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# Three-Dimensional Brownian Motion and the Golden Ratio Rule

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Let $X = (X_t)_{t \geq 0}$ be a transient diffusion process in $(0, \infty)$ with the diffusion coefficient $\sigma > 0$ and the scale function $L$ such that $X_t \to \infty$ as $t \to \infty$, let $I_t$ denote its running minimum for $t \geq 0$, and let $\theta$ denote the time of its ultimate minimum $I_\infty$. Setting $c(i,x) = 1 - 2L(x)/L(i)$ we show that the stopping time

$$\tau_* = \inf \left\{ t \geq 0 \mid X_t \geq f_*(I_t) \right\}$$

minimises $\mathsf{E}(|\theta - \tau| - \theta)$ over all stopping times $\tau$ of $X$ (with finite mean) where the optimal boundary $f_*$ can be characterised as the minimal solution to

$$f'(i) = -\frac{\sigma^2(f(i))\, L'(f(i))}{c(i, f(i))\, [L(f(i)) - L(i)]} \int_i^{f(i)} \frac{c_i'(i,y)\, [L(y) - L(i)]}{\sigma^2(y)\, L'(y)}\, dy$$

staying strictly above the curve $h(i) = L^{-1}(L(i)/2)$ for $i > 0$. In particular, when $X$ is the radial part of three-dimensional Brownian motion, we find that

$$\tau_* = \inf \left\{ t \geq 0 \mid \frac{X_t - I_t}{I_t} \geq \varphi \right\}$$

where $\varphi = (1+\sqrt{5})/2 = 1.61\ldots$ is the golden ratio. The derived results are applied to problems of optimal trading in the presence of bubbles where we show that the golden ratio rule offers a rigourous optimality argument for the choice of the well-known golden retracement in technical analysis of asset prices.

## 1. Introduction

The *golden ratio* has fascinated people of diverse interests for at least 2,400 years (see e.g. [24]). In mathematics (and the arts) two quantities $a$ and $b$ are in the golden ratio if the ratio of the sum of the quantities $a+b$ to the larger quantity $a$ is equal to the ratio of the larger quantity $a$ to the smaller quantity $b$. This amounts to setting $(a+b)/a = a/b =: \varphi$ and solving $\varphi^2 - \varphi - 1 = 0$ which yields $\varphi = (1+\sqrt{5})/2 = 1.61\ldots$ Apart from being abundant in nature, and finding diverse applications ranging from architecture to music, the golden ratio has also found more recent uses in *technical analysis* of asset prices (in strategies such as *Fibonacci retracement* representing an ad-hoc method for determining *support* and *resistance* levels). Despite its universal presence and canonical role in diverse applied areas, we

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are not aware of any more *exact* connections between the golden ratio and *stochastic processes* (including any proofs of optimality in particular).

One of the aims of the present paper is to disclose the appearance of the golden ratio in an optimal stopping strategy related to the radial part of three-dimensional Brownian motion. More specifically, denoting the radial part by $X$ it is well known that $X$ is transient in the sense that $X_t \to \infty$ as $t \to \infty$. After starting at some $x > 0$ the ultimate minimum of $X$ will therefore be attained at some time $\theta$ that is not predictable through the sequential observation of $X$ (in the sense that it is only revealed at the end of time). The question we are addressing is to determine a (predictable) stopping time of $X$ that is as close as possible to $\theta$. We answer this question by showing that *the time at which the excursion of* $X$ *away from the running minimum* $I$ *and the running minimum* $I$ *itself form the golden ratio* is as close as possible to $\theta$ in a normalised mean deviation sense. We consider this problem by embedding it into transient Bessel processes of dimension $d > 2$ and in this context we derive similar optimal stopping rules. We also disclose further/deeper extensions of these results to transient diffusion processes. The relevance of these questions in financial applications is motivated by the problem of optimal trading in the presence of bubbles. In this context we show that the golden ratio rule offers a rigourous optimality argument for the choice of the well-known *golden retracement* in technical analysis of asset prices. To our knowledge this is the first time that such an argument has been found/given in the literature.

The problem considered in the present paper belongs to the class of optimal prediction problems (within optimal stopping). Similar optimal prediction problems have been studied in recent years by many authors (see e.g. [1], [2], [9], [10], [11], [12], [14], [19], [25], [30], [39], [40], [41]). Once the 'unknown' future is projected to the 'known' present we find that the resulting optimal stopping problem takes a novel integral form that has not been studied before. The appearance of the minimum process in this context makes the problem related to optimal stopping problems for the maximum process that were initially studied and solved in important special cases of diffusion processes in [7], [8] and [23]. The general solution to problems of this kind was derived in the form of the maximality principle in [31] (see also Section 13 and Chapter V in [35] and the other references therein). More recent contributions and studies of related problems include [4], [15], [16], [20], [22], [26], [27], [29]. Close three-dimensional relatives of these problems also appear in the recent papers [6] and [43] where the problems were effectively solved by guessing and finding the optimal stopping boundary in a closed form. The maximality principle has been extended to three-dimensional problems in the recent paper [34].

Although the structure of the present problem is similar to some of these problems, it turns out that none of these results is applicable in the present setting. Governed by these particular features in this paper we show how the problem can be solved when (i) no closed-form solution for the candidate stopping boundary is available and (ii) the loss function takes an integral form where the integrand is a functional of both the process $X$ and its running minimum $I$. This is done by extending the arguments associated with the maximality principle to the setting of the present problem and disclosing the general form of the solution that is valid in all particular cases. The key novel ingredient revealed in the solution is the replacement of the diagonal and its role in the maximality principle by a nonlinear curve in the two-dimensional state space of $X$ and $I$. We believe that this methodology is of general interest and the arguments developed in the proof should be applicable in similar two/multi-dimensional integral settings.

## 2. Optimal prediction problem

1. We consider a non-negative diffusion process $X=(X_t)_{t\ge 0}$ solving

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \tag{2.1}$$

where $\mu$ and $\sigma>0$ are continuous functions satisfying (2.4)+(2.5) below and $B=(B_t)_{t\ge 0}$ is a standard Brownian motion. By $\mathsf{P}_x$ we denote the probability measure under which the process $X$ starts at $x>0$. Recalling that the scale function of $X$ is given by

$$L(x) = \int^x \exp\Big(-\int^y \frac{\mu(z)}{(\sigma^2/2)(z)}\,dz\Big)\,dy \tag{2.2}$$

and the speed measure of $X$ is given by

$$m(dx) = \frac{dx}{(\sigma^2/2)(x)\,L'(x)} \tag{2.3}$$

we assume that the following conditions are satisfied:

$$L(0+) = -\infty \quad \& \quad L(\infty-) = 0 \tag{2.4}$$

$$\textstyle\int_{0+}^1 L(dy) = \infty \quad \& \quad \int_{0+}^1 m(dy) < \infty \quad \& \quad \int_{0+}^1 |L(y)|\,m(dy) < \infty\,. \tag{2.5}$$

From (2.4) we read that $X$ is a transient diffusion process in the sense that $X_t \rightarrow \infty$ $\mathsf{P}_x$-a.s. as $t\rightarrow\infty$, and from (2.5) we read that $0$ is an entrance boundary point for $X$ in the sense that the process $X$ could start at $0$ but will never return to it (implying also that $X$ will never visit $0$ after starting at $x>0$ ).

2. The main example we have in mind is the $d$-dimensional Bessel process $X$ solving

$$dX_t = \frac{d-1}{2X_t}\,dt + dB_t \tag{2.6}$$

where $d>2$. Recalling that the scale function is determined up to an affine transformation we can choose the scale function (2.2) and hence the speed measure (2.3) to read

$$L(x) = -\frac{1}{x^{d-2}} \tag{2.7}$$

$$m(dx) = \tfrac{2}{d-2}\,x^{d-1}\,dx \tag{2.8}$$

for $x>0$. It is well known that when $d\in\{3,4,\ldots\}$ one can realise $X$ as the radial part of $d$-dimensional standard Brownian motion. Similar interpretations of (2.6) are also valid when $d=1$ (with an addition of the local time at zero) and $d=2$ but $X$ is not transient in these cases (but recurrent) and hence the problem considered below will have a trivial solution. Other examples of (2.1) are obtained by composing Bessel processes solving (2.6) with strictly decreasing and smooth functions. This is of interest in financial applications and will be discussed below. There are also many other examples of transient diffusion processes solving (2.1) that are not related to Bessel processes.

3. To formulate the problem to be studied below consider the diffusion process $X$ solving (2.1) and introduce its running minimum process $I = (I_t)_{t \ge 0}$ by setting

$$I_t = \inf_{0 \le s \le t} X_s \tag{2.9}$$

for $t \ge 0$. Due to the facts that $X$ is transient (converging to $+\infty$) and $0$ is an entrance boundary point for $X$, we see that the ultimate infimum $I_\infty = \inf_{t \ge 0} X_t$ is attained at some random time $\theta$ in the sense that

$$X_\theta = I_\infty \tag{2.10}$$

with $\mathsf{P}_x$-probability one for $x > 0$ given and fixed (the case $x = 0$ being trivial and therefore excluded). It is well known that $\theta$ is unique up to a set of $\mathsf{P}_x$-probability zero (cf. [42, Theorem 2.4]). The random time $\theta$ is clearly unknown at any given time and cannot be detected through sequential observations of the sample path $t \mapsto X_t$ for $t \ge 0$. In many applied situations of this kind we want to devise sequential strategies which will enable us to come as 'close' as possible to $\theta$. Most notably, the main example we have in mind is the problem of optimal trading in the presence of bubbles to be addressed below. In mathematical terms this amounts to finding a stopping time of $X$ that is as 'close' as possible to $\theta$. A first step towards this goal is provided by the following lemma. We recall that stopping times of $X$ refer to stopping times with respect to the natural filtration of $X$ that is defined by $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$ for $t \ge 0$.

