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Bounding the Maximal Height of a Diffusion by the Time Elapsed

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Let $X = (X_t)_{t\geq 0}$ be a one-dimensional time-homogeneous diffusion process associated with the infinitesimal generator

$$\mathbb{L}_X = \mu(x)\, \frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\, \frac{\partial^2}{\partial x^2}$$

where $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ are continuous. We show how the question of finding a function $x \mapsto H(x)$ such that

$$c_1 E\big(H(\tau)\big) \leq E\Big(\max_{0\leq t\leq \tau} |X_t|\Big) \leq c_2 E\big(H(\tau)\big)$$

holds for all stopping times τ of X relates to solutions of the equation:

$$\mathbb{L}_X(F) = 1 \ .$$

Explicit expressions for H are derived in terms of μ and σ. The method of proof relies upon a domination principle established by Lenglart and Itô calculus.

1. Introduction

If $B = (B_t)_{t\geq 0}$ is a *standard Brownian motion*, then the famous *Burkholder-Gundy inequality* [1] states that there exist universal constants $a_1 > 0$ and $a_2 > 0$ such that

$$a_1 E\big(\sqrt{\tau}\big) \leq E\Big(\max_{0\leq t\leq \tau} |B_t|\Big) \leq a_2 E\big(\sqrt{\tau}\big) \tag{1.1}$$

for all stopping times τ of B. It is well-known that this results extends to a large class of martingales or submartingales (see e.g. [6] pp.153-163) sometimes creating a false picture that the martingale property of the process is indispensable.

Recently it was shown in [2] that if $V = (V_t)_{t\geq 0}$ is the Ornstein-Uhlenbeck velocity process solving the *Langevin* stochastic differential equation

$$dV_t = -\beta\, V_t\, dt + dB_t \tag{1.2}$$

where $\beta > 0$ and $V_0 = 0$, then there exist universal constants $b_1 > 0$ and $b_2 > 0$ such that

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(1.3) $$\frac{b_1}{\sqrt{\beta}}\, E\Big(\sqrt{\log(1+\beta\tau)}\Big) \le E\bigg(\max_{0\le t\le\tau}|V_t|\bigg) \le \frac{b_2}{\sqrt{\beta}}\, E\Big(\sqrt{\log(1+\beta\tau)}\Big)$$

for all stopping times τ of V. While the inequality (1.1) can be obtained from (1.3) by passing to the limit when $\beta \downarrow 0$, the process V is far removed from a martingale.

Our main aim in this paper is to show that the preceding result about the Ornstein-Uhlenbeck velocity process is not just an isolated example but admits extensions to quite general diffusions. More specifically, assuming that $X = (X_t)_{t\ge0}$ is a diffusion process solving the stochastic differential equation (2.1) driven by a standard Brownian motion, we explain how the problem of finding a function $x \mapsto H(x)$ such that

(1.4) $$c_1 E\big(H(\tau)\big) \le E\bigg(\max_{0\le t\le\tau}|X_t|\bigg) \le c_2 E\big(H(\tau)\big)$$

holds for all stopping times τ of X with some universal constants $c_1 > 0$ and $c_2 > 0$, can be naturally related to solutions of the equation

(1.5) $$\mathbb{L}_X(F) = 1$$

where $\mathbb{L}_X$ is the infinitesimal generator of X given in (2.2).

The main result of the paper is contained in Theorems 2.3-2.5. The list of examples started with Example 2.6 and 2.7 is easily continued (see also [5]).

2. The results and proof

Let $X = (X_t)_{t\ge0}$ be a one-dimensional time-homogeneous diffusion process solving the stochastic differential equation

(2.1) $$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t$$

where $B=(B_t)_{t\ge0}$ is a standard Brownian motion (see e.g. [3] or [6]). We will assume that the drift coefficient $x \mapsto \mu(x)$ and diffusion coefficient $x \mapsto \sigma(x) > 0$ are continuous.

