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Optimal Stopping and Maximal Inequalities for Geometric Brownian Motion

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Explicit formulas are found for the payoff and the optimal stopping strategy of the optimal stopping problem:

$$\sup_{\tau} E\left(\max_{0\le t\le\tau} X_t - c\tau \right)$$

where $X = (X_t)_{t\ge 0}$ is geometric Brownian motion with drift μ and volatility $\sigma > 0$, and the supremum is taken over all stopping times for X . The payoff is shown to be finite, if and only if $\mu < 0$. The optimal stopping time is given by:

$$\tau_* = \inf\left\{ t > 0 \mid X_t = g_*\left(\max_{0\le s\le t} X_s \right) \right\}$$

where $s \mapsto g_*(s)$ is the *maximal* solution of the (nonlinear) differential equation:

$$\frac{\partial g}{\partial s} = K\,\frac{g^{\Delta+1}}{s^{\Delta}-g^{\Delta}} \quad (s > 0)$$

under the condition $0 < g(s) < s$, where $\Delta = 1 - 2\mu/\sigma^2$ and $K = \Delta\sigma^2/2c$. The estimate is established:

$$g_*(s) \sim \left(\frac{\Delta-1}{K\Delta} \right)^{1/\Delta} s^{1-1/\Delta}$$

as $s \to \infty$. Applying these results we prove the following maximal inequality:

$$E\left(\max_{0\le t\le\tau} X_t \right) \le 1 - \frac{\sigma^2}{2\mu} + \frac{\sigma^2}{2\mu}\exp\left(-\frac{(\sigma^2-2\mu)^2}{2\sigma^2}\,E(\tau) - 1 \right)$$

where τ may be any stopping time for X . This extends the well-known identity:

$$E\left(\sup_{t>0} X_t \right) = 1 - \frac{\sigma^2}{2\mu}$$

and is shown to be sharp. The method of proof relies upon a smooth pasting guess (for the Stephan problem with moving boundary) and Itô-Tanaka's formula (being applied two-dimensionally). The key point and main novelty in our approach is the maximality principle for the moving boundary (the optimal stopping boundary is the maximal solution of the differential equation obtained by a smooth pasting guess). We think that this principle is by itself of theoretical and practical interest.

1. Introduction

The main purpose of the paper is to describe the structure and derive explicit formulas for the payoff and the optimal stopping strategy in the optimal stopping problem associated with a geometric Brownian motion, where the gain is given by the maximum of the process, while the

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cost is proportional to the duration of the observation time. The main interest for such a "regretless" class of optional stopping problems comes from option pricing theory (see [9], [6], [2]).

The results obtained are used in the last section to derive a maximal inequality for geometric Brownian motion. To the best of our knowledge it appears to be the first of its kind.

The method of proof relies upon a smooth pasting guess (for the Stephan problem with moving boundary) and a maximality principle for the moving boundary (the optimal stopping boundary turns out to be a maximal solution of the differential equation obtained from the smooth pasting guess). The smooth pasting guess allows us to apply Itô-Tanaka's formula (in a two-dimensional setting), while the maximality principle enables us to pick out the optimal stopping boundary from amongst all possible ones in a unique way. It is this maximality principle which is the key point and main novelty in our approach (compare with [5] and [9]), and it is not clear how the problem could be solved without its use. It should be noted that the optimal stopping boundary found here is nontrivial, and it is probably impossible in a closed form to find even a particular (nonzero) solution of the differential equation it solves, thus showing the full power of the method. Moreover, we feel that this maximality principle might be fulfilled in a similar setting involving other diffusions as well, and therefore find it by itself of theoretical and practical interest.

The problem is more precisely formulated in the beginning of Section 2. The remaining part of Section 2 is devoted to the smooth pasting guess and formulation of the maximality principle. In Section 3 we prove the existence and uniqueness of the maximal solution to the differential equation obtained by the smooth pasting guess, and derive sharp estimates for this solution. These facts are used in Section 4 where the proof is presented for the prescribed form of the payoff and the optimal stopping strategy. Applying these results in Section 5, we derive a new type of maximal inequality for geometric Brownian motion.

2. The maximality principle

In this section we shall introduce the setting and formulate the main problem under consideration. Applying the so-called *principle of smooth fit* (smooth pasting) which was enunciated by A. Kolmogorov in the 1950's and used later by many authors (see [5] and [9] for more details), we shall first guess a solution (up to the choice of the optimal stopping boundary satisfying a (nonlinear) differential equation obtained by the smooth pasting guess). Then we will formulate the *maximality principle* which will enable us to select the optimal stopping boundary in a unique way. The work started here will be completed in the following two sections. The next section is devoted to the differential equation itself, while the following section contains a proof that the principle of smooth fit and the maximality principle are indeed satisfied.

