i-1}}^b \Big) \to \int_0^t g(s)\; d\ell_s^b \end{aligned}$$

for some s_i^ε and $s_i^* = \limsup_{\varepsilon \downarrow 0} s_i^\varepsilon$ from $[t_{i-1}, t_i]$ where $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ so that $\max_{1 \le i \le n} (t_i - t_{i-1}) \to 0$ as $n \to \infty$. In exactly the same way using a liminf instead of the limsup one derives the reverse inequality, and this establishes (2.74) for continuous g . (The identity then can also be extended further to measurable g by standard means.)

Applying (2.74) to $g(s) = f(s, b(s)+) - f(s, b(s)-)$ we see that (2.72) reads:

(2.76) $$[f(\,\cdot\,, B), B]_t = \int_0^t f_x(s, B_s)\, I\big(B_s \ne b(s)\big)\, ds + \int_0^t \Big(f(s, b(s)+) - f(s, b(s)-) \Big)\, d\ell_s^b .$$

Recalling finally that $f(t,x) = F_x(t,x)$ we see that (2.44) follows from (2.53) upon using (2.76) to identify the final term. This establishes (2.44) under (2.37)-(2.43).

18. Having established the formula (2.44) we can return to Subsection 9 above and note that with $(t,x) \in [0,1] \times I\!R$ given and fixed the function $F(s,y) = e^{-\lambda s} V_c(t+s, x+y)$ satisfies the conditions (2.37)-(2.43) under regularity conditions on G . It thus follows by (2.44) that

(2.77) $$\begin{aligned} e^{-\lambda s} V_c(t+s, B_{t+s}) \;&=\; V_c(t,x) \\ &+ \int_0^s e^{-\lambda u} \Big((V_c)_t + \tfrac{1}{2} (V_c)_{xx} - \lambda V_c \Big)(t+u, B_{t+u}) \cdot I\Big(B_{t+u} \ne c(t+u) \Big)\, du + M_s^c \\ &+ \frac{1}{2} \int_0^s e^{-\lambda u} \Delta_x (V_c)_x (t+u, c(t+u))\, d\ell_u^c \end{aligned}$$

where $M_s^c = \int_0^s e^{-\lambda u} (V_c)_x (t+u, B_{t+u})\, I(B_{t+u} \ne c(t+u))\, dB_u$ for $0 \le s \le 1-t$ is a continuous (local) martingale.

Setting $s = 1 - t$, using that $V_c(1,x) = G(1,x)$ and that V_c satisfies (2.12) and (2.16), upon taking the $P_{t,x}$ expectation in (2.77) we get:

$$(2.78)\qquad e^{-\lambda(1-t)}E_{t,x}\big(G(1,B_1)\big) \;=\; V_c(t,x)$$
$$+\int_0^{1-t} e^{-\lambda u}E_{t,x}\Big(\Big(G_t+\tfrac{1}{2}G_{xx}-\lambda G\Big)(t+u,B_{t+u})\cdot I\Big(B_{t+u}>c(t+u)\Big)\Big)\,du$$
$$+\,\frac{1}{2}\int_0^{1-t} e^{-\lambda u}\Delta_x(V_c)_x(t+u,c(t+u))\;d_uE_{t,x}(\ell_u^c)$$

for all $(t,x)\in[0,1]\times I\!R$.

Comparing this expression with (2.29), we see that the final term must be zero, i.e.

$$(2.79)\qquad \int_0^{1-t} e^{-\lambda u}\Delta_x(V_c)_x(t+u,c(t+u))\;d_uE_{t,x}(\ell_u^c)\;=\;0$$

for all $(t,x)\in[0,1]\times I\!R$. Hence it follows by a *uniqueness theorem for local times on curved boundaries* that $\Delta_x(V_c)_x(t,c(t))=0$, or in other words:

$$(2.80)\qquad x\mapsto (V_c)_x(t,x)\;\;\text{is continuous at}\;\;c(t)$$

for all $t\in[0,1]$.