For further reference we recall that the *infinitesimal generator* of X is given by

(2.2) $$\mathbb{L}_X = \mu(x)\,\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\,\frac{\partial^2}{\partial x^2}\ .$$

The *scale function* of X is given by

(2.3) $$S(x) = \int_0^x \exp\bigg(-\int_0^y \frac{2\mu(z)}{\sigma^2(z)}\, dz\bigg)\, dy\ .$$

The *speed measure* of X is given by

(2.4) $$m(dx) = \frac{2\, dx}{S'(x)\sigma^2(x)}\ .$$

For more information on these characteristics of a diffusion see e.g. [6].

1. The proof of the theorems below is based upon the following *domination principle* which was initially proved in [4] when $H(x) = x^p$ for $0 < p < 1$. The extension to more general functions $x \mapsto H(x)$ given in the next lemma follows along the same lines and can be found in [6] (p.155-156). We present the proof for completeness.

Lemma 2.1 (Lenglart)
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space, let $Z = (Z_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted non-negative continuous process, let $A = (A_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted increasing continuous process satisfying $A_0 = 0$, and let $H : \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing continuous function satisfying $H(0) = 0$. Suppose that it is known that

$$E(Z_\tau) \leq E(A_\tau) \tag{2.5}$$

for all bounded $(\mathcal{F}_t)$-stopping times τ such that $\big(Z_{t\wedge\tau}\big)_{t\geq 0}$ is bounded. Then we have:

$$E\Big(\sup_{0\leq t\leq\tau} H(Z_t)\Big) \leq E\Big(\widetilde{H}(A_\tau)\Big) \tag{2.6}$$

for all $(\mathcal{F}_t)$-stopping times τ , where

$$\widetilde{H}(x) = x\int_x^\infty \frac{1}{s}\, dH(s) + 2H(x) \tag{2.7}$$

for all $x \geq 0$.

Proof. By Fubini's theorem we find:

$$\begin{aligned} E\Big(\sup_{0\leq t\leq\tau} H(Z_t)\Big) &= E\Big(H\Big(\sup_{0\leq t\leq\tau} Z_t\Big)\Big) = E\Big(\int_0^\infty 1_{\{\sup_{0\leq t\leq\tau} Z_t\, \geq\, s\}}\, dH(s)\Big) \\ &\leq \int_0^\infty \Big(P\Big\{\sup_{0\leq t\leq\tau} Z_t \geq s\,,\ A_\tau \leq s\Big\} + P\Big\{A_\tau > s\Big\}\Big)\, dH(s) \end{aligned} \tag{2.8}$$

since $s \mapsto H(s)$ is increasing and continuous. Consider the stopping times

$$\begin{aligned} \tau_1 &= \inf\,\{\, t\geq 0 \mid Z_t \geq s\,\} \\ \tau_2 &= \inf\,\{\, t\geq 0 \mid A_t \geq s\,\}\ . \end{aligned} \tag{2.9}$$

Then Markov's inequality and (2.5) imply:

$$\begin{aligned} P\Big\{\sup_{0\leq t\leq\tau} Z_t \geq s\,,\ A_\tau \leq s\Big\} &\leq P\Big\{\,\tau_1 \leq \tau\,,\ \tau_2 \geq \tau\,\Big\} \leq P\Big\{\, Z_{\tau_1\wedge\tau_2\wedge\tau} \geq s\,\Big\} \\ &\leq \frac{1}{s}\, E\big(A_{\tau_1\wedge\tau_2\wedge\tau}\big) \end{aligned} \tag{2.10}$$

whenever τ is bounded. From (2.8) and (2.10) we conclude:

$$E\Big(\sup_{0\leq t\leq\tau} H(Z_t)\Big) \leq \int_0^\infty \Big(\frac{1}{s}\, E\Big(A_\tau\, 1_{\{A_\tau \leq s\}}\Big) + 2P\Big\{A_\tau > s\Big\}\Big)\, dH(s) \tag{2.11}$$

$$= E\Big(A_\tau \int_{A_\tau}^\infty \frac{1}{s}\; dH(s)\Big) + 2E\Big(H(A_\tau)\Big) = E\Big(\widetilde{H}(A_\tau)\Big)$$

for all bounded τ . Finally, observe that $x \mapsto \widetilde{H}(x)$ is increasing, and pass to the limit when $k \to \infty$ to reach any τ through bounded ones $\tau \wedge k$. This completes the proof. □