**Lemma 1.** *We have*

$$|\theta - \tau| = \theta + \int_0^\tau \big(2I(\theta \le t) - 1\big)\, dt \tag{2.11}$$

*for all stopping (random) times $\tau$ of $X$.*

**Proof.** The identity is well known (see e.g. [35, p. 450]) and can be derived by noting that

$$\begin{aligned} |\theta - \tau| &= (\theta - \tau)^+ + (\tau - \theta)^+ = \int_0^\theta I(\tau \le t)\, dt + \int_0^\tau I(\theta \le t)\, dt \\ &= \int_0^\theta \big(1 - I(\tau > t)\big)\, dt + \int_0^\tau I(\theta \le t)\, dt \\ &= \theta - \int_0^\tau I(\theta > t)\, dt + \int_0^\tau I(\theta \le t)\, dt \\ &= \theta - \int_0^\tau \big(1 - I(\theta \le t)\big)\, dt + \int_0^\tau I(\theta \le t)\, dt \\ &= \theta + \int_0^\tau \big(2I(\theta \le t) - 1\big)\, dt \end{aligned} \tag{2.12}$$

for all stopping (random) times $\tau$ of $X$ as claimed. $\square$

4. Taking $\mathsf{E}_x$ on both sides in (2.11) yields a non-trivial measure of error (from $\tau$ to $\theta$) as long as $\mathsf{E}_x \theta < \infty$ for $x > 0$ given and fixed. The latter condition, however, may not

always be fulfilled. For example, when $X$ is a transient Bessel process of dimension $d>2$ it is known (see [38, Lemma 1]) that $\mathsf{P}_x(\theta>t) \sim t^{-(d/2-1)}$ as $t \rightarrow \infty$. Hence we see that $\mathsf{E}_x\theta = \int_0^\infty \mathsf{P}_x(\theta>t)\,dt < \infty$ if and only if $d/2-1>1$ or equivalently $d>4$. It is clear from (2.11) however that the pointwise minimisation of the Euclidean distance on the left-hand side is equivalent to the pointwise minimisation of the integral on the right-hand side. To preserve the generality we therefore 'normalise' $|\theta-\tau|$ on the left-hand side by subtracting $\theta$ from it. After taking $\mathsf{E}_x$ on both sides of the resulting identity we obtain

$$\mathsf{E}_x\big(|\theta-\tau|-\theta\big) = \mathsf{E}_x \int_0^\tau \big(2I(\theta \le t)-1\big)\,dt \tag{2.13}$$

for all stopping times $\tau$ of $X$ (for which the right-hand side is well defined). The optimal prediction problem therefore becomes

$$V(x) = \inf_\tau \mathsf{E}_x\big(|\theta-\tau|-\theta\big) \tag{2.14}$$

where the infimum is taken over all stopping times $\tau$ of $X$ (with finite mean) and $x>0$ is given and fixed. Note that the problem (2.14) is equivalent to the problem of minimising $\mathsf{E}_x|\theta-\tau|$ over all stopping times $\tau$ of $X$ (with finite mean) whenever $\mathsf{E}_x\theta<\infty$. To tackle the problem (2.14) we first focus on the right-hand side in (2.13) above.

**Lemma 2.** *We have*

$$\mathsf{E}_x \int_0^\tau \big(2I(\theta \le t)-1\big)\,dt = \mathsf{E}_x \int_0^\tau \Big(1-2\,\frac{L(X_t)}{L(I_t)}\Big)\,dt \tag{2.15}$$

*for all stopping times $\tau$ of $X$ (with finite mean) and all $x>0$.*

**Proof.** Using a well-known argument (see e.g. [35, p. 450]) we find that

$$\begin{aligned}
\mathsf{E}_x \int_0^\tau \big(2I(\theta \le t)-1\big)\,dt &= \mathsf{E}_x \int_0^\infty \big(2I(\theta \le t)-1\big)\,I(t<\tau)\,dt \\
&= \int_0^\infty \mathsf{E}_x\Big(\mathsf{E}_x\big[\big(2I(\theta \le t)-1\big)\,I(t<\tau)\,|\,\mathcal{F}_t^X\big]\Big)\,dt \\
&= \int_0^\infty \mathsf{E}_x\Big(I(t<\tau)\,\mathsf{E}_x\big[\big(2I(\theta \le t)-1\big)\,|\,\mathcal{F}_t^X\big]\Big)\,dt \\
&= \mathsf{E}_x \int_0^\tau \big(2\mathsf{P}_x(\theta \le t\,|\,\mathcal{F}_t^X)-1\big)\,dt \\
&= \mathsf{E}_x \int_0^\tau \big(1-2\mathsf{P}_x(\theta > t\,|\,\mathcal{F}_t^X)\big)\,dt
\end{aligned} \tag{2.16}$$

for any stopping time $\tau$ of $X$ (with finite mean) and any $x>0$ given and fixed. Setting $I^t = \inf_{s \ge t} X_s$ and recalling that $I_t = \inf_{0 \le s \le t} X_s$ we find by the Markov property that

$$\begin{aligned}
\mathsf{P}_x(\theta>t\,|\,\mathcal{F}_t^X) &= \mathsf{P}_x(I^t<I_t\,|\,\mathcal{F}_t^X) = \mathsf{P}_x(I^t<i\,|\,\mathcal{F}_t^X)\big|_{i=I_t} \\
&= \mathsf{P}_x(I_\infty \circ \theta_t < i\,|\,\mathcal{F}_t^X)\big|_{i=I_t} = \mathsf{P}_{X_t}(I_\infty < i)\big|_{i=I_t}
\end{aligned} \tag{2.17}$$

for $t>0$. To compute the latter probability we recall that $M := L(X)$ is a continuous local martingale and note that $I_\infty < i$ if and only if $L(I_\infty) < L(i)$ where $L(I_\infty) = \inf_{t\ge 0} L(X_t) = \inf_{t\ge 0} M_t$. This shows that the set $\{I_\infty < i\}$ coincides with the set $\{\inf_{t\ge 0} M_t < L(i)\}$ which in turn can be expressed as $\{\sigma < \infty\}$ where $\sigma = \inf\{\, t\ge 0 \,|\, M_t < L(i)\,\}$. Taking $x \ge i$ we see that the continuous local martingale $M^\sigma = (M_{\sigma\wedge t})_{t\ge 0}$ is bounded above by $0$ and bounded below by $L(i)$ with $M_0^\sigma = L(x)$ under $\mathsf{P}_x$. It follows therefore that $M^\sigma$ is a uniformly integrable martingale and hence by the optional sampling theorem we find that

$$L(x) = \mathsf{E}_x M_\sigma = \mathsf{E}_x[L(i)\, I(\sigma<\infty)] + \mathsf{E}_x[M_\infty\, I(\sigma=\infty)] = L(i)\,\mathsf{P}_x(\sigma<\infty) \tag{2.18}$$

upon using that $M_\infty := \lim_{t\to\infty} M_t = 0$ $\mathsf{P}_x$-a.s. on $\{\sigma=\infty\}$. Combining (2.18) with the previous conclusions we obtain

$$\mathsf{P}_x(I_\infty < i) = \mathsf{P}_x(\sigma<\infty) = \frac{L(x)}{L(i)} \tag{2.19}$$

for $i \le x$ in $(0,\infty)$. From (2.17) and (2.19) we see that

$$\mathsf{P}_x(\theta > t \,|\, \mathcal{F}_t^X) = \frac{L(X_t)}{L(I_t)} \tag{2.20}$$

for all $x>0$ and $t\ge 0$ (for the underlying three-dimensional law see [5, Theorem A]). Inserting this expression back into (2.16) we obtain (2.15) and the proof is complete. $\square$

5. From (2.13) and (2.15) we see that the problem (2.14) is equivalent to

$$V(x) = \inf_\tau \mathsf{E}_x \int_0^\tau \Big(1-2\,\frac{L(X_t)}{L(I_t)}\Big)\, dt \tag{2.21}$$

where the infimum is taken over all stopping times $\tau$ of $X$ (with finite mean) and $x>0$ is given and fixed. Passing from the initial diffusion process $X$ to the scaled diffusion process $L(X)$ we see that there is no loss of generality in assuming that $\mu=0$ in (2.1) or equivalently that $L(x)=x$ for $x>0$ (with $L(X_t)\to 0$ $\mathsf{P}_x$-a.s. as $t\to\infty$). Note that the time of the ultimate minimum $\theta$ is the same for both $X$ and $L(X)$ since $L$ is strictly increasing. Note also that $\tau$ is a stopping time of $X$ if and only if $\tau$ is a stopping time of $L(X)$. To keep the track of the general formulae throughout we will continue with considering the general case (when $\mu$ is not necessarily zero and $L$ is not necessarily the identity function). This problem will be tackled in the next section below.

6. For future reference we recall that the infinitesimal generator of $X$ equals

$$\mathbb{L}_X = \mu(x)\,\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\,\frac{\partial^2}{\partial x^2} \tag{2.22}$$

for $x>0$. Throughout we denote $\tau_a = \inf\{\, t\ge 0 \mid X_t = a\,\}$ and set $\tau_{a,b} = \tau_a \wedge \tau_b$ for $a<b$ in $(0,\infty)$. It is well known that

$$\mathsf{P}_x\big(X_{\tau_{a,b}} = a\big) = \frac{L(b)-L(x)}{L(b)-L(a)} \quad \& \quad \mathsf{P}_x\big(X_{\tau_{a,b}} = b\big) = \frac{L(x)-L(a)}{L(b)-L(a)} \tag{2.23}$$

for $a \le x \le b$ in $(0,\infty)$. The Green function of $X$ is given by

$$G_{a,b}(x,y) = \frac{(L(b)-L(y))\,(L(x)-L(a))}{L(b)-L(a)} \quad \text{if} \quad a \le x \le y \le b \tag{2.24}$$
$$= \frac{(L(b)-L(x))\,(L(y)-L(a))}{L(b)-L(a)} \quad \text{if} \quad a \le y \le x \le b\,.$$

If $f:(0,\infty)\to I\!R$ is a measurable function, then it is well known that

$$\mathsf{E}_x \int_0^{\tau_{a,b}} f(X_t)\,dt = \int_a^b f(y)\,G_{a,b}(x,y)\,m(dy) \tag{2.25}$$

for $a \le x \le b$ in $(0,\infty)$. This identity holds in the sense that if one of the integrals exists so does the other one and they are equal.

## 3. Optimal stopping problem

It was shown in the previous section that the optimal prediction problem (2.14) is equivalent to the optimal stopping problem (2.21). The purpose of this section is to present the solution to the latter problem. Using the fact that the two problems are equivalent this also leads to the solution of the former problem.