1. Suppose we are given a geometric Brownian motion $X = (X_t)_{t \geq 0}$ with drift $\mu < 0$ and volatility $\sigma > 0$, being defined on the probability space $(\Omega, \mathcal{F}, P)$, and X under P starts at $x > 0$. Thus, we have:

$$X_t = x \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right) \;,\; t \geq 0 \tag{2.1}$$

where $B = (B_t)_{t \geq 0}$ is standard Brownian motion which under P starts at 0. The process X satisfies the stochastic differential equation:

$$(2.2) \qquad dX_t = \mu X_t\, dt + \sigma X_t\, dB_t \ .$$

The infinitesimal operator of X in $]0,\infty[$ equals:

$$(2.3) \qquad \mathbf{L_X} = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} \ .$$

Given $s \geq x > 0$, introduce the maximum process associated with X :

$$(2.4) \qquad S_t = \Big(\max_{0 \leq r \leq t} X_r \Big) \vee s$$

for $t \geq 0$. Then under $P_{x,s} := P$ the process (X,S) starts at (x,s) .

2. The main problem under consideration in this paper is to find explicit formulas for the payoff and the optimal stopping strategy in the optimal stopping problem:

$$(2.5) \qquad V(x,s) := \sup_{\tau} E_{x,s}\big(S_\tau - c\tau\big)$$

where the supremum is taken over all stopping times for X . Recalling Doob's theorem:

$$(2.6) \qquad P\Big\{ \sup_{t>0} \big(B_t - \alpha t\big) \geq \beta \Big\} = e^{-2\alpha\beta}$$

where α , $\beta > 0$, we easily find out that:

$$(2.7) \qquad E\Big(\sup_{t>0} X_t \Big) = 1 - \frac{\sigma^2}{2\mu}$$

for $\mu < 0$, while this expectation equals $+\infty$ if $\mu \geq 0$. Thus $V(x,s) < \infty$ for all $\mu < 0$. Moreover, we will obtain as a consequence of Theorem 4.1 below that the converse is also true. In other words, we have $V(x,s) < \infty$ if and only if $\mu < 0$. This explains why we do assume from the very beginning that the drift μ is strictly negative (for $\mu \geq 0$ we have $V(x,s) = +\infty$). Finally, note that since $V(x,s) \geq E_{x,s}(S_0 - c\,0) = s > -\infty$, by (2.7) we see that the supremum in (2.5) could equivalently be taken over all stopping times τ for X for which $E(\tau) < \infty$.

3. In order to guess candidates for the payoff $V(x,s)$ and the optimal stopping strategy τ_* , we should note that the state space of the Markov process (X,S) equals $E = \{\, (x,s) \mid 0 < x \leq s \,\}$, and inside E the process moves horizontally (only the first coordinate changes) when being off the diagonal $x = s$, while it could increase (in the second coordinate) only after hitting the diagonal. Moreover, if analyzing the structure of the expression for $V(x,s)$ in (2.5) we may note that there is a very strong intuitive argument for the existence of a point $g_*(s) \in \,]0,s[$ (for the given vertical level $s > 0$) at which (or being on the left from it) we should stop the process instantly. Otherwise, we could wait too long (before stopping the process) which eventually would make the payoff negative, due (in part) to its cost which is proportional to the duration of the observation time and (on the other hand) to a rather moderate increase of the process X . (Considering the exit times of X from small enough intervals around the given starting point $x > 0$, one may easily find that $V(x,x) > x$. Thus we cannot stop the process at the diagonal, and therefore the map $s \mapsto g_*(s)$ is to lie strictly below the diagonal in E .) For the reasons just explained it seems reasonable to assume that the optimal stopping strategy for the problem (2.5) is of the form:

(2.8) $$\tau_* = \inf \left\{ \, t > 0 \mid X_t \leq g_*(S_t) \, \right\}$$

where $G_* = \left\{ \, \big(g_*(s), s\big) \mid s > 0 \, \right\} \subset E \setminus \{\text{diagonal}\}$ is the optimal stopping boundary. Thus, the state space E of the process (X, S) is to be divided into two parts: the *domain of the continued observation* $C_* = \{ \, (x, s) \mid 0 < g_*(s) < x \leq s \, \}$ and the *stopping domain* $D_* = \{ \, (x, s) \mid 0 < x \leq g_*(s) < s \, \}$. The border line between C_* and D_* is the optimal stopping boundary G_* which is to be determined.

4. If our premises were true, then the payoff would be given by:

(2.9) $$V(x, s) = E_{x,s}\big(S_{\tau_*} - c\tau_*\big) \ .$$

Since τ_* may be viewed as the exit time of the "diffusion" (X, S) from an open set, and since the infinitesimal operator of (X, S) at each horizontal level $g_*(s) < x < s$ coincides with the infinitesimal operator of X (at each such level only the first coordinate of (X, S) changes), by general Markov process theory it is clear that $V(x, s)$ is to satisfy the differential equation:

(2.10) $$\mathbf{L_X} V(x, s) = c$$

for $x \in \,]g_*(s), s[$. It is a *Stephan problem with moving (free) boundary* (the boundary $s \mapsto g_*(s)$ of the problem is unknown as well). In order to approach possible solutions to (2.10), we should determine some smoothness conditions at the boundaries. One condition is obvious: if the process (X, S) starts at the optimal stopping boundary G_* , we must stop *instantly*:

(2.11) $$V(x, s)\Big|_{x = g_*(s)+} = s \ .$$

The next condition might be viewed as its refinement. It is the condition of *smooth pasting* at the optimal stopping boundary:

(2.12) $$\frac{\partial V}{\partial x}(x, s)\Big|_{x = g_*(s)+} = 0 \ .$$

The last condition we shall make use of is the condition of *normal reflection* at the diagonal:

(2.13) $$\frac{\partial V}{\partial s}(x, s)\Big|_{x = s-} = 0 \ .$$

5. In this context it is very instructive to look below into the proof of Theorem 4.1 in order to see how (2.10)-(2.13) are matched together when applying Itô-Tanaka's formula to the process $V_*(X_t, S_t)$ for $t \geq 0$ (see (4.16)+(4.17)). It should be noted that the candidate $V_*(x, s)$ (which will shortly be given) is twice continuously differentiable in $E \setminus G_*$, while the smooth pasting condition (2.12) implies that $x \mapsto V(x, s)$ at the vertical level $s > 0$ is convex and continuously differentiable at $g_*(s)$. Thus, when the process (X, S) is "away" from the optimal stopping boundary G_* , Itô's formula can be applied. Moreover, when the process approaches the "dangerous" place $g_*(s)$ at the vertical level $s > 0$, then only the first coordinate X of the process changes, and thus Itô-Tanaka's formula may be applied (the measure associated with the

first derivative, due to the smooth pasting condition, has no atoms).

6. The equation (2.10) with $\mathbf{L_X}$ from (2.3) is of Cauchy's type:

$$\frac{\sigma^2}{2} x^2 y'' + \mu x y' = c \ . \tag{2.14}$$

Since $\Delta := 1 - 2\mu/\sigma^2 \neq 0$, the general solution to (2.14) is easily verified to be given by:

$$y = A x^{\Delta} - \frac{2c}{\Delta\sigma^2} \log(x) + B \ . \tag{2.15}$$

In other words, the payoff is of the form:

$$V(x,s) = A(s) x^{\Delta} - \frac{2c}{\Delta\sigma^2} \log(x) + B(s) \tag{2.16}$$

for $0 < g_*(s) < x < s$. By (2.11) and (2.12) we readily find:

$$A(s) = \frac{2c}{\Delta^2 \sigma^2 g_*(s)^{\Delta}} \tag{2.17}$$

$$B(s) = s + \frac{2c}{\Delta\sigma^2} \log\Big(g_*(s)\Big) - \frac{2c}{\Delta^2\sigma^2} \ . \tag{2.18}$$

Inserting this into (2.16) we get:

$$V(x,s) = \frac{2c}{\Delta^2\sigma^2} \left(\left(\frac{x}{g_*(s)} \right)^{\Delta} - \log \left(\frac{x}{g_*(s)} \right)^{\Delta} - 1 \right) + s \tag{2.19}$$

for $0 < g_*(s) < x < s$. Moreover, it is clear from our reasoning above that $V(x,s) = s$ for $0 < x \le g_*(s)$. Finally, by using condition (2.13) we easily find that the optimal stopping boundary $s \mapsto g_*(s)$ is to satisfy the differential equation:

$$g'(s) = K \, \frac{g(s)^{\Delta+1}}{s^{\Delta} - g(s)^{\Delta}} \quad (s > 0) \tag{2.20}$$

where $K = \Delta\sigma^2/2c$. This is the maximum information obtained by (2.10)-(2.13).

7. In this way we have arrived at the crucial point of our approach: how to determine the optimal stopping boundary $s \mapsto g_*(s)$? All we know so far is that $g_*(s)$ is to satisfy the equation (2.20). But how to choose the right solution from (2.20)? A priori this is not clear. In a similar context of [5] and [9], the authors were gifted enough to be able to pick up the right solution by guessing, but this approach breaks down here, since it is impossible to guess even a particular (nonzero) solution to (2.20) under our condition $\Delta > 1$ (see Section 3). For these reasons we were forced to search for an argument which would help us to select the right solution from (2.20). A very simple observation in this direction is already stated above when deriving (2.8). Namely, it is clear from the structure of the payoff that we shouldn't wait too long before stopping the process. (By analytic methods it is possible to clarify this intuitive argument, and make it more quantitative, but for simplicity we shall refrain from presenting the details.) Such an argument leads us to the conclusion that *the optimal stopping boundary* $s \mapsto g_*(s)$ *should be the maximal solution to (2.20) which lies below the diagonal*. (In this context it is instructive to examine the last part of the proof

of Theorem 4.1 below. There, it is easily seen that we are forced to choose the optimal stopping boundary $s \mapsto g_*(s)$ as large as possible in value (see (4.18)+(4.20)+(4.21) and (4.22)+(4.24)). In such a way the expected waiting time $E(\tau_*)$ to hit the boundary is minimal (see (4.20)).