Remark 2.2

If $H(x) = x^p$ with $0<p<1$, then $\widetilde{H}(x) = ((2-p)/(1-p))\, x^p$; if $H(x) = x$, then $\widetilde{H}(x) \equiv +\infty$, and the bound on the right-hand side in (2.6) is non-interesting. Generally, the right-hand side in (2.6) gives a non-trivial bound if $H(x)$ tends to infinity as slow as x^p for some $0<p<1$; the bound is better (asymptotically optimal) if the error in (2.5) is smaller (negligible).

2. We first treat a symmetric case when the drift coefficient $x \mapsto \mu(x)$ is odd and the diffusion coefficient $x \mapsto \sigma(x)$ is even.

Theorem 2.3

Let $X = (X_t)_{t\geq 0}$ be a one-dimensional time-homogeneous diffusion process solving (2.1) with $X_0 = 0$ under P , where $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ are continuous and satisfy:

$$\mu(-x) = -\mu(x) \quad \textit{and} \quad \sigma(-x) = \sigma(x) \tag{2.12}$$

for all $x \geq 0$. Let the map F be defined by

$$F(x) = \int_0^x m\big((0,z]\big)\, S'(z)\; dz \tag{2.13}$$

for $x \geq 0$, where $S = S(x)$ is the scale function of X given in (2.3), and $m = m(dx)$ is the speed measure of X given in (2.4).

Suppose that the following condition is satisfied:

$$\sup_{x>0} \left(\frac{F(x)}{x} \int_x^\infty \frac{dz}{F(z)} \right) < \infty \tag{2.14}$$

and let $H(x) = F^{-1}(x)$ denote the inverse of F for $x \geq 0$ (both being strictly increasing).

Then there exist universal constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 E\big(H(\tau)\big) \leq E\Big(\max_{0\leq t\leq\tau} |X_t| \Big) \leq c_2 E\big(H(\tau)\big) \tag{2.15}$$

for all stopping times τ of X .

Proof. If $x \mapsto G(x)$ is C^2 everywhere on $\mathbb{R}$ but at 0 where it is C^1 , then Itô formula (see e.g. [6]) can be applied to $G(X_t)$ and this yields:

$$G(X_t) = G(X_0) + \int_0^t \big(\mathbb{L}_X G\big)(X_s)\, ds + \int_0^t G'(X_s)\, \sigma(X_s)\, dB_s \tag{2.16}$$

where $\mathbb{L}_X$ is given in (2.2). Setting in this representation:

$$(2.17) \qquad M_t = \int_0^t G'(X_s)\, \sigma(X_s)\, dB_s$$

for $t \geq 0$, it is well-known that $M = (M_t)_{t \geq 0}$ is a continuous local martingale (see e.g. [6]).

Motivated by these facts consider the *initial value problem*:

$$(2.18) \qquad \mathbb{L}_X(G) = 1$$

$$(2.19) \qquad G(0) = G'(0) = 0 \; .$$

The general solution of (2.18) is given by

$$(2.20) \qquad G(x) = \int_0^x e^{-A(u)} \Big(C_1 + \int_0^u b(v)\, e^{A(v)}\, dv \Big) du + C_2$$

for $x \in \mathbb{R}$ where $A(x) = \int_0^x a(t)\, dt$ with $a(x) = 2\mu(x)/\sigma^2(x)$ and $b(x) = 2/\sigma^2(x)$. The initial conditions (2.19) imply that $C_1 = C_2 = 0$. Using (2.3) and (2.4) it is then easily verified that the unique solution $G = G(x)$ of the problem (2.18)-(2.19) is given by (2.13) for $x \geq 0$, and due to (2.12) we have $G(x) = G(-x)$ for $x < 0$. Thus G is even on $\mathbb{R}$, and therefore

$$(2.21) \qquad G(X_t) = F\big(|X_t|\big)$$

for all $t \geq 0$ where F is given in (2.13). Observe also that G is strictly increasing on $\mathbb{R}_+$.