In the setting of (2.1)-(2.5) consider the optimal stopping problem (2.21). This problem is two-dimensional and the underlying Markov process equals $(I,X)$. Setting $I_t^i = i \wedge \inf_{0\le s\le t} X_s$ for $t \ge 0$ enables $(I,X)$ to start at $(i,x)$ under $\mathsf{P}_x$ for $i \le x$ in $(0,\infty)$ and we will denote the resulting probability measure on the canonical space by $\mathsf{P}_{i,x}$. Thus under $\mathsf{P}_{i,x}$ the canonical process $(I,X)$ starts at $(i,x)$. The problem (2.21) then extends as follows

$$V(i,x) = \inf_\tau \mathsf{E}_{i,x} \int_0^\tau \Big(1-2\,\frac{L(X_t)}{L(I_t)}\Big)\,dt \tag{3.1}$$

for $i \le x$ in $(0,\infty)$ where the infimum is taken over all stopping times $\tau$ of $X$ (with finite mean). In addition to $\sigma$ and $L$ from (2.1) and (2.2) above let us set

$$c(i,x) = 1 - 2\,\frac{L(x)}{L(i)} \tag{3.2}$$

for $i \le x$ in $(0,\infty)$. The main result of this section may then be stated as follows.

**Theorem 3.** *The optimal stopping time in the problem* (3.1) *is given by*

$$\tau_* = \inf\,\{\, t \ge 0 \mid X_t \ge f_*(I_t)\,\} \tag{3.3}$$

*where the optimal boundary $f_*$ can be characterised as the minimal solution to*

$$f'(i) = -\frac{\sigma^2(f(i))\,L'(f(i))}{c(i,f(i))\,[L(f(i))-L(i)]} \int_i^{f(i)} \frac{c_i'(i,y)\,[L(y)-L(i)]}{\sigma^2(y)\,L'(y)}\,dy \tag{3.4}$$

*staying strictly above the curve* $h(i) = L^{-1}(L(i)/2)$ *for* $i > 0$ *(in the sense that if the minimal solution does not exist then there is no optimal stopping time). The value function is given by*

$$V(i,x) = -\int_x^{f_*(i)} \frac{c(i,y)\,[L(y)-L(x)]}{(\sigma^2/2)(y)\,L'(y)}\,dy \tag{3.5}$$

*for* $i \le x \le f_*(i)$ *and* $V(i,x) = 0$ *for* $x \ge f_*(i)$ *with* $i > 0$ *.*

**Proof.** 1. It is evident from the integrand in (3.1) that the excursions of $X$ away from the running minimum $I$ play a key role in the analysis of the problem. In particular, recalling the definition (3.2) we see from (3.1) that the process $(I,X)$ can never be optimally stopped in the set $C_0 := \{\, (i,x) \in S \mid c(i,x) < 0 \,\}$ where we let $S = \{\, (i,x) \in (0,\infty) \times (0,\infty) \mid i \le x \,\}$ denote the state space of the process $(I,X)$. Indeed, if $(i,x) \in C_0$ is given and fixed, then the first exit time of $(I,X)$ from a sufficiently small ball with the centre at $(i,x)$ (on which $c$ is strictly negative) will produce a value strictly smaller than $0$ (the value corresponding to stopping at once). Defining

$$h(i) = L^{-1}\big(\tfrac{1}{2}L(i)\big) \tag{3.6}$$

for $i > 0$ we see that $c(i,x) < 0$ for $x < h(i)$ and $c(i,x) > 0$ for $x > h(i)$ whenever $i \le x$ in $(0,\infty)$ are given and fixed. Note that the mapping $i \mapsto h(i)$ is increasing and continuous as well as that $h(i) > i$ for $i > 0$ with $h(0+) = 0$ and $h(+\infty) = +\infty$. This shows that $C_0 = \{\, (i,x) \in S \mid i \le x < h(i) \,\}$. Note in particular that $C_0$ contains the diagonal $\{\, (i,x) \in S \mid i = x \,\}$ in the state space.

2. Before we formalise further conclusions let us recall that the general theory of optimal stopping for Markov processes (see [35, Chapter 1]) implies that the continuation set in the problem (3.1) equals $C = \{\, (i,x) \in S \mid V(i,x) < 0 \,\}$ and the stopping set equals $D = \{\, (i,x) \in S \mid V(i,x) = 0 \,\}$. It means that the first entry time of $(I,X)$ into $D$ is optimal in the problem (3.1) whenever well defined. It follows therefore that $C_0$ is contained in $C$ and the central question becomes to determine the remainder of the set $C$. Since $X_t \to \infty$ $\mathsf{P}_x$-a.s. as $t \to \infty$ it follows that $L(X_t) \to 0$ $\mathsf{P}_x$-a.s. as $t \to \infty$ so that the integrand in (3.1) becomes strictly positive eventually and this reduces the incentive to continue (given also that the 'favourable' set $C_0$ becomes more and more distant). This indicates that there should exist a point $f(i)$ at or above which the process $X$ should be optimally stopped under $\mathsf{P}_{i,x}$ where $i \le x$ in $(0,\infty)$ are given and fixed. This yields the following candidate

$$\tau_f = \inf\{\, t \ge 0 \mid X_t \ge f(I_t) \,\} \tag{3.7}$$

for an optimal stopping time in (3.1) where the function $i \mapsto f(i)$ is to be determined.

3. *Free-boundary problem.* To compute the value function $V$ and determine the optimal function $f$ we are led to formulate the free-boundary problem

$$(\mathbb{L}_X V)(i,x) = -c(i,x) \quad \text{for } i < x < f(i) \tag{3.8}$$

$$V_i'(i,x)\Big|_{x=i+} = 0 \quad \text{(normal reflection)} \tag{3.9}$$

$$V(i,x)\Big|_{x=f(i)-} = 0 \quad \text{(instantaneous stopping)} \tag{3.10}$$

$$V'_x(i,x)\Big|_{x=f(i)-} = 0 \quad \text{(smooth fit)} \tag{3.11}$$

for $i>0$ where $I\!L_X$ is the infinitesimal generator of $X$ given in (2.22) above. For the rationale and further details regarding free-boundary problems of this kind we refer to [35, Section 13] and the references therein (we note in addition that the condition of normal reflection (3.9) dates back to [18]).

4. *Nonlinear differential equation.* To solve the free-boundary problem (3.8)-(3.11) consider the stopping time $\tau_f$ defined in (3.7) and (formally) the resulting function

$$V_f(i,x) = \mathsf{E}_{i,x} \int_0^{\tau_f} c(I_t, X_t)\, dt \tag{3.12}$$

for $i \le x \le f(i)$ in $(0,\infty)$ given and fixed. Applying the strong Markov property of $(I,X)$ at $\tau_{i,f(i)} = \inf\{\, t \ge 0 \mid X_t \notin (i, f(i))\,\}$ and using (2.23)-(2.25) we find that

$$V_f(i,x) = V_f(i,i)\, \frac{L(f(i))-L(x)}{L(f(i))-L(i)} + \int_i^{f(i)} c(i,y)\, G_{i,f(i)}(x,y)\, m(dy)\,. \tag{3.13}$$

It follows from (3.13) that

$$V_f(i,i) = \frac{L(f(i))-L(i)}{L(f(i))-L(x)}\, V_f(i,x) - \frac{L(f(i))-L(i)}{L(f(i))-L(x)} \int_i^{f(i)} c(i,y)\, G_{i,f(i)}(x,y)\, m(dy)\,. \tag{3.14}$$

Using (3.10) and (3.11) we find after dividing and multiplying with $x-f(i)$ that

$$\lim_{x \uparrow f(i)} \frac{V_f(i,x)}{L(f(i))-L(x)} = -\frac{1}{L'(f(i))}\, \frac{\partial V_f}{\partial x}(i,x)\Big|_{x=f(i)-} = 0\,. \tag{3.15}$$

Moreover, it is easily seen by (2.24) that

$$\lim_{x \uparrow f(i)} \frac{L(f(i))-L(i)}{L(f(i))-L(x)} \int_i^{f(i)} c(i,y)\, G_{i,f(i)}(x,y)\, m(dy) = \int_i^{f(i)} c(i,y)\, [L(y)-L(i)]\, m(dy)\,. \tag{3.16}$$

Combining (3.14)-(3.16) we see that

$$V_f(i,i) = -\int_i^{f(i)} c(i,y)\, [L(y)-L(i)]\, m(dy)\,. \tag{3.17}$$

Inserting this back into (3.13) and using (2.24)+(2.25) we conclude that

$$V_f(i,x) = -\int_x^{f(i)} c(i,y)\, [L(y)-L(x)]\, m(dy) \tag{3.18}$$

for $i \le x \le f(i)$ in $(0,\infty)$. Finally, using (3.9) we find that

$$f'(i) = -\frac{\sigma^2(f(i))\, L'(f(i))}{2\, c(i,f(i))\, [L(f(i))-L(i)]} \int_i^{f(i)} c'_i(i,y)\, [L(y)-L(i)]\, m(dy) \tag{3.19}$$

for $i > 0$ . Recalling that $C_0$ is contained in $C$ we see that there is no restriction to assume that each candidate function $f$ solving (3.19) satisfies $f(i) \geq h(i)$ for all $i > 0$ . In addition we will also show below that all points $(i, h(i))$ belong to $C$ for $i > 0$ so that (at least in principle) there would be no restriction to assume that each candidate function $f$ solving (3.19) also satisfies $f(i) > h(i)$ for all $i > 0$ . These candidate functions will be referred to as admissible. We will also see below however that solutions to (3.19) 'starting' at $h$ play a crucial role in finding/describing the solution.

Summarising the preceding considerations we can conclude that to each candidate function $f$ solving (3.19) there corresponds the function (3.18) solving the free-boundary problem (3.8)-(3.11) as is easily verified by direct calculation. Note however that this function does not necessarily admit the stochastic representation (3.12) (even though it was formally derived from this representation). The central question then becomes how to select the optimal boundary $f$ among all admissible candidates solving (3.19). To answer this question we will invoke the subharmonic characterisation of the value function (see [35, Chapter 1]) for the three-dimensional Markov process $(I, X, A)$ where $A_t = \int_0^t c(I_s, X_s)\, ds$ for $t \geq 0$ . Fuller details of this argument will become clearer as we progress below. It should be noted that among all admissible candidate functions solving (3.19) only the optimal boundary will have the power of securing the stochastic representation (3.12) for the corresponding function (3.18). This is a subtle point showing the full power of the method (as well as disclosing limitations of the optimal stopping problem itself).