8. It is our pleasure to report that this maximality principle turns out to be true. We moreover feel that such a principle could be fulfilled in a similar setting for other diffusions as well, but will not enter into such a discussion here. Instead, motivated by the maximality principle in our particular case, we shall in the following section devote our attention to the equation (2.20) itself. It is a first order nonlinear differential equation, and we were unable to find any disscusion of it in the existing literature.

3. Existence and uniqueness of the maximal solution

1. Throughout we shall consider the first order *nonlinear* differential equation:

$$\frac{\partial y}{\partial x} = K \, \frac{y^{\Delta+1}}{x^{\Delta} - y^{\Delta}} \quad (x > 0) \tag{3.1}$$

where $K > 0$ and $\Delta > 1$ are given and fixed constants. Motivated by the maximality principle explained in the preceding section, we shall show that there exists a maximal non-negative solution to (3.1) which lies below the diagonal $y = x$. This solution is given by:

$$y_* = \sup \left\{ \, y \mid y \text{ solves (3.1) and satisfies } 0 < y < x \text{ for all } x > 0 \right\} \tag{3.2}$$

where the supremum is attained and therefore may be written as a maximum. A more constructive way of obtaining y_* is to choose any $x_0 > 0$ and find a sequence of solutions y_n to (3.1) satisfying $y_*(x_0) = \lim_{n \to \infty} y_n(x_0)$. Then we get:

$$y_*(x) = \lim_{n \to \infty} y_n(x) \tag{3.3}$$

for all $x > 0$. (In the end of this section we indicate how to compute the value of y_* at any given point as exactly as desired.) Finally, we shall show that the following estimates are valid:

$$\frac{x}{\left(1 + \frac{K\Delta}{\Delta-1} \, x\right)^{1/\Delta}} \le y_*(x) \le \left(\frac{\Delta-1}{K\Delta}\right)^{1/\Delta} x^{1-1/\Delta} \tag{3.4}$$

for all $x > (\Delta-1)/K\Delta$ (the left-hand inequality holds for all $x > 0$ as well). These estimates are sharp and the following consequences are valid:

$$y_*(x) \sim \left(\frac{\Delta-1}{K\Delta}\right)^{1/\Delta} x^{1-1/\Delta} \quad (x \to \infty) \tag{3.5}$$

$$\lim_{\Delta \to 1+} y_{*,\Delta}(x) = 0 \quad (x > 0) \tag{3.6}$$

(compare (3.6) with the solution for the case $\Delta = 1$ below). To conclude we shall note that any solution Y_K to (3.1) with $K > 0$ given and fixed is *self-similar* in the following sense:

$$Y_K(x) = \frac{1}{K} \, Y_1(Kx) \tag{3.7}$$

for all $x > 0$. Thus in our proof below it is no restriction to assume in (3.1) that $K = 1$.

2. The proof of the facts just presented is essentially based upon a well-known theorem on the existence and uniqueness of solutions for the first order nonlinear differential equations (of normal form). For convenience we shall recall its statement. *The initial value problem*:

$$\frac{\partial y}{\partial x} = f(x,y) \quad ; \quad y(x_0) = y_0 \tag{3.8}$$

has a unique solution defined on an interval containing x_0 *, whenever* $f(x,y)$ *and* $(\partial f/\partial y)(x,y)$ *are continuous on some open rectangle containing* (x_0,y_0) . The method of successive approximations due to Picard is used in its proof, and therefore this theorem will be referred to as *Picard's theorem* below.

First note that Picard's theorem applies to the case of (3.1) above, whenever x_0 and y_0 are taken strictly positive, and this explains the constructive way for obtaining y_* as described following (3.2) above. Moreover, to verify that y_* defines a solution to (3.1), it is enough to replace the equation (3.1) by an equivalent integral equation and use monotone convergence theorem (the sequence y_n converging to y_* in (3.3) above may be taken increasing). The arguments just described are to be completed by showing that there exists at least one (global) solution to (3.1) satisfying $0 < y < x$ for all $x > 0$. To show its existence, we find it convenient to replace the equation (3.1) under $0 < y < x$ with the equivalent system:

$$\frac{\partial z}{\partial x} = \frac{\Delta z^2}{x^{\Delta} - z} \qquad (\, 0 < z < x^{\Delta} \,) \tag{3.9}$$

upon identifying $z = y^{\Delta}$. In order to show the global existence of a solution for (3.9), we shall make use of the following maps:

$$w_{\alpha}(x) = \alpha x^{\Delta - 1} \tag{3.10}$$

being defined for all $x > 0$ with $\alpha > 0$. Since $w'_{\alpha}(x) = \alpha(\Delta - 1)x^{\Delta-2}$, in view of applying Picard's theorem, we shall determine those $\alpha > 0$ for which there exists $x_0(\alpha) > 0$ such that:

$$\frac{\Delta \, (w_{\alpha}(x))^2}{x^{\Delta} - w_{\alpha}(x)} \leq \alpha \, (\Delta - 1) \, x^{\Delta - 2} \tag{3.11}$$