More explicitly, this claim can be verified as follows. Using (2.74) it is easily established that (2.79) is equivalent to the fact that

$$(2.81)\qquad \int_t^1 \frac{h(s)}{\sqrt{s-t}}\,\varphi\Big(\frac{c(s)-x}{\sqrt{s-t}}\Big)\,ds\;=\;0$$

for all $(t,x)\in[0,1]\times I\!R$, where $\varphi(z)=(1/\sqrt{2\pi})\,e^{-z^2/2}$, and we set:

$$(2.82)\qquad h(s)\;=\;e^{-\lambda s}\Delta_x(V_c)_x(s,g(s))$$

for $s\in[t,1]$. With $t\in[0,1]$ given and fixed, the left-hand side in (2.81) can be rewritten as:

$$(2.83)\qquad -\frac{\partial}{\partial x}\Big(\int_t^1 h(s)\,\Phi\Big(\frac{c(s)-x}{\sqrt{s-t}}\Big)\,ds\Big)\;=\;0$$

where $\Phi(z)=\int_{-\infty}^z\varphi(y)\,dy$, and hence we find that

$$(2.84)\qquad \int_t^1 h(s)\,\Phi\Big(\frac{c(s)-x}{\sqrt{s-t}}\Big)\,ds\;=\;H(t)$$

for all $x\in I\!R$, where H is some function of t only. Letting x first to $+\infty$ in (2.84) we see that $H(t)$ must be zero; letting x then to $-\infty$ in (2.84) we see that

$$(2.85)\qquad \int_t^1 h(s)\,ds\;=\;0$$

for all $t\in[0,1]$, and hence $h=0$ on $[0,1]$, thus proving the claim.

Thus V_c satisfies (2.14) and as already remarked in Subsection 9 above, the rest of the proof then can be repeated word by word to show that $V_c=V$ and $c=b$.

19. In this way we have established the following result.

Theorem 2.1
Under the regularity assumptions imposed above, the optimal stopping boundary b from (2.3) is characterized as a unique solution of the instantaneous-stopping equation (2.27).

It is a complicated statement which is best tested by specific examples. The equation (2.27) is a *nonlinear Volterra integral equation of the second kind.* The result establishes a basic connection between the class of these equations with analytic methods (fixed-point theorems) on one side and the optimal stopping problem (2.2) with probabilistic methods on the other.

3. Examples

In this section we study two concrete examples of the optimal stopping problem (2.1). It is of interest to continue this list of examples and classify the equations obtained.

1. As already mentioned in the introduction, the optimal stopping boundary for the problem (2.1) is rarely know explicitly and we need to use numerical methods to get out numbers. The following simple method can be used to calculate the optimal boundary b numerically by means of the integral equation (2.27). The method will be applicable in most situations but better methods can of course be developed in each specific example.

Set $t_i = ih$ for $i = 0, 1, \ldots, n$ where $h = 1/n$ and denote:

$$J(t, b(t)) = e^{-\lambda(1-t)} E\Big(G\big(1, b(t) + \sqrt{1-t}\, B_1\big)\Big) - G\big(t, b(t)\big) \tag{3.1}$$

$$K(t, b(t); t+u, b(t+u)) = E\Big(H(t+u, b(t) + \sqrt{u}\, B_1)\, I\big(B_1 > (b(t+u) - b(t))/\sqrt{u}\big)\Big) \tag{3.2}$$

where $H = G_t + \frac{1}{2} G_{xx} - \lambda G$ and $B_1 \sim N(0,1)$ so that these functions are easily computed. Then the following discrete approximation of the integral equation (2.27) is valid:

$$J(t_i, b(t_i)) = \sum_{j=i}^{n-1} e^{-\lambda(t_{j+1}-t_i)}\, K\big(t_i, b(t_i); t_{j+1}, b(t_{j+1})\big) \cdot h \tag{3.3}$$