5. *The minimal solution.* Motivated by the previous question we note from (3.18) that $f \mapsto V_f$ is decreasing over admissible solutions to (3.19). This suggests to select the candidate function among admissible solutions to (3.19) that is as far as possible from $h$ . The subharmonic characterisation of the value function suggests to proceed in the opposite direction and this is the lead that we will follow in the sequel.

To address the existence and uniqueness of solutions to (3.19), denote the right-hand side of (3.19) by $\Phi(i, f(i))$ . From the general theory of nonlinear differential equations we know that if the direction field $(i, f) \mapsto \Phi(i, f)$ is (locally) continuous and (locally) Lipschitz in the second variable, then the equation (3.19) admits a (locally) unique solution. For instance, this will be the case if along a (local) continuity of $(i, f) \mapsto \Phi(i, f)$ we also have a (local) continuity of $(i, f) \mapsto \Phi'_f(i, f)$ . In particular, we see from the structure of $\Phi$ that the equation (3.19) admits a (locally) unique solution whenever $x \mapsto \sigma^2(x)$ is (locally) continuously differentiable. It is important to realise that the preceding arguments apply only away from $h$ since each point $(i, h(i))$ is a singularity point of the equation (3.19) in the sense that $f'(i+) = \infty$ when $f(i+) = h(i)$ due to $c(i, h(i)) = 0$ for $i > 0$ . In this case it is also important to note that the preceding arguments can be applied to the equivalent equation for the inverse of $i \mapsto f(i)$ since this singularity gets removed (the derivative of the inverse being zero).

To construct the minimal solution to (3.19) staying strictly above $h$ we can proceed as follows (see *Figure* 1). For any $i_n > 0$ such that $i_n \downarrow 0$ as $n \to \infty$ let $i \mapsto f_n(i)$ denote the solution to (3.19) on $(i_n, \infty)$ such that $f_n(i_n+) = h(i_n)$ . Note that $i \mapsto f_n(i)$ is singular at $i_n$ and that passing to the equivalent equation for the inverse of $i \mapsto f_n(i)$ this singularity gets removed as explained above. (Note that the solution to the equivalent equation for the inverse can be continued below $h(i_n)$ as well until hitting the diagonal at some strictly positive point at which the derivative is $-\infty$ . This yields another solution to (3.19) staying below $f_n$ and

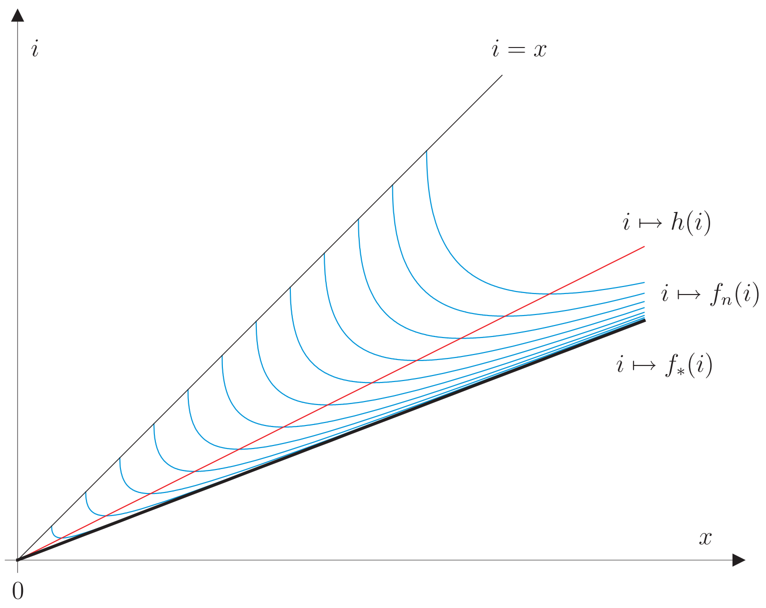


**Figure 1.** Solutions $f_n$ and $f_*$ to the differential equation (3.4) from Theorem 3. The optimal stopping boundary $f_*$ is the minimal solution staying strictly above the curve $h$ . This is a genuine drawing corresponding to the golden ratio rule of Corollary 5 when $X$ is the radial part of three-dimensional Brownian motion and the optimal stopping boundary $f_*$ is linear.

providing its 'physical' link to the diagonal. We will not make use of this part of the solution in the sequel.) Note that the right-hand side of the equation (3.19) is positive for $f(i) > h(i)$ so that $i \mapsto f_n(i)$ is strictly increasing on $[i_n, \infty)$ . By the uniqueness of the solution we know that the two curves $i \mapsto f_n(i)$ and $i \mapsto f_m(i)$ cannot intersect for $n \neq m$ and hence we see that $(f_n)_{n \geq 1}$ is increasing. It follows therefore that $f_* := \lim_{n \to \infty} f_n$ exists on $(0, \infty)$ . Passing to an integral equation equivalent to (3.19) it is easily verified that $i \mapsto f_*(i)$ solves (3.19) wherever finite. This $f_*$ represents the minimal solution to (3.19) staying strictly above the curve $h$ on $(0, \infty)$ . We will first consider the case when $f_*$ is finite valued on $(0, \infty)$ .

6. *Stochastic representation.* We show that the function (3.18) associated with the minimal solution $f_*$ admits the stochastic representation (3.12). For this, let $i \mapsto f_n(i)$ be the solution to (3.19) on $(i_n, \infty)$ such that $f_n(i_n+) = h(i_n)$ for $i_n > 0$ with $i_n \downarrow 0$ as $n \to \infty$ . Consider the function $(i, x) \mapsto V_{f_n}(i, x)$ defined by (3.18) for $i \leq x \leq f_n(i)$ and $i \geq i_n$ with $n \geq 1$ given and fixed. Recall that $V_{f_n}$ solves the free-boundary problem (3.8)-(3.11) for $i \geq i_n$ . Consider the stopping time $\tau_n := \tau_{i_n} \wedge \tau_{f_n}$ where $\tau_{i_n} = \inf \{ t \geq 0 \mid X_t = i_n \}$ and $\tau_{f_n} = \inf \{ t \geq 0 \mid X_t \geq f_n(I_t) \}$ . Applying Itô's formula and using (3.8) we find that

$$
\begin{aligned}
(3.20) \qquad V_{f_n}(I_{\tau_n}, X_{\tau_n}) &= V_{f_n}(i, x) + \int_0^{\tau_n} \frac{\partial V_{f_n}}{\partial i}(I_t, X_t)\, dI_t + \int_0^{\tau_n} \frac{\partial V_{f_n}}{\partial x}(I_t, X_t)\, dX_t \\
&\quad + \frac{1}{2} \int_0^{\tau_n} \frac{\partial^2 V_{f_n}}{\partial x^2}(I_t, X_t)\, d\langle X, X \rangle_t
\end{aligned}
$$

$$= V_{f_n}(i,x) + \int_0^{\tau_n} \mathbb{L}_X(V_{f_n})(I_t,X_t)\,dt + \int_0^{\tau_n} \sigma(X_t)\,\frac{\partial V_{f_n}}{\partial x}(I_t,X_t)\,dB_t$$
$$= V_{f_n}(i,x) - \int_0^{\tau_n} c(I_t,X_t)\,dt + M_{\tau_n}$$

where we also use (3.9) to conclude that the integral with respect to $dI_t$ is equal to zero and $M_t = \int_0^{t\wedge\tau_n} \sigma(X_s)\,(\partial V_{f_n}/\partial x)(I_s,X_s)\,dB_s$ is a continuous local martingale for $t \ge 0$.

Since the process $(I,X)$ remains in the compact set $\{\,(j,y)\in S \mid i_n \le j \le y \le f_n(i)\,\}$ up to time $\tau_n$ under $\mathsf{P}_{i,x}$, and both $\sigma$ and $\partial V_{f_n}/\partial x$ are continuous (and thus bounded) on this set, we see that $M$ is a uniformly integrable martingale and hence by the optional sampling theorem we have $\mathsf{E}_{i,x}M_{\tau_n} = 0$. Taking $\mathsf{E}_{i,x}$ on both sides of (3.20) we therefore obtain

$$V_{f_n}(i,x) = \mathsf{E}_{i,x}V_{f_n}(I_{\tau_n},X_{\tau_n}) + \mathsf{E}_{i,x}\int_0^{\tau_n} c(I_t,X_t)\,dt \tag{3.21}$$
$$= V_{f_n}(i_n,i_n)\;\mathsf{P}_{i,x}(\tau_{i_n}\!<\!\tau_{f_n}) + \mathsf{E}_{i,x}\int_0^{\tau_n} c(I_t,X_t)\,dt$$

since $(I_{\tau_n},X_{\tau_n}) = (i_n,i_n)$ on $\{\tau_{i_n}\!<\!\tau_{f_n}\}$ and $V_{f_n}(I_{\tau_n},X_{\tau_n}) = 0$ on $\{\tau_{f_n}\!<\!\tau_{i_n}\}$. Using that $|c| \le 1$ we find by (3.17), (2.23) and (3.6) that

$$|V_{f_n}(i_n,i_n)|\;\mathsf{P}_{i,x}(\tau_{i_n}\!<\!\tau_{f_n}) \le \int_{i_n}^{f(i_n)} |c(i,y)|\;|L(y)\!-\!L(i_n)|\;m(dy)\;\mathsf{P}_{i,x}(\tau_{i_n}\!<\!\tau_{f_n}) \tag{3.22}$$
$$\le |L(h(i_n))\!-\!L(i_n)| \int_{i_n}^{h(i_n)} m(dy)\;\frac{L(f_*(i))\!-\!L(x)}{L(f_*(i))\!-\!L(i_n)}$$
$$= \frac{1}{2}\,|L(i_n)|\;\frac{L(f_*(i))\!-\!L(x)}{L(f_*(i))\!-\!L(i_n)}\int_{i_n}^{h(i_n)} m(dy) \rightarrow 0$$

as $n\rightarrow\infty$ since $L(i_n)\rightarrow -\infty$ and $h(i_n)\rightarrow 0$ so that $\int_{i_n}^{h(i_n)} m(dy)\rightarrow 0$ due to (2.5) above. Hence letting $n\rightarrow\infty$ in (3.21) and using that $V_{f_n}\rightarrow V_{f_*}$ by the monotone convergence theorem, as well as that $\tau_n \uparrow \tau_{f_*}$ since $f_n \uparrow f_*$ and $i_n \downarrow 0$, we find noting that $\mathsf{E}_{i,x}\tau_{f_*} < \infty$ and using the dominated convergence theorem that

$$V_{f_*}(i,x) = \mathsf{E}_{i,x}\int_0^{\tau_{f_*}} c(I_t,X_t)\,dt \tag{3.23}$$

for all $i \le x$ in $(0,\infty)$ as claimed.