for all $x \geq x_0(\alpha)$. From (3.11) we find that such α's must satisfy:

$$0 < \alpha \leq \left(\frac{\Delta}{\Delta - 1} + \frac{1}{x} \right)^{-1} \tag{3.12}$$

for all $x \geq x_0(\alpha)$. Thus, for $0 < \alpha < (\Delta - 1)/\Delta$ given, we may take:

$$x_0(\alpha) = \left(\frac{1}{\alpha} - \frac{\Delta}{\Delta - 1} \right)^{-1} \tag{3.13}$$

and (3.11) will hold. Note that $\alpha < x_0(\alpha)$, thus $w_{\alpha}(x) < x^{\Delta}$ for $x \geq x_0(\alpha)$. Now applying Picard's theorem repeatedly and using the fact that $z \mapsto \Delta z^2/(x^{\Delta} - z)$ is increasing, the solution z to (3.9) satisfying $z(x_0(\alpha)) = \alpha(x_0(\alpha))^{\Delta - 1}$ can be continued for all $x \geq x_0(\alpha)$ to stay below

$w_\alpha(x)$ and thus below x^Δ as well. Moreover, this solution extends to $]0, x_0(\alpha)[$ as well, since the "local extensions" obtained by Picard's theorem when moving to the left from $x_0(\alpha)$ cannot hit either zero nor diagonal, due to the specific form of the direction field $f(x,z) = \Delta z^2/(x^\Delta - z)$ in (3.9). This proves the existence of a solution for (3.9), and thus there exists at least one (positive) solution to (3.1) which stays below the diagonal as required.

Moreover, from the construction just presented it is clear that the maximal solution z_* to (3.9) satisfies the inequality:

$$z_*\big(x_0(\alpha)\big) \,\geq\, w_\alpha\big(x_0(\alpha)\big) \,=\, \frac{\big(x_0(\alpha)\big)^\Delta}{1 + \frac{\Delta}{\Delta-1}\, x_0(\alpha)}\;. \tag{3.14}$$

This proves the left-hand inequality in (3.4) for $K = 1$. The general case follows by using (3.7). By the same argument, it remains to prove the right-hand inequality in (3.4) only for $K = 1$.

To do this, we shall note that:

$$\frac{\Delta z^2}{x^2 - z} \,\geq\, \frac{\Delta z^2}{x^2} \tag{3.15}$$

for all $x > 0$ with $z \geq 0$. Moreover, it is easily verified that the general solution to the equation:

$$\frac{\partial z}{\partial x} \,=\, \frac{\Delta z^2}{x^2} \quad (\, x > 0 \,) \tag{3.16}$$

is given by the following formula:

$$Z_a(x) \,=\, \left(\frac{\Delta}{\Delta - 1}\, \frac{1}{x^{\Delta-1}} + a \right)^{-1} \tag{3.17}$$

where a is a constant, subject to a condition. Hence we see that Z_0 given by:

$$x \,\longmapsto\, \frac{\Delta - 1}{\Delta}\, x^{\Delta-1} \tag{3.18}$$

is the maximal non-negative solution to (3.16). Suppose now that:

$$z_*(\tilde{x}) \,>\, Z_0(\tilde{x}) \tag{3.19}$$

for some $\tilde{x} > 0$. Fix $\beta \in\,]Z_0(\tilde{x}), z_*(\tilde{x})[$ and consider the initial value problem (3.16) with $z(\tilde{x}) = \beta$. Then by Picard's theorem we find:

$$z_*(x) \,>\, Z_{\tilde{a}}(x) \tag{3.20}$$

for all $x \geq \tilde{x}$ with some $\tilde{a} < 0$ (which is easily found explicitly from $Z_{\tilde{a}}(\tilde{x}) = \beta$). Hence:

$$\frac{1}{z_*(x)} \,\leq\, \frac{1}{Z_{\tilde{a}}(x)} \,\rightarrow\, \frac{1}{\tilde{a}} \,<\, 0 \tag{3.21}$$

as $x \rightarrow \infty$. This contradicts the fact that z_* is non-negative, and completes the proof of the right-hand inequality in (3.4).

3. We shall conclude this section with two supplements. The first is devoted to the equation (3.1) in the case $\Delta = 1$. We shall show that in this case there is no strictly positive solution which stays below the diagonal. (In the context of our main problem in Section 2, this corresponds to the case when the drift μ equals zero and the payoff (2.5) is infinite). In the second remark we shall indicate how estimates (3.4) can be used to compute the maximal solution y_* of (3.1) lying below the diagonal as close to the exact values as desired. This is directly applied to the exact computation of the optimal stopping boundary $s \mapsto g_*(s)$ for our main problem in Section 2.

4. The equation (3.1) for $\Delta = 1$ is equivalently (up to dividing by zero) written as follows (for simplicity we shall assume that $K = 1$ as well):

$$y + \left(1 - \frac{x}{y}\right) \frac{\partial y}{\partial x} = 0 \quad (x > 0) \ . \tag{3.22}$$

It is easily seen that this equation is not exact. However, multiplying through in (3.22) by $(1/y)\exp(1/y)$, the equation obtained becomes exact, due to the identity:

$$\frac{\partial}{\partial y}\left(\exp\left(y^{-1}\right)\right) = \frac{\partial}{\partial x}\left(y^{-1}\exp\left(y^{-1}\right)\left(1 - xy^{-1}\right)\right) . \tag{3.23}$$

Hence by a standard method we find that each (nonzero) solution y to (3.22) satisfies the formula:

$$xe^{1/y} + \int_{t_0}^{y} \frac{e^{1/t}}{t}\, dt = C \tag{3.24}$$

for some constant C and any $t_0 > 0$. Thus, the equation (3.1) with $\Delta = 1$ admits a closed form for its solutions. (This is very likely false in the main case $\Delta > 1$ treated above.)

Moreover, let us show that there is no solution to (3.22) satisfying $0 < y < x$ for all $x > 0$. Indeed, if so, since $y' > 0$ and thus y is increasing, by (3.24) we would have:

$$e^{1/y_1} = xe^{1/y} + \int_{y_1}^{y} \frac{e^{1/t}}{t}\, dt \geq xe^{1/x} + \log(y/y_1) \geq x \tag{3.25}$$

for all $x \geq 1$ where $y_1 = y(1)$. Letting $x \to \infty$ this leads to a contradiction.

5. Here we indicate how estimates (3.4) can be used to approximate the value of y_* evaluated at any given point as close as desired. Let x_0 and $\varepsilon > 0$ be given and fixed. Suppose we want to compute $y_*(x_0)$ within an error up to the size of ε . For this, we shall first find a point $x_\varepsilon > x_0$ such that the difference of the two bounds found in (3.4) when being evaluated at x_ε is smaller than ε . (Note that this difference tends to zero when the argument tends to infinity.) Next we shall solve (numerically) the differential equation (3.1) by going from x_ε to x_0 backward, firstly under the initial condition $y(x_\varepsilon)$ being equal to the left-hand bound in (3.4) evaluated at x_ε , and then to the right-hand bound, respectively. In this way we get two solutions y_1 and y_2 of (3.1) satisfying:

$$y_1(x) \leq y_*(x) \leq y_2(x) \tag{3.26}$$

for all $x_0 \leq x \leq x_\varepsilon$. It remains to be noted from (3.1) that:

$$(3.27) \qquad y_2'(x) - y_1'(x) \geq 0$$

for all $x > 0$. Thus the difference $x \mapsto y_2(x) - y_1(x)$ is increasing on $]0, \infty[$. Therefore:

$$(3.28) \qquad \max \{ y_*(x_0) - y_1(x_0) \, , \, y_2(x_0) - y_*(x_0) \} \leq y_2(x_0) - y_1(x_0) \leq y_2(x_\varepsilon) - y_1(x_\varepsilon) \leq \varepsilon \, .$$

Thus, either $y_1(x_0)$ or $y_2(x_0)$ is within ε of $y_*(x_0)$. This completes the claim.

4. The payoff and the optimal stopping strategy

In this section we prove that the explicit formulas for the payoff and the optimal stopping strategy guessed in Section 2 are correct. This establishes the validity of the principle of smooth fit and the maximality principle for the optimal stopping problem under consideration. The results obtained here are further applied in the following section. The fundamental result of the paper is formulated in the theorem as follows.

Theorem 4.1

Let $X = (X_t)_{t \geq 0}$ be geometric Brownian motion with drift $\mu < 0$ and volatility $\sigma > 0$ as defined in (2.1), and let $S = (S_t)_{t \geq 0}$ be the maximum process associated with X as defined in (2.4). Consider the optimal stopping problem with the payoff given by:

$$(4.1) \qquad V(x,s) := \sup_\tau E_{x,s} \big(S_\tau - c\tau \big)$$

for $s \geq x > 0$ given and fixed (under $P_{x,s}$ the process (X,S) starts at (x,s)), where the supremum is taken over all stopping times for X .

1. Then $V(x,s) < \infty$ if and only if $\mu < 0$. The optimal stopping strategy in (4.1) (the stopping time at which the supremum is attained) is given by the following formula:

$$(4.2) \qquad \tau_* = \inf \{ t > 0 \mid X_t \leq g_*(S_t) \}$$

where $s \mapsto g_*(s)$ is the maximal solution of the differential equation:

$$(4.3) \qquad g'(s) = K \frac{g(s)^{\Delta+1}}{s^\Delta - g(s)^\Delta} \quad (s > 0)$$

under the condition $0 < g(s) < s$, with $\Delta = 1 - 2\mu/\sigma^2$ and $K = \Delta\sigma^2/2c$. The payoff is given by the following formula:

$$(4.4) \qquad V(x,s) = \frac{2c}{\Delta^2\sigma^2} \left(\left(\frac{x}{g_*(s)} \right)^\Delta - \log \left(\frac{x}{g_*(s)} \right)^\Delta - 1 \right) + s \;\; , \quad \textit{if} \;\; g_*(s) < x \leq s$$

$$= s \;\; , \quad \textit{if} \;\; 0 < x \leq g_*(s) \, .$$

2. The optimal stopping boundary $s \mapsto g_*(s)$ is strictly increasing on $]0, \infty[$ and satisfies $0 < g_*(s) < s$ for all $s > 0$, as well as the following limiting conditions:

$$g_*(0+) \;=\; g_*'(+\infty) \;=\; 0 \tag{4.5}$$

$$g_*'(0+) \;=\; 1\;. \tag{4.6}$$

Moreover, the following estimates are shown to be valid:

$$\frac{s}{\Big(1+\frac{K\Delta}{\Delta-1}\,s\Big)^{1/\Delta}} \;\le\; g_*(s) \;\le\; \left(\frac{\Delta-1}{K\Delta}\right)^{1/\Delta} s^{1-1/\Delta} \tag{4.7}$$

for all $s > (\Delta-1)/K\Delta$ *(the left-hand inequality holds for all* $s>0$ *as well). Finally, the optimal stopping strategy* τ_* *is exponentially integrable, and the following estimate holds true:*

$$E_{x,s}\Bigg(\exp\left(\frac{\sigma^2\Delta^2}{8}\,\tau_*\right)\Bigg) \;\le\; \left(\frac{x}{g_*(s)}\right)^{\Delta/2} \tag{4.8}$$

whenever $g_*(s) < x \le s$ *. (The left-hand side equals* 1 *for* $0 < x \le g_*(s)$ *.)*

Proof. Since the facts from the second part of the theorem will be needed in the proof of the first part, we shall begin with the second part.

For the *second part* it should be noted that (4.5)-(4.7) follow directly from our construction of the maximal solution which is presented in Section 3. (The estimates (4.7) are nothing but the estimates (3.4) just rewritten, while (4.5)+(4.6) is easily deduced by (4.7).)

To show (4.8) assume that $g_*(s) < x \le s$ are given and fixed. The key point is to note that:

$$\tau_* \;\le\; \tau_{g_*(s)} := \inf\;\big\{\, t>0 \;|\; X_t \;=\; g_*(s)\,\big\} \tag{4.9}$$

since $s \mapsto g_*(s)$ is increasing. Moreover, by (2.1) we find:

$$\tau_{g_*(s)} \;=\; \inf\;\big\{\, t>0 \;|\; B_t \;=\; at-b\,\big\} \;:=\; \tau_{a,b} \tag{4.10}$$

with $a = (\sigma^2 - 2\mu)/2\sigma$ and $b = \log(x/g_*(s))^{1/\sigma}$, both strictly positive. Finally, it is well-known (see [8], p.70) that we have:

$$E\Bigg(\exp\left(\frac{a^2}{2}\,\tau_{a,b}\right)\Bigg) \;=\; \exp\big(ab\big)\;. \tag{4.11}$$

Now, matching (4.9)-(4.11) together, we get (4.8). This completes the proof for the second part.

For the *first part* note that $V(x,s) < \infty$ for $\mu < 0$ follows by (2.7). To prove the converse, it is enough to show that $V(x,s) = +\infty$ when $\mu = 0$. For this, denote the payoff associated with the given μ by $V_\mu(x,s)$, and the corresponding optimal stopping boundary by $g_{*,\mu}(s)$. Then clearly $V_0(x,s) \ge V_\mu(x,s)$, while from (4.7) we find that $\lim_{\mu\uparrow 0} g_{*,\mu}(s) = 0$. Therefore from (4.4) (being proved below) we get $\lim_{\mu\uparrow 0} V_\mu(x,s) = +\infty$, thus proving the claim.

It remains to show the *validity of (4.2) and (4.4)*. For this, we shall denote the function on the right-hand side in (4.4) by $V_*(x,s)$. It should be recalled that this function satisfies (2.10)-(2.13). Since clearly $V_*(x,s) \ge s$, we may write:

$$(4.12) \qquad V(x,s) = \sup_\tau E_{x,s}\big(S_\tau - c\tau\big)$$
$$\le \sup_\tau E_{x,s}\big(S_\tau - V_*(X_\tau, S_\tau)\big) + \sup_\tau E_{x,s}\big(V_*(X_\tau, S_\tau) - c\tau\big)$$
$$\le \sup_\tau E_{x,s}\big(V_*(X_\tau, S_\tau) - c\tau\big)\ .$$

Hence, to complete the proof, it is enough to show the following two facts:

$$(4.13) \qquad \sup_\tau E_{x,s}\big(V_*(X_\tau, S_\tau) - c\tau\big) \le V_*(x,s)$$

$$(4.14) \qquad E_{x,s}\big(S_{\tau_*} - c\tau_*\big) = V_*(x,s)\ .$$

Proof of (4.13)+(4.14): By Itô's formula (see Remark 4.2 below) we have:

$$(4.15) \qquad V_*(X_t, S_t) = V_*(X_0, S_0) + \int_0^t \frac{\partial V_*}{\partial x}(X_r, S_r)\ dX_r + \int_0^t \frac{\partial V_*}{\partial s}(X_r, S_r)\ dS_r$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r, S_r)\ d\big\langle X, X\big\rangle_r$$

where we set $(\partial^2 V_*/\partial x^2)(g_*(s), s) = 0$. By (2.2), (2.3), and the fact that $d\big\langle X, X\big\rangle_t = \sigma^2 X_t^2\, dt$, this can be written as follows:

$$(4.16) \qquad V_*(X_t, S_t) = V_*(X_0, S_0) + \int_0^t \mathbf{L_X} V_*(X_r, S_r)\ dr + \int_0^t \frac{\partial V_*}{\partial s}(X_r, S_r)\ dS_r$$
$$+ \int_0^t \frac{\partial V_*}{\partial x}(X_r, S_r)\, \sigma X_r\ dB_r\ .$$

Next note that $\mathbf{L_X} V_*(x,s) = c$ for $g_*(s) < x < s$ and $\mathbf{L_X} V_*(x,s) = 0$ for $0 < x \le g_*(s)$. Moreover, due to the normal reflection of X , the set of those $r > 0$ for which $X_r = S_r$ is of Lebesgue measure zero. Finally, since the increment dS_r equals 0 outside of the diagonal $x = s$, and $V_*(x,s)$ at the diagonal satisfies (2.13), we see that the second integral in (4.16) is identically zero. These facts matched together in (4.16) show that:

$$(4.17) \qquad V_*(X_\tau, S_\tau) \le V_*(x,s) + c\tau + M_\tau$$

for any (bounded) stopping time τ for X , where $M = (M_t)_{t \ge 0}$ is the local martingale given by:

$$(4.18) \qquad M_t = \int_0^t \frac{\partial V_*}{\partial x}(X_r, S_r)\, \sigma X_r\ dB_r$$

for $t \ge 0$. Moreover, this also shows that:

$$(4.19) \qquad V_*(X_\tau, S_\tau) = V_*(x,s) + c\tau + M_\tau\ .$$

for any stopping time τ for X satisfying $\tau \le \tau_*$.

Since the suprema in (4.12) are equivalently taken over all bounded stopping times, and since by (4.8) we see that τ_* has all moments finite, by (4.17) and (4.19) respectively, the proof of (4.13) and (4.14) will be completed if we are able to show that:

$$E_{x,s}\big(M_\tau\big) = 0 \tag{4.20}$$

for any stopping time τ for X with all moments finite.

To complete the proof, we shall show that (4.20) holds for any stopping time τ for which there is $r > \Delta/2(\Delta-1)$ such that $E(\tau^r) < \infty$. So, let such a stopping time τ be given and fixed. Note from (4.4) that:

$$\frac{\partial V_*}{\partial x}(x,s) = \frac{2c}{\Delta\sigma^2}\left(\frac{x^{\Delta-1}}{g_*(s)^\Delta} - \frac{1}{x}\right) \quad , \text{ if } \; g_*(s) < x \le s \tag{4.21}$$

$$= 0 \quad , \text{ if } \; 0 < x \le g_*(s) \; .$$

Thus, to get (4.20), by Burkholder-Gundy's inequality for continuous local martingales (see [3] and [4]) it is enough to show that:

$$E_{x,s}\left(\int_0^\tau \left(\frac{X_r^\Delta}{g_*(S_r)^\Delta} - 1\right)^2 1_{\{X_r \ge g_*(S_r)\}}\, dr\right)^{1/2} := I < \infty \; . \tag{4.22}$$

Now recall the left-hand estimate in (4.7). Since $X_r \le S_r$, this gives:

$$I \le E_{x,s}\left(\int_0^\tau \left(\frac{K\Delta}{\Delta-1}\, S_r\right)^2 dr\right)^{1/2} \le \frac{K\Delta}{\Delta-1}\, E_{x,s}\Big(S_\tau\sqrt{\tau}\Big) \; . \tag{4.23}$$

Further, by the Hölder inequality we get:

$$I \le \frac{K\Delta}{\Delta-1}\Big(E_{x,s}\big(S_\tau^p\big)\Big)^{1/p}\Big(E_{x,s}\big(\tau^{q/2}\big)\Big)^{1/q} \tag{4.24}$$

for any $p > 1$ with $1/p + 1/q = 1$. By Doob's theorem (2.6) we easily find that:

$$E\left(\sup_{t>0} X_t\right)^p < \infty \tag{4.25}$$

provided that $\mu < \sigma^2(1-p)/2$. Thus, if p is taken close enough to 1 to satisfy the last inequality, then $E_{x,s}\big(S_\tau^p\big) < \infty$ in (4.24). This imposes a condition on q which is easily verified to be $q > \Delta/(\Delta-1)$. Since by our hypothesis $E(\tau^{q/2}) < \infty$ for all such q , this completes the proof.

□

Remark 4.2

Here we explain in more detail how to get (4.15) by means of Itô's formula. Note that