for $i = 0, 1, \ldots, n-1$. Setting $i = n-1$ and $b(t_n) = \gamma(1)$ with γ from (2.7)-(2.9) above, we can solve the equation (3.3) numerically and get a number $b(t_{n-1})$. Setting $i = n-2$ and using the values of $b(t_{n-1})$ and $b(t_n)$, we can solve (3.3) numerically and get a number $b(t_{n-2})$. Continuing the recursion we obtain $b(t_n), b(t_{n-1}), \ldots, b(t_1), b(t_0)$ as an approximation of the optimal boundary b at points $1, 1-1/n, \ldots, 1/n, 0$.

Example 3.1
Let $G(t, x) = x$ and $\lambda > 0$. Then the optimal stopping problem (2.1) reads as follows:

$$V = \sup_{0 \le \tau \le 1} E\big(e^{-\lambda\tau} B_\tau\big) \ . \tag{3.4}$$

Note that $\gamma(t) = 0$ for all $t \in [0, 1]$ and the conditions (2.4)-(2.9) are satisfied. Simple probabilistic arguments show that the stopping time τ_b from (2.3) with b from the class $\mathcal{C}$ is optimal.

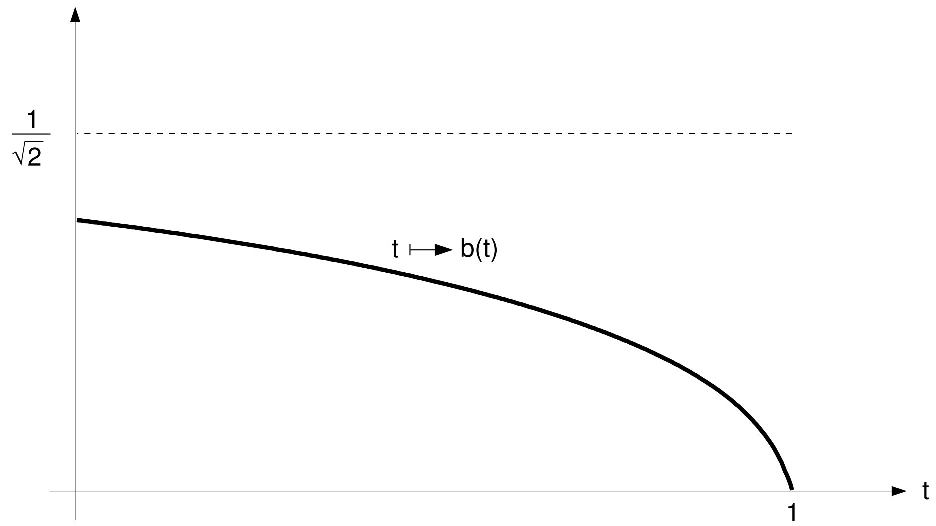

Figure 1. A computer drawing of the optimal stopping boundary in Example 3.1 calculated by the numerical method based on (3.3) with $n = 400$.

The integral equation (2.27) in this example takes the form:

$$\int_0^{1-t} \sqrt{u}\, e^{-\lambda u} \varphi\Big(\frac{b(t+u)-b(t)}{\sqrt{u}}\Big)\, du \;=\; b(t) \int_0^{1-t} e^{-\lambda u}\, \Phi\Big(\frac{b(t+u)-b(t)}{\sqrt{u}}\Big)\, du \tag{3.5}$$

for $t \in [0,1]$ where $\varphi(x) = (1/\sqrt{2\pi})\, e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(z)\, dz$ for $x \in I\!R$. By the result above we know that b is the unique solution of this equation in the class $\mathcal{C}$.