7. *Non-positivity.* We show that for every solution $f$ to (3.19) such that $f \ge f_*$ on $(0,\infty)$ and the function $V_f$ defined by (3.18) above we have

$$V_f(i,x) \le 0 \tag{3.24}$$

for all $i \le x$ in $(0,\infty)$. Clearly, since $c(i,y) \ge 0$ for $y \ge h(i)$ in (3.18), it is enough to prove (3.24) for $f_*$ and $i \le x < h(i)$ with $i > 0$. For this, consider the stopping time $\tau_h = \inf\{\,t\ge 0 \mid X_t \ge h(I_t)\,\}$ and note that $\tau_{f_*} = \tau_h \!+\! \tau_{f_*}\circ\theta_{\tau_h}$. Hence by the strong Markov property of $(I,X)$ applied at $\tau_h$ we find using (3.23) that

$$V_{f_*}(i,x) = \mathsf{E}_{i,x}\int_0^{\tau_h} c(I_t,X_t)\,dt + \mathsf{E}_{i,x}\int_{\tau_h}^{\tau_h+\tau_{f_*}\circ\theta_{\tau_h}} c(I_t,X_t)\,dt \tag{3.25}$$

$$
\begin{aligned}
&= \mathsf{E}_{i,x} \int_0^{\tau_h} c(I_t, X_t)\, dt + \mathsf{E}_{i,x} \int_0^{\tau_{f_*} \circ \theta_{\tau_h}} c(I_{t+\tau_h}, X_{t+\tau_h})\, dt \\
&= \mathsf{E}_{i,x} \int_0^{\tau_h} c(I_t, X_t)\, dt + \mathsf{E}_{i,x}\, \mathsf{E}_{i,x} \Big[ \int_0^{\tau_{f_*}} c(I_t, X_t)\, dt \circ \theta_{\tau_h} \mid \mathcal{F}^X_{\tau_h} \Big] \\
&= \mathsf{E}_{i,x} \int_0^{\tau_h} c(I_t, X_t)\, dt + \mathsf{E}_{i,x}\, \mathsf{E}_{I_{\tau_h}, X_{\tau_h}} \Big[ \int_0^{\tau_{f_*}} c(I_t, X_t)\, dt \Big] \\
&= \mathsf{E}_{i,x} \int_0^{\tau_h} c(I_t, X_t)\, dt + \mathsf{E}_{i,x}\, V_{f_*}(I_{\tau_h}, X_{\tau_h}) \le 0
\end{aligned}
$$

where the final inequality follows from the facts that $c(I_t, X_t) \le 0$ for all $t \in [0, \tau_h]$ and $V_{f_*}(I_{\tau_h}, X_{\tau_h}) \le 0$ due to $X_{\tau_h} = h(I_{\tau_h})$ upon recalling (3.18) as already indicated above. This completes the proof of (3.24).

8. *Optimality of the minimal solution.* We will begin by disclosing the subharmonic characterisation of the value function (3.1) in terms of the solutions to (3.19) staying strictly above $h$. For this, let $i \mapsto f(i)$ be any solution to (3.19) satisfying $f(i) > h(i)$ for all $i > 0$. Consider the function $(i, x) \mapsto V_f(i, x)$ defined by (3.18) for $i \le x \le f(i)$ in $(0, \infty)$ and set $V_f(i, x) = 0$ for $x \ge f(i)$ in $(0, \infty)$. Let $i \le x$ in $(0, \infty)$ be given and fixed. Due to the 'double-deck' structure of $V_f$ we can apply the change-of-variable formula from [32] that in view of (3.11) reduces to standard Itô's formula and gives

$$
\begin{aligned}
V_f(I_t, X_t) &= V_f(i, x) + \int_0^t \frac{\partial V_f}{\partial i}(I_s, X_s)\, dI_s + \int_0^t \frac{\partial V_f}{\partial x}(I_s, X_s)\, dX_s \\
&\quad + \frac{1}{2} \int_0^t \frac{\partial^2 V_f}{\partial x^2}(I_s, X_s)\, d\langle X, X \rangle_s \\
&= V_f(i, x) + \int_0^t I\!\!L_X(V_f)(I_s, X_s)\, ds + \int_0^t \sigma(X_s) \frac{\partial V_f}{\partial x}(I_s, X_s)\, dB_s
\end{aligned}
\tag{3.26}
$$

where we also use (3.9) to conclude that the integral with respect to $dI_s$ is equal to zero. The process $M = (M_t)_{t \ge 0}$ defined by

$$
M_t = \int_0^t \sigma(X_s) \frac{\partial V_f}{\partial x}(I_s, X_s)\, dB_s \tag{3.27}
$$

is a continuous local martingale. Introducing the increasing process $P = (P_t)_{t \ge 0}$ by setting

$$
P_t = \int_0^t c(I_s, X_s)\, I(X_s \ge f(I_s))\, ds \tag{3.28}
$$

and using the fact that the set of all $s$ for which $X_s$ equals $f(I_s)$ is of Lebesque measure zero, we see by (3.8) that (3.26) can be rewritten as follows

$$
V_f(I_t, X_t) + \int_0^t c(I_s, X_s)\, ds = V_f(i, x) + M_t + P_t\,. \tag{3.29}
$$

From this representation we see that the process $V_f(I_t, X_t) + \int_0^t c(I_s, X_s)\, ds$ is a local submartingale for $t \ge 0$.

Let $\tau$ be any stopping time of $X$ (with finite mean). Choose a localisation sequence $(\sigma_n)_{n\ge 1}$ of bounded stopping times for $M$. Then by (3.24) and (3.29) we can conclude using the optional sampling theorem that

$$\begin{aligned}(3.30)\qquad \mathsf{E}_{i,x}\int_0^{\tau\wedge\sigma_n} c(I_t,X_t)\,dt &\ge \mathsf{E}_{i,x}\Big[V_f(I_{\tau\wedge\sigma_n},X_{\tau\wedge\sigma_n})+\int_0^{\tau\wedge\sigma_n} c(I_t,X_t)\,dt\Big]\\ &\ge V_f(i,x)+\mathsf{E}_{i,x}M_{\tau\wedge\sigma_n}=V_f(i,x)\,.\end{aligned}$$

Letting $n\to\infty$ and using the dominated convergence theorem (upon recalling that $|c|\le 1$ as already used above) we find that

$$(3.31)\qquad \mathsf{E}_{i,x}\int_0^{\tau} c(I_t,X_t)\,dt \ge V_f(i,x)\,.$$

Taking first the infimum over all $\tau$, and then the supremum over all $f$, we conclude that

$$(3.32)\qquad V(i,x)\ge \sup_f V_f(i,x)=V_{f_*}(i,x)$$

upon recalling that $f\mapsto V_f$ is decreasing over $f\ge f_*$ so that the supremum is attained at $f_*$. Combining (3.32) with (3.23) we see that (3.3) and (3.5) hold as claimed.

Note that (3.30) implies that the function $(i,x)\mapsto V_f(i,x)+a$ is subharmonic for the Markov process $(I,X,A)$ where $A_t=\int_0^t c(I_s,X_s)\,ds$ for $t\ge 0$. Recalling that $f\mapsto V_f$ is decreasing over $f\ge f_*$, and that $V_f(i,x)\le 0$ for all $i\le x$ in $(0,\infty)$ by (3.24) above, we see that selecting the minimal solution $f_*$ staying strictly above $h$ is equivalent to invoking the subharmonic characterisation of the value function (according to which the value function is the largest subharmonic function lying below the loss function). For more details on the latter characterisation in a general setting we refer to [35, Chapter 1]. It is also useful to know that the subharmonic characterisation of the value function represents the dual problem to the primal problem (3.1) (for more details on the meaning of this claim including connections to the Legendre transform see [33]).

Consider finally the case when $f_*$ is not finite valued on $(0,\infty)$. Since $i\mapsto f_*(i)$ is increasing we see that there is $i_*\ge 0$ such that $f_*(i)<\infty$ for all $i\in(0,i_*)$ when $i_*>0$ and $f_*(i)=\infty$ for all $i\ge i_*$ with $i\ne 0$ when $i_*=0$. If $i_*>0$ then the proof above can be applied in exactly the same way to show that (3.3) and (3.5) hold as claimed under $\mathsf{P}_{i,x}$ for all $i\le x$ in $(0,\infty)$ with $i<i_*$. If $i\ge i_*$ with $i\ne 0$ when $i_*=0$ then the same proof shows that (3.5) still holds with $\infty$ in place of $f_*(i)$, however, the stopping time (3.3) can no longer be optimal in (3.1). This is easily seen by noting that the value in (3.5) is non-positive (it could also be $-\infty$) for any $x\ge h(i)$ for instance, while the $\mathsf{P}_{i,x}$-probability for $X$ hitting $i$ before drifting away to $\infty$ is strictly smaller than $1$ so that the $\mathsf{P}_{i,x}$-expectation over this set in (3.1) equals $\infty$ (since the integrand tends to $1$ as $t$ tends to $\infty$) showing that the stopping time (3.3) cannot be optimal. The proof above shows that the optimality of (3.5) in this case is obtained through $\tau_n=\tau_{i_n}\wedge\tau_{f_n}$ which play the role of approximate stopping times (obtained by passing to the limit when $n$ tends to $\infty$ in (3.21) above). This completes the proof of the theorem. $\square$

## 4. The golden ratio rule

In this section we show that the minimal solution to (3.4) admits a simple closed-form expression when $X$ is a transient Bessel process (Theorem 4). In the case when $X$ is the radial part of three-dimensional Brownian motion this leads to the golden ratio rule (Corollary 5). We also show that $X$ stopped according to the golden ratio rule has what we refer to as the golden ratio distribution (Corollary 8).