Let τ be a bounded stopping time of X such that $\big(G(X_{t \wedge \tau})\big)_{t \geq 0}$ is bounded. Passing to a localising sequence of stopping times for M if needed, we find from (2.16)-(2.19) that

$$(2.22) \qquad E\big(G(X_\tau)\big) = E(\tau)$$

by means of the optional sampling theorem (see e.g. [6]).

Recalling that H denotes the inverse of F on $\mathbb{R}_+$, it is a matter of routine to verify that under (2.14) we can find a constant $c > 0$ such that

$$(2.23) \qquad x \int_x^\infty \frac{dH(s)}{s} \leq c\, H(x)$$

for all $x > 0$. This fact taken together with (2.7) implies:

$$(2.24) \qquad \widetilde{H}(x) \leq d\, H(x)$$

for all $x \geq 0$ where $d = (c+2)$.

Setting $Z_t = G(X_t)$ and $A_t = t$ we see by (2.22) that all hypotheses in Lemma 2.1 are satisfied, and thus by (2.6) we find:

$$(2.25) \qquad E\Big(\max_{0 \leq t \leq \tau} |X_t| \Big) = E\Big(\max_{0 \leq t \leq \tau} H\big(Z_t\big) \Big) \leq E\Big(\widetilde{H}(A_\tau)\Big) \leq d\, E\Big(H(A_\tau)\Big) = d\, E\Big(H(\tau)\Big)$$

by means of (2.21) and (2.24). This establishes the right-hand side inequality in (2.15).

On the other hand, setting $Z_t = t$ and $A_t = \max_{0 \leq s \leq t} G(X_s)$ we see by (2.22) that all hypotheses in Lemma 2.1 are satisfied, and thus by (2.6) we find:

$$(2.26)\qquad E\Big(H(\tau)\Big) = E\Big(\max_{0\le t\le \tau} H(Z_t)\Big) \le E\Big(\widetilde{H}(A_\tau)\Big) \le d\,E\Big(H(A_\tau)\Big) = d\,E\Big(\max_{0\le t\le \tau} |X_t|\Big)$$

by means of (2.24) and (2.21). This establishes the left-hand side inequality in (2.15), and the proof is complete. □

3. If the symmetry condition (2.12) is not present, then the analogue of the preceding result can be established but the method does not produce the same function H on both sides in (2.15). This is due to the fact that the solution of the initial value problem (2.18)-(2.19) is no longer even.

Theorem 2.4

Let $X=(X_t)_{t\ge 0}$ be a one-dimensional time-homogeneous diffusion process solving (2.1) with $X_0=0$ under P, where $x\mapsto \mu(x)$ and $x\mapsto \sigma(x)>0$ are continuous.

Let the map F be defined by

$$(2.27)\qquad F(x) = \int_0^x m\big((0,z]\big)\, S'(z)\, dz$$

for $x\in I\!R$, where $S=S(x)$ is the scale function of X given in (2.3), and $m=m(dx)$ is the speed measure of X given in (2.4). Define the maps F_1 and F_2 as follows:

$$(2.28)\qquad F_1(x) = F(-x)\vee F(x)$$

$$(2.29)\qquad F_2(x) = F(-x)\wedge F(x)$$

for $x\ge 0$, and let $H_i(x)=F_i^{-1}(x)$ denote the inverse of F_i for $x\ge 0$ with $i=1,2$.

Suppose that the following condition is satisfied:

$$(2.30)\qquad \sup_{x>0}\left(\frac{F_i(x)}{x}\int_x^\infty \frac{dz}{F_i(z)}\right) < \infty$$

for $i=1,2$. Then there exist universal constants $c_1>0$ and $c_2>0$ such that

$$(2.31)\qquad c_1 E\big(H_1(\tau)\big) \le E\Big(\max_{0\le t\le \tau}|X_t|\Big) \le c_2 E\big(H_2(\tau)\big)$$

for all stopping times τ of X.

Proof. Consider the initial value problem (2.18)-(2.19). Then in exactly the same way as in the proof of Theorem 2.3 we find that the unique solution $G=G(x)$ of this problem is given by (2.27) for $x\in I\!R$. Observe that the map is strictly decreasing on $I\!R_-$ and strictly increasing on $I\!R_+$. Thus both maps F_1 and F_2 given in (2.28)-(2.29) are strictly increasing on $I\!R_+$, and the same is true for their inverses H_1 and H_2.

In exactly the same way as in the proof of Theorem 2.3 we then find that (2.22) holds. Similarly, the condition (2.30) implies that (2.23) and (2.24) are valid with H being replaced by H_1 and H_2 respectively. Noting further that $H_1(G(x))\le |x|\le H_2(G(x))$ for all $x\in I\!R$, we find:

$$(2.32)\qquad E\Big(\max_{0\le t\le \tau} H_1\big(G(X_t)\big)\Big) \le E\Big(\max_{0\le t\le \tau}|X_t|\Big) \le E\Big(\max_{0\le t\le \tau} H_2\big(G(X_t)\big)\Big)$$

for all τ . The proof is then easily completed by establishing the following two inequalities:

$$E\Big(\max_{0\le t\le\tau} H_2\big(G(X_t)\big)\Big) \le d\,E\Big(H_2(\tau)\Big) \tag{2.33}$$

$$E\Big(H_1(\tau)\Big) \le d\,E\Big(\max_{0\le t\le\tau} H_1\big(G(X_t)\big)\Big) \tag{2.34}$$

that can be done using the same arguments as in (2.25) and (2.26) respectively.

□

4. If the diffusion X from the preceding theorem starts at a point x_0 different from zero, the result is still valid if one replaces the infinitesimal characteristics $\mu = \mu(x)$ and $\sigma = \sigma(x)$ in the statement by the new ones $\widetilde{\mu}(x) = \mu(x+x_0)$ and $\widetilde{\sigma}(x) = \sigma(x+x_0)$ respectively. Moreover, in this case a simplification can be achieved both in the statement and proof if one is only interested in giving the upper bound. The result is especially applicable to non-negative diffusions when there is also no restriction to allow that the diffusion coefficient takes value 0 at 0 .

Theorem 2.5

Let $X = (X_t)_{t\ge0}$ be a one-dimensional time-homogeneous diffusion process solving (2.1) with $X_0 = x_0$ under P , where $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ are continuous.

Let the map F be defined by

$$F(x) = \int_{x_0}^{x} m\big((x_0,z]\big)\, S'(z)\; dz \tag{2.35}$$

for $x \ge x_0$, where $S = S(x)$ is the scale function of X given by (2.3) with x_0 instead of 0 in the two integrals, and $m = m(dx)$ is the speed measure of X given in (2.4).

Suppose that the following condition is satisfied:

$$\sup_{x>x_0}\left(\frac{F(x)}{x}\int_x^\infty \frac{dz}{F(z)}\right) < \infty \tag{2.36}$$

and let $H(x) = F^{-1}(x)$ denote the inverse of F for $x \ge x_0$ (both being strictly increasing).

Then there exists a universal constant $c > 0$ such that

$$E\Big(\max_{0\le t\le\tau} X_t\Big) \le c\,E\big(H(\tau)\big) \tag{2.37}$$

for all stopping times τ of X .

Proof. Motivated by (2.16) consider the initial value problem:

$$\mathbb{L}_X(G) = 1 \tag{2.38}$$

$$G(x_0) = G'(x_0) = 0\ . \tag{2.39}$$

Then in exactly the same way as in the proof of Theorem 2.3 we find that the unique solution $G = G(x)$ of this problem is given by (2.35) for $x \ge x_0$. Extend this solution to points smaller than x_0 by setting $G(x) = 0$ for $x < x_0$. Then from (2.16) we see that

(2.40) $$G(X_t) \le t + M_t$$

for all $t \ge 0$, where $M = (M_t)_{t\ge 0}$ is a continuous local martingale given in (2.17).

In exactly the same way as in (2.22) we then find that

(2.41) $$E\big(G(X_\tau)\big) \le E(\tau)$$

for all bounded stopping times τ of X such that $\big(G(X_{t\wedge\tau})\big)_{t\ge 0}$ is bounded. Noting further that $x \le H(G(x))$ for all $x \in I\!R$, we find that

(2.42) $$E\Big(\max_{0\le t\le\tau} X_t\Big) \le E\Big(\max_{0\le t\le\tau} H\big(G(X_t)\big)\Big)$$

for all τ . The proof is then easily completed by establishing the following inequality:

(2.43) $$E\Big(\max_{0\le t\le\tau} H\big(G(X_t)\big)\Big) \le d\,E\Big(H(\tau)\Big)$$

that can be done using the same arguments as in (2.25).

□

5. A real scope of the preceding results can be better understood by considering a few examples. We shall begin by noting that Theorem 2.3 applies in the setting of (1.1) and (1.3) respectively.

In the case of (1.1) the diffusion X equals B , and from (2.13) we find that

(2.44) $$F(x) = x^2$$

for $x \ge 0$. The condition (2.14) is verified straightforwardly, and therefore (1.1) follows from (2.15) upon noting that $H(x) = F^{-1}(x) = \sqrt{x}$ for $x \ge 0$.

Similarly, in the case of (1.2) the diffusion X equals V , and from (2.13) we find that

(2.45) $$F(x) = 2\int_0^x e^{\beta y^2} \int_0^y e^{-\beta z^2}\,dz\,dy$$

for $x \ge 0$. A successive application of L'Hospital's rule then shows that the condition (2.14) holds, and (1.3) follows from (2.15) upon estimating the inverse $H = F^{-1}$.

Example 2.6 (Extending (1.1) to a Brownian motion with drift)

Let $X_t = B_t - \mu t$ for $t \ge 0$ where $\mu \ge 0$ is given and fixed. Then $dX_t = -\mu\,dt + dB_t$ and from (2.35) with $x_0 = 0$ we find that

(2.46) $$F_\mu(x) = \frac{e^{2\mu x} - 2\mu x - 1}{2\mu^2}$$

for $x \ge 0$. The condition (2.36) is then easily verified, and thus (2.37) implies:

(2.47) $$E\Big(\max_{0\le t\le\tau} (B_t - \mu t)\Big) \le c\,E\big(H_\mu(\tau)\big)$$

for all stopping times τ of B , where $H_\mu(x) = F_\mu^{-1}(x)$ denotes the inverse of F_μ for $x \ge 0$.

Observe that $F_\mu(x) \to x^2$ and $H_\mu(x) \to \sqrt{x}$ as $\mu \downarrow 0$ (cf. (2.44) above). Note also that (2.37) fails if $\mu < 0$. This indicates a typical limitation of this condition.

Example 2.7 (Branching diffusion)
Consider a simple branching diffusion solving

$$dX_t = \mu X_t\, dt + \sigma \sqrt{X_t}\, dB_t \tag{2.48}$$

with $X_0 = x_0 > 0$ under P , where $\mu \in I\!R$ and $\sigma > 0$. From (2.35) we find that

$$F(x) = \kappa \int_{x_0}^{x} e^{\lambda y} \int_{x_0}^{y} \frac{e^{-\lambda z}}{z}\; dz\; dy \tag{2.49}$$

for $x \ge x_0$, where $\kappa = 2/\sigma^2$ and $\lambda = -2\mu/\sigma^2$. Applying successively L'Hospital's rule it is then possible to verify that (2.36) holds if and only if $\mu < 0$. In this case (2.37) implies:

$$E\Big(\max_{0 \le t \le \tau} X_t \Big) \le c\, E\big(H(\tau)\big) \tag{2.37}$$

for all stopping times τ of X , where $H(x) = F^{-1}(x)$ denotes the inverse of F for $x \ge x_0$.

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