Since it does not seem possible to find a simple explicit expression for the solution of this equation, we have used the numerical method suggested above and the result for $\lambda = 1$ is shown in *Figure 1* above. Using moreover that the stopping time:

$$\tau_* = \inf \{\, t > 0 \mid B_t = 1/\sqrt{2}\, \} \tag{3.6}$$

is optimal for the infinite-horizon problem (when the supremum in (3.4) is taken over all stopping times), we see that the optimal stopping boundary must satisfy $0 < b(t) < 1/\sqrt{2}$ for all $0 < t < 1$.

Example 3.2

Let $G(t,x) = (1-t)\,x$ and $\lambda = 0$. Then the optimal stopping problem (2.1) reads as follows:

$$V = \sup_{0 \le \tau \le 1} E\big((1-\tau) B_\tau\big) \; . \tag{3.7}$$

Note that $\gamma(t) = 0$ for all $t \in [0,1]$ and the conditions (2.4)-(2.9) are satisfied. This example is one of the very few where the optimal stopping boundary b can be determined explicitly.

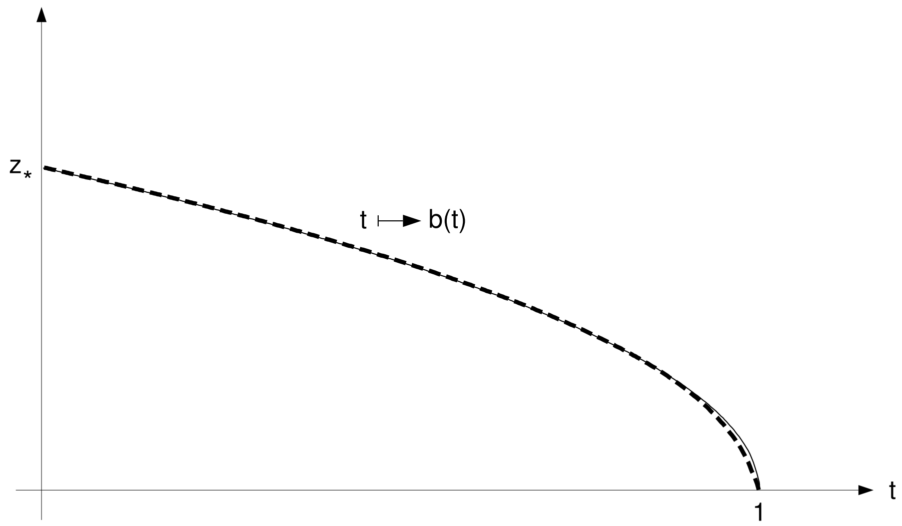

Figure 2. A computer drawing of the optimal stopping boundary in Example 3.2. The full line is the actual stopping boundary $t \mapsto z_* \sqrt{1-t}$ and the dashed line is the approximating boundary calculated by the numerical method based on (3.3) with $n = 400$.

For example, by the method of time-change (see [12]) it is possible to verify that the optimal stopping time in (3.7) is given by

$$\tau_* = \inf \{ \, 0 \le t \le 1 \mid B_t \ge z_* \sqrt{1-t} \, \} \tag{3.8}$$

where z_* is the root of the equation:

$$z_* \, D_{-3}(-z_*) = D_{-4}(-z_*) \tag{3.9}$$

and D_{-n} is a parabolic cylinder function given by

$$D_{-n}(z) = \frac{e^{-z^2/2}}{\Gamma(n)} \int_0^\infty y^{n-1} e^{-zy-y^2/2} \, dy \tag{3.10}$$

for $n \ge 1$. A numerical calculation shows that $z_* = 0.63\ldots$ which suggests that there is a misprint in the formula of the main theorem in [10] where the problem (3.7) was initially solved.

The integral equation (2.27) in this example takes the form:

$$\int_0^{1-t} \sqrt{u} \; \varphi\Big(\frac{b(t+u)-b(t)}{\sqrt{u}} \Big) \, du \; = \; b(t) \int_0^{1-t} \Phi\Big(\frac{b(t+u)-b(t)}{\sqrt{u}} \Big) \, du \tag{3.11}$$

for $t \in [0,1]$ where φ and Φ are defined following (3.5) above. By the result above we know that $b(t) = z_* \sqrt{1-t}$ is the unique solution of this equation in the class $\mathcal{C}$.