In the setting of (2.6)-(2.8) consider the optimal prediction problem (2.14). Recall that this problem is equivalent to the optimal stopping problem (2.21) which further extends as (3.1). The main result of this section can now be stated as follows.

**Theorem 4.** *If $X$ is the $d$-dimensional Bessel process solving* (2.6) *with $d > 2$, then the optimal stopping time in* (2.14) *is given by*

$$\tau_* = \inf\{\, t \geq 0 \mid X_t \geq \lambda\, I_t \,\} \tag{4.1}$$

*where $\lambda$ is the unique solution to*

$$\lambda^d - (1+d)\,\lambda^2 + \tfrac{4}{4-d}\,\lambda^{4-d} - \tfrac{(d-2)^2}{4-d} = 0 \quad \textit{if} \quad d \neq 4 \tag{4.2}$$

$$\lambda^4 - 5\,\lambda^2 + 4\,\log\lambda + 4 = 0 \quad \textit{if} \quad d = 4 \tag{4.3}$$

*belonging to $(2^{1/(d-2)}, \infty)$. The value function* (3.1) *is given explicitly by*

$$\begin{aligned} V(i,x) &= \tfrac{2}{d-2}\Big[ x^2 \Big(\tfrac{1}{2} + \big(\tfrac{i}{x}\big)^{d-2}\Big)\Big(\big(\tfrac{\lambda i}{x}\big)^2 - 1\Big) \\ &\qquad - \tfrac{x^2}{d}\Big(\big(\tfrac{\lambda i}{x}\big)^d - 1\Big) - \tfrac{2\lambda^{4-d}}{d-4}\, i^2 \Big(\big(\tfrac{\lambda i}{x}\big)^{d-4} - 1\Big)\Big] \quad \textit{if} \quad d \neq 4 \\ &= \Big[ x^2 \Big(\tfrac{1}{2} + \big(\tfrac{i}{x}\big)^2\Big)\Big(\big(\tfrac{\lambda i}{x}\big)^2 - 1\Big) \\ &\qquad - \tfrac{x^2}{4}\Big(\big(\tfrac{\lambda i}{x}\big)^4 - 1\Big) - 2\, i^2 \log\big(\tfrac{\lambda i}{x}\big)\Big] \quad \textit{if} \quad d = 4 \end{aligned} \tag{4.4}$$

*for $i \leq x \leq \lambda i$ and $V(i,x) = 0$ for $x \geq \lambda i$ with $i > 0$.*

**Proof.** By the result of Theorem 3 we know that the optimal stopping time $\tau_*$ is given by (3.3) above where the optimal boundary $f_*$ can be characterised as the minimal solution to (3.4) staying strictly above the curve $h(i) = L^{-1}(L(i)/2)$ for $i > 0$. Using (2.7) and (3.2) it can be verified that (3.4) reads as follows

$$\begin{aligned} f'(i) &= \frac{\frac{d-2}{4-d}\big(\frac{f(i)}{i}\big)\Big[(4-d)\big(\frac{f(i)}{i}\big)^{d-2} + (d-2)\big(\frac{f(i)}{i}\big)^{d-4} - 2\Big]}{\Big(\big(\frac{f(i)}{i}\big)^{d-2} - 1\Big)\Big(\big(\frac{f(i)}{i}\big)^{d-2} - 2\Big)} \quad \text{if} \quad d \neq 4 \\ &= \frac{2\big(\frac{f(i)}{i}\big)\Big[\big(\frac{f(i)}{i}\big)^2 - 2\log\big(\frac{f(i)}{i}\big) - 1\Big]}{\Big(\big(\frac{f(i)}{i}\big)^2 - 1\Big)\Big(\big(\frac{f(i)}{i}\big)^2 - 2\Big)} \quad \text{if} \quad d = 4 \end{aligned} \tag{4.5}$$

and $h(i) = 2^{1/(d-2)} i$ for $i > 0$ . Hence it is enough to show that $f_*(i) = \lambda i$ is the minimal solution to (4.5) staying strictly above the curve $h(i) = 2^{1/(d-2)} i$ for $i > 0$ .

To show that $f_*$ is a solution to (4.5) staying strictly above $h$ , insert $f(i) = \lambda i$ into (4.5) with $\lambda > 0$ to be determined. Multiplying both sides of the resulting identity by $\lambda^{4-d}$ (to be able to derive the factorisation (4.7) below) it is easy to see that this yields the equation $F(\lambda) = 0$ where we set

$$F(\lambda) = \lambda^d - (1+d)\,\lambda^2 + \tfrac{4}{4-d}\,\lambda^{4-d} - \tfrac{(d-2)^2}{4-d} \quad \text{if} \quad d \neq 4 \tag{4.6}$$

$$= \lambda^4 - 5\,\lambda^2 + 4\,\log\lambda + 4 \quad \text{if} \quad d = 4$$

for $\lambda > 0$ . After some algebraic manipulations we find that

$$F'(\lambda) = d\,\lambda^{3-d}\big(\lambda^{d-2} - \tfrac{2}{d}\big)\big(\lambda^{d-2} - 2\big) \tag{4.7}$$

for $\lambda > 0$ and $d > 2$ . Hence we see that the equation $F'(\lambda) = 0$ has two roots $\lambda_0 = (2/d)^{1/(d-2)}$ and $\lambda_1 = 2^{1/(d-2)}$ where $0 < \lambda_0 < 1 < \lambda_1 < \infty$ . It is easy to check that $F''(\lambda_0) < 0$ and $F''(\lambda_1) > 0$ showing that $F$ has a local maximum at $\lambda_0$ and $F$ has a local minimum at $\lambda_1$ . Noting that $F(0+) < 0$ , $F(1) = 0$ and $F(\infty-) = \infty$ this shows that (i) $F$ is strictly increasing on $(0, \lambda_0)$ with $F(0+) < 0$ and $F(\lambda_0) > 0$ ; (ii) $F$ is strictly decreasing on $(\lambda_0, \lambda_1)$ with $F(1) = 0$ and $F(\lambda_1) < 0$ ; and (iii) $F$ is strictly increasing on $(\lambda_1, \infty)$ with $F(\infty-) = \infty$ . It follows therefore that the equation $F(\lambda) = 0$ has exactly three roots $\lambda_0^* < 1 < \lambda_1^*$ where $\lambda_0^* \in (0, \lambda_0)$ and $\lambda_1^* \in (\lambda_1, \infty)$ . Setting $\lambda = \lambda_1^*$ this shows that $f_*(i) = \lambda i$ is a solution to (4.5) staying strictly above the curve $h(i) = 2^{1/(d-2)} i$ for $i > 0$ as claimed.

To show that $f_*$ is the minimal solution satisfying this property, set $\kappa(i) = f(i)/i$ and note that (4.5) can then be rewritten as follows

$$i\kappa'(i) = -\frac{F\big(\kappa(i)\big)}{\kappa^{3-d}(i)\big(\kappa^{d-2}(i) - 1\big)\big(\kappa^{d-2}(i) - 2\big)} \tag{4.8}$$

for $i > 0$ . Since $F(\kappa(i)) < 0$ for $\kappa(i) \in (2^{1/(d-2)}, \lambda)$ we see from (4.8) that $i \mapsto \kappa(i)$ is increasing for $\kappa(i) \in (2^{1/(d-2)}, \lambda)$ . Noting that (4.8) implies that

$$-\int_{\kappa(i)}^{\kappa(i_0)} \frac{\kappa^{3-d}\big(\kappa^{d-2} - 1\big)\big(\kappa^{d-2} - 2\big)}{F(\kappa)}\, d\kappa = \int_i^{i_0} \frac{di}{i} = \log\Big(\frac{i_0}{i}\Big) \tag{4.9}$$

it follows therefore that the integrand on the left-hand side is bounded by a constant (not dependent on $i$ ) as long as $\kappa(i) \in (2^{1/(d-2)}, \lambda)$ for $i \in (0, i_0)$ with any $i_0 > 0$ given and fixed. Letting then $i \downarrow 0$ in (4.9) we see that the left-hand side remains bounded while the right-hand side tends to $\infty$ leading to a contradiction. Noting that $\kappa(i) \in (2^{1/(d-2)}, \lambda)$ if and only if $f(i) \in (h(i), f_*(i))$ we can therefore conclude that there is no solution $f$ to (4.5) satisfying $f(i) \in (h(i), f_*(i))$ for $i > 0$ . Thus $f_*$ is the minimal solution to (4.5) staying strictly above $h$ and the proof is complete. $\square$

**Corollary 5 (The golden ratio rule).** *If $X$ is the radial part of three-dimensional Brownian motion, then the optimal stopping time in (2.14) is given by*

$$\tau_* = \inf\Big\{\, t \geq 0 \mid \frac{X_t - I_t}{I_t} \geq \varphi \,\Big\} \tag{4.10}$$

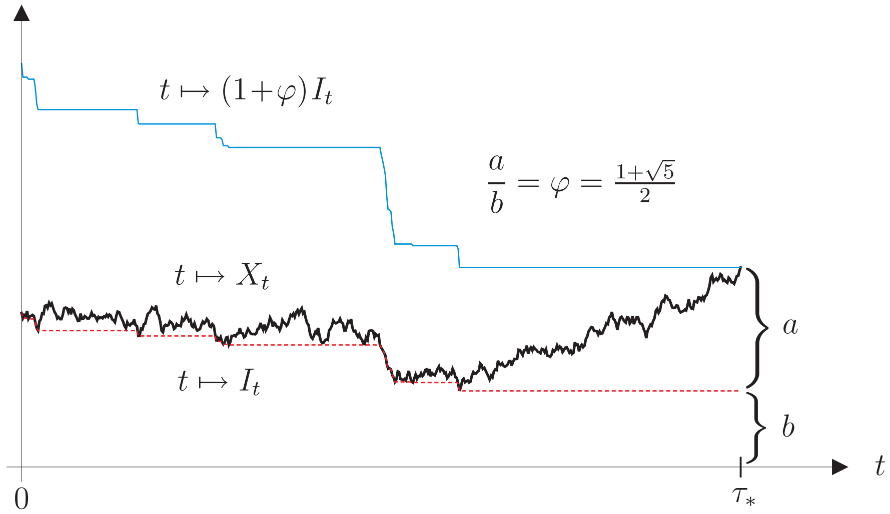


**Figure 2.** The golden ratio rule for the radial part $X$ of three-dimensional Brownian motion.

*where* $\varphi = (1+\sqrt{5})/2 = 1.61\ldots$ *is the golden ratio (see Figure 2).*

**Proof.** In this case $d=3$ and the equation (4.2) reads

$$\lambda^3 - 4\lambda^2 + 4\lambda - 1 = (\lambda-1)(\lambda^2 - 3\lambda + 1) = 0 \tag{4.11}$$

for $\lambda > 0$. Solving the latter quadratic equation and choosing the root strictly greater than $1$ we find that $\lambda = 1+\varphi$ where $\varphi = (1+\sqrt{5})/2 = 1.61\ldots$ is the golden ratio. The optimality of (4.10) then follows from (4.1) and the proof is complete. $\square$

Returning to the result of Theorem 3 above we now determine the law of the transient diffusion process $X$ stopped at the optimal stopping time $\tau_*$ (for related results on the Skorokhod embedding problem see [36, pp. 269-277] and the references therein).

**Proposition 6.** *In the setting of Theorem 3 we have*

$$\mathsf{P}_x(X_{\tau_*} \le y) = \exp\left( -\int_{f_*^{-1}(y)}^{x} \frac{dL(z)}{L(f_*(z)) - L(z)} \right) \tag{4.12}$$

*for* $0 < y \le f_*(x)$ *with* $x > 0$ .

**Proof.** Note that

$$\begin{aligned} \tau_* &= \inf\{\, t \ge 0 \mid X_t \ge f_*(I_t) \,\} \\ &= \inf\{\, t \ge 0 \mid L(X_t) \ge (L \circ f_* \circ L^{-1})(L(I_t)) \,\} \\ &= \inf\{\, t \ge 0 \mid X_t^L \ge f_*^L(I_t^L) \,\} \end{aligned} \tag{4.13}$$

where we set $X_t^L = L(X_t)$ , $f_*^L = L \circ f_* \circ L^{-1}$ and $I_t^L = L(I_t) = \inf_{0 \le s \le t} L(X_s) = \inf_{0 \le s \le t} X_s^L$ for $t \ge 0$. Let $x > 0$ be given and fixed. For $j \le L(x)$ set $G(j) = \int_{-\infty}^{j} g(k)\, dk$ where

$g : (-\infty, L(x)] \rightarrow I\!R$ is a continuously differentiable function with bounded support. Using the fact that $dI_t^L = 0$ when $X_t^L \neq I_t^L$ it is easily verified by Itô's formula that the process $M^L = (M_t^L)_{t \geq 0}$ defined by

$$M_t^L = G(I_t^L) + (X_t^L - I_t^L) G'(I_t^L) \tag{4.14}$$

is a continuous local martingale. Moreover, since $G' = g$ is continuous and has bounded support we see that $M^L$ is bounded and therefore uniformly integrable. By the optional sampling theorem we thus find that

$$\begin{aligned} G^L(x) &= \mathsf{E}_x M_0^L = \mathsf{E}_x M_{\tau_*}^L = \mathsf{E}_x G(I_{\tau_*}^L) + \mathsf{E}_x \big[ (X_{\tau_*}^L - I_{\tau_*}^L) G'(I_{\tau_*}^L) \big] \\ &= \int_{-\infty}^{L(x)} G(j)\, dF(j) + \mathsf{E}_x \big[ (f_*^L(I_{\tau_*}^L) - I_{\tau_*}^L) G'(I_{\tau_*}^L) \big] \\ &= G(j) F(j) \big|_{-\infty}^{L(x)} - \int_{-\infty}^{L(x)} F(j)\, dG(j) + \int_{-\infty}^{L(x)} (f_*^L(j) - j)\, G'(j)\, dF(j) \\ &= G^L(x) - \int_{-\infty}^{L(x)} F(j)\, g(j)\, dj + \int_{-\infty}^{L(x)} (f_*^L(j) - j)\, g(j)\, dF(j) \end{aligned} \tag{4.15}$$

where we set $G^L(x) = G(L(x))$ and $F$ denotes the distribution function of $I_{\tau_*}^L$ under $\mathsf{P}_x$ . Since (4.15) holds for all functions $g$ of this kind, it follows that

$$F'(j) = \frac{F(j)}{f_*^L(j) - j} \tag{4.16}$$

for $j < L(x)$ with $F(L(x)) = 1$ . Solving (4.16) under this boundary condition we find that

$$F(j) = \exp\left( - \int_j^{L(x)} \frac{dk}{f_*^L(k) - k} \right) \tag{4.17}$$

for $j \leq L(x)$ . Recalling that $f_*^L = L \circ f_* \circ L^{-1}$ and substituting $k = L(z)$ it follows that

$$\begin{aligned} \mathsf{P}_x(I_{\tau_*} \leq i) &= \mathsf{P}_x(L(I_{\tau_*}) \leq L(i)) = \mathsf{P}_x(I_{\tau_*}^L \leq L(i)) = F(L(i)) \\ &= \exp\left( - \int_{L(i)}^{L(x)} \frac{dk}{f_*^L(k) - k} \right) = \exp\left( - \int_i^x \frac{dL(z)}{L(f_*(z)) - L(z)} \right) \end{aligned} \tag{4.18}$$

for $i \leq x$ in $(0, \infty)$ . Hence we find that

$$\mathsf{P}_x(X_{\tau_*} \leq y) = \mathsf{P}_x(f_*(I_{\tau_*}) \leq y) = \mathsf{P}_x(I_{\tau_*} \leq f_*^{-1}(y)) = \exp\left( - \int_{f_*^{-1}(y)}^x \frac{dL(z)}{L(f_*(z)) - L(z)} \right) \tag{4.19}$$

for $0 < y \leq f_*(x)$ with $x > 0$ . This completes the proof. $\square$

Specialising this result to the $d$-dimensional Bessel process $X$ of Theorem 4 we obtain the following consequence.

**Corollary 7.** *In the setting of Theorem 4 we have*

$$\mathsf{P}_x(X_{\tau_*} \leq y) = \left( \frac{y}{\lambda x} \right)^{\frac{d-2}{1-(1/\lambda)^{d-2}}} \tag{4.20}$$

*for* $0 < y \le \lambda x$ *with* $x > 0$ .

**Proof.** In this case $f_*(i) = \lambda i$ for $i > 0$ where $\lambda$ is the unique solution to either (4.2) when $d \neq 4$ or (4.3) when $d = 4$ and $L$ is given by (2.7). Inserting these expressions into the right-hand side of (4.12) it is easily verified that this yields (4.20). □

Specialising this further to the radial part $X$ of three-dimensional Brownian motion in Corollary 5 we obtain the following conclusion.

**Corollary 8 (The golden ratio distribution).** *In the setting of Corollary 5 we have*

$$\mathsf{P}_x(X_{\tau_*} \le y) = \Big( \frac{y}{(1+\varphi)\,x} \Big)^{\varphi} \tag{4.21}$$

*for* $0 < y \le (1+\varphi)\,x$ *with* $x > 0$ .

**Proof.** In this case $d = 3$ and $\lambda = 1+\varphi$ so that $(d-2)/(1-(1/\lambda)^{d-2}) = 1/(1-1/(1+\varphi)) = (1+\varphi)/\varphi = \varphi^2/\varphi = \varphi$ . Hence we see that (4.20) reduces to (4.21). □

Note from (4.21) that the density function of $X_{\tau_*}$ under $\mathsf{P}_x$ is given by

$$f_{X_{\tau_*}}(y) = \frac{\varphi}{((1+\varphi)\,x)^{\varphi}}\, y^{\varphi - 1} \tag{4.22}$$

for $0 < y < (1+\varphi)\,x$ with $x > 0$ and equals zero otherwise. We refer to (4.21)+(4.22) as the *golden ratio distribution*. It is easy to see that

$$\mathsf{E}_x X_{\tau_*} = \varphi x \tag{4.23}$$

for $x > 0$ . The fact that this number is strictly greater than $x$ (the initial point corresponding to stopping at once) is not surprising since $X$ is a submartingale. It needs to be recalled moreover that the aim of applying the golden ratio rule $\tau_*$ is to be as close as possible to the time $\theta$ at which the ultimate minimum is attained. We will see in the next section that the golden ratio distribution provides insight as to what extent the golden ratio rule has the power of capturing the ultimate maximum of a strict local martingale.

## 5. Applications in optimal trading

In this section we present some applications of the previous results in problems of optimal trading. We also outline some remarkable connections between such problems and the practice of technical analysis. These applications and connections rest on three basic ingredients that we describe first.

1. *Fibonnaci retracement.* We begin by explaining a few technical terms from the field of applied finance. *Technical analysis* is a financial term used to describe methods and techniques for forecasting the direction of asset prices through the study of past market data (primarily prices themselves plus the volume of their trade). *Support* and *resistance* are concepts in technical analysis associated with the expectation that the movement of the asset price will tend to

cease and reverse its trend of decrease/increase at certain predetermined price levels. A *support/resistance level* is a price level at which the price will tend to find support/resistance when moving down/up. This means that the price is more likely to bounce off this level rather than break through it. One may also think of these levels as turning points of the prices. *Fibonacci retracement* is a method of technical analysis for determining support and resistance levels. The name comes after its use of *Fibonacci numbers* $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ with $F_0 = 0$ and $F_1 = 1$. Fibonacci retracement is based on the idea that after reversing the trend at a support/resistance level, the price will retrace a predictable portion of the past downward/upward move by advancing in the opposite direction until finding a new resistance/support level, after which it will return to the initial trend of moving downwards/upwards. Fibonacci retracement is created by taking two extreme points on a chart showing the asset price as a function of time and dividing the vertical distance between them by the key Fibonacci ratios ranging from $0\%$ (start of the retracement) to $100\%$ (end of the retracement representing a complete reversal to the original trend). The other key Fibonacci ratios are $23.6\%$ (shallow retracement), $38.2\%$ (moderate retracement) and $61.8\%$ (golden retracement). They are obtained by formulae $(F_n/F_{n+3}) \times 100 \approx \varphi^{-3} \times 100$, $(F_n/F_{n+2}) \times 100 \approx \varphi^{-2} \times 100$ and $(F_n/F_{n+1}) \times 100 \approx \varphi^{-1} \times 100$ respectively (see the next paragraph). These retracement levels serve as alert points for a potential reversal at which traders may employ other methods of technical analysis to identify and confirm a reversal. Despite its widespread use in technical analysis of asset prices, there appears to be no (rigourous) explanation of any kind as to why the Fibonacci ratios should be used to this effect. We will show below that the golden ratio rule derived in the previous section offers a rigourous optimality argument for the choice of the golden retracement ( $61.8\%$ ). To our knowledge this is the first time that such an argument has been found/given in the literature.

2. *Golden ratio and Fibonacci numbers.* The link between the two is well known and is expressed by Binet's formula

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{5.1}$$

where $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2 = 1-\varphi = -1/\varphi$. It follows that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi \,. \tag{5.2}$$

This fact is used in the description of Fibonacci retracement above.

3. *The CEV model.* One of the simplest/tractable models for asset price movements that is capable of reproducing the implied volatility smile/frown effect and the (inverse) leverage effect (both observed in the empirical data) is the *Constant Elasticity of Variance* (CEV) model in which the (non-negative) asset price process $Z = (Z_t)_{t \geq 0}$ solves

$$dZ_t = \mu Z_t \, dt + \sigma Z_t^{1+\beta} \, dB_t \tag{5.3}$$

where $\mu \in I\!R$ is the appreciation rate, $\sigma > 0$ is the volatility coefficient, and $\beta \in I\!R$ is the elasticity parameter. If $\beta = 0$ then $Z$ is a geometric Brownian motion which was initially considered in [28] and [37]. For $\beta \neq 0$ this model was firstly considered in [3] for $\beta < 0$ and then in [13] for $\beta > 0$. Due to its predictive power and tractability, the CEV model is widely used by practitioners in the financial industry, especially for modelling prices of equities and commodities. If $\beta < 0$ then the model embodies the *leverage effect* (commonly observed in equity markets) where the volatility of the asset price increases as its price decreases. If $\beta > 0$ then the model embodies the *inverse leverage effect* (often observed in commodity markets) where the volatility of the asset price increases when its price increases. For example, it is reported in [17] that the elasticity coefficient $\beta$ for Gold on the London Bullion Market in the period from 2000 to 2007 was approximately 0.49. Similar elasticity coefficients have also been observed for other precious metals (such as Copper for instance).

In the remainder of this section we focus on the case when $\mu = 0$ and $\beta > 0$. It is well known (cf. [13]) that $Z$ solving (5.3) is a strict local martingale (a local martingale which is not a true martingale) in this case due to the fact that $t \mapsto \mathsf{E}_z(Z_t)$ is strictly decreasing on $I\!R_+$ for any $z > 0$. This also implies that $Z$ does not admit an equivalent martingale measure so that the CEV model may admit arbitrage opportunities. One way of looking at the models of this type is to associate them with asset price bubbles (see [21]). After soaring to a finite ultimate maximum (bubble) at a finite time, the asset price will tend to zero as time goes to infinity, and the central question for a holder of the asset becomes when to sell so as to be as close as possible to the time at which the ultimate maximum is attained. More precisely, introducing the running maximum process $S = (S_t)_{t \ge 0}$ associated with $Z$ by setting

$$S_t = \sup_{0 \le s \le t} Z_s \tag{5.4}$$

and recalling that $Z_t \to 0$ as $t \to \infty$, we see that the ultimate supremum $S_\infty = \sup_{t \ge 0} Z_t$ is attained at some random time $\theta$ in the sense that

$$Z_\theta = S_\infty \tag{5.5}$$

with $\mathsf{P}_z$-probability one for $z > 0$ given and fixed. The optimal selling problem addressed above then becomes the optimal prediction problem

$$V(z) = \inf_\tau \mathsf{E}_z\big(|\theta - \tau| - \theta\big) \tag{5.6}$$

where the infimum is taken over all stopping times $\tau$ of $Z$ (with finite mean) and $z > 0$ is given and fixed. We will now show that due to the well-known connection between CEV and Bessel processes (dating back to similar transformations in [3] and [13]) the problem (5.6) can be reduced to the problem (2.14) solved above.

4. *The golden ratio rule for the CEV process.* For $d > 2$ given and fixed consider the $d$-dimensional Bessel process $X$ solving (2.6) under $\mathsf{P}_x$ with $x > 0$. Recall that the scale function $L$ is given by (2.7) and set $K(x) = -c_\sigma L(x)$ for $x > 0$ with $c_\sigma > 0$ given and fixed. Then the process $Z = K(X)$ defined by

$$Z_t = K(X_t) = \frac{c_\sigma}{X_t^{d-2}} \tag{5.7}$$

is on natural scale and Itô's formula shows that $Z$ solves

$$dZ_t = \sigma Z_t^{1+\frac{1}{d-2}}\, d\tilde{B}_t \tag{5.8}$$

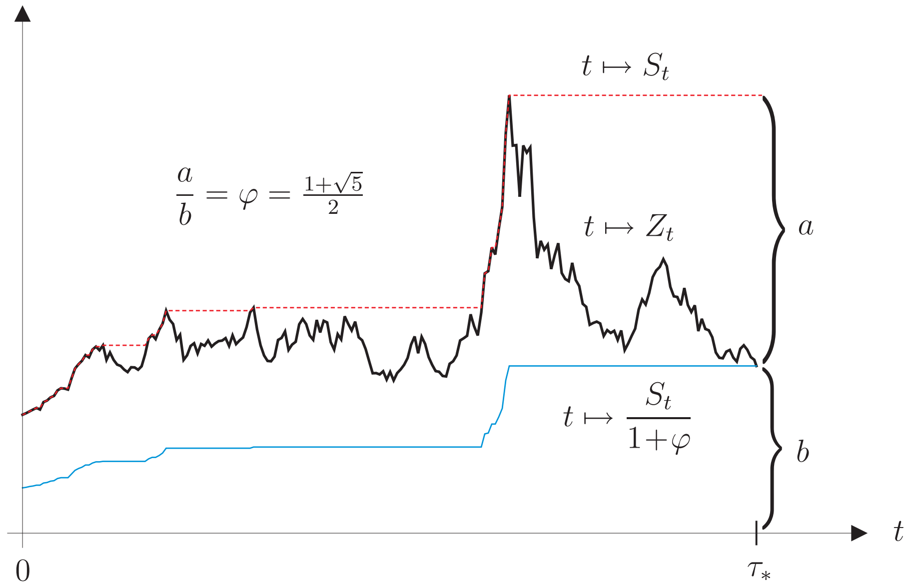


**Figure 3.** The golden ratio rule for the CEV process $Z = 1/X$ where $X$ is the radial part of three-dimensional Brownian motion. Note the presence of a bubble and its relation to the golden ratio.

where $\sigma = (d-2)/c_\sigma^{1/(d-2)}$ and $\tilde{B} = -B$ is a standard Brownian motion. Note that the equation (5.8) coincides with the equation (5.3) for $\mu = 0$ and $\beta = 1/(d-2)$. By the uniqueness in law for this equation (among positive solutions) it follows that $Z = K(X)$ is a CEV process. From the properties of $X$ it follows that after starting at $z = K(x) > 0$, the process $Z$ stays strictly positive (without exploding at a finite time) and $Z_t \to 0$ with $\mathsf{P}_z$-probability one as $t \to \infty$. This shows that $\theta$ in (5.5) is well defined. Moreover, due to the reciprocal relationship (5.7) we see that the time of the ultimate maximum $\theta$ for $Z$ in (5.5) coincides with the time of the ultimate minimum $\theta$ for $X$ in (2.10) and hence the problem (5.6) has the same solution as the problem (2.14) (note also that the natural filtrations of $Z$ and $X$ coincide so that $\tau$ is a stopping time of $Z$ if and only if $\tau$ is a stopping time of $X$). Since $X_t \geq \lambda I_t$ if and only if $Z_t/c_\sigma = X_t^{2-d} \leq \lambda^{2-d} I_t^{2-d} = \lambda^{2-d} S_t/c_\sigma$ it follows from (4.1) in Theorem 4 that the optimal stopping time in (5.6) is given by

$$\tau_* = \inf \{ \, t \geq 0 \mid S_t \geq \lambda^{d-2} Z_t \, \} \tag{5.9}$$

where $\lambda$ is the unique solution to either (4.2) or (4.3) belonging to $(2^{1/(d-2)}, \infty)$. In particular, if $d = 3$ then we know from (4.11) that $\lambda = 1+\varphi$ so that (5.9) reads

$$\tau_* = \inf \left\{ t \geq 0 \;\Big|\; \frac{S_t - Z_t}{Z_t} \geq \varphi \right\}. \tag{5.10}$$

This is the *golden ratio rule* for the CEV process $Z = 1/X$ where $X$ is the radial part of three-dimensional Brownian motion (see *Figure* 3).

To relate the golden ratio rule (5.10) to Fibonacci retracement discussed above, let $a = S_{\tau_*} - Z_{\tau_*}$ denote the larger quantity and let $b = Z_{\tau_*}$ denote the smaller quantity in the golden ratio rule. To determine the percentage of $a$ in $a+b$ we need to calculate

$$\frac{a}{a+b} = \frac{S_{\tau_*} - Z_{\tau_*}}{S_{\tau_*}} = 1 - \frac{Z_{\tau_*}}{S_{\tau_*}} = 1 - \frac{1}{1+\varphi} = \frac{\varphi}{1+\varphi} = \frac{1}{\varphi}\,. \tag{5.11}$$

Multiplying this expression by 100 gives 61.8% and this is exactly the *golden retracement* discussed above. In view of the optimality of (5.10) in (5.6) we see that the golden retracement of 61.6% for the CEV process $Z = 1/X$ (starting close to zero) where $X$ is the radial part of three-dimensional Brownian motion can be seen as a rational support level (in the sense that rational investors who aim at selling the asset at the time of the ultimate maximum will sell the asset at the time of the golden retracement and therefore the asset price could be expected to raise afterwards). To our knowledge this is the first time that such a rational optimality argument for the golden retracement has been established.

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