It can also be verified directly that the boundary $b(t) = z_* \sqrt{1-t}$ solves the integral equation, but it is not straightforward. For the purpose of comparison, we have used the numerical method suggested above and the result is shown in *Figure 2* above. This example illustrates that while it is very difficult to solve the integral equation (2.27) explicitly (even if we know the solution), the numerical method seems to work satisfactorily. The discrepancy of the two solutions around the point 1 on *Figure 2* is caused by a singular behaviour of the optimal boundary at that point. This anomaly requires a finer mesh than just uniform in order to achieve higher precision. Similar refinements apply quite generally to the same effect.

Acknowledgment. The second author gratefully acknowledges financial support from the Forschungsinstitut für Mathematik, ETH Zürich.

REFERENCES

[1] DYNKIN, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Math. Dokl.* 4 (627-629).

[2] FÖLLMER, H., PROTTER, P. *and* SHIRYAYEV, A. N. (1995). Quadratic covariation and an extension of Itô's formula. *Bernoulli* 1 (149-169).

[3] FRIEDMAN, A. (2000). Free boundary problems in science and technology. *Notices Amer. Math. Soc.* 47 (854-861).

[4] GRIGELIONIS, B. I. *and* SHIRYAEV, A. N. (1966). On Stefan's problem and optimal stopping rules for Markov processes. *Theory Probab. Appl.* 11 (541-558).

[5] JACKA, S. D. (1991). Optimal stopping and the American put. *Math. Finance* 1 (1-14).

[6] JACKA, S. D. *and* LYNN, J. R. (1992). Finite-horizon optimal stopping, obstacle problems and the shape of the continuation region. *Stochastics Stochastics Rep.* 39 (25-42).

[7] KARATZAS, I. *and* SHREVE, S. E. (1998). *Methods of Mathematical Finance*. Springer-Verlag, New York.

[8] MCKEAN, H. P. Jr. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Ind. Management Rev.* 6 (32-39).

[9] MIKHALEVICH, V. S. (1958). A Bayes test of two hypotheses concerning the mean of a normal process. *Visnik Kiiv. Univ.* No. 1 (Ukrainian) (101-104).

[10] MIROSHNICHENKO, T. P. (1975). Optimal stopping of the integral of a Wiener process. *Theory Probab. Appl.* 20 (387-391).

[11] MYNENI, R. (1992). The pricing of the American option. *Ann. Appl. Probab.* 2 (1-23).

[12] PEDERSEN, J. L. *and* PESKIR, G. (1997). Solving non-linear optimal stopping problems by the method of time-change. *Research Report* No. 390, *Dept. Theoret. Statist. Aarhus* (24 pp). *Stochastic Anal. Appl.* 18, 2000 (811-835).

[13] PESKIR, G. (2001). On integral equations arising in the first-passage problem for Brownian motion. *Research Report* No. 421, *Dept. Theoret. Statist. Aarhus* (20 pp). *J. Integral Equations Appl.* 14, 2002 (397-423).

[14] REVUZ, D. *and* YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer-Verlag, Berlin.

[15] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules.* Springer-Verlag, New York.

[16] SNELL, J. L. (1952). Applications of martingale system theorems. *Trans. Amer. Math. Soc.* 73 (293-312).

[17] VAN MOERBEKE, P. (1976). On optimal stopping and free boundary problems. *Arch. Rational Mech. Anal.* 60 (101-148).

Jesper Lund Pedersen
RiskLab, Department of Mathematics
ETH-Zentrum, CH-8092 Zürich
Switzerland
www.math.ethz.ch/∼pedersen
pedersen@math.ethz.ch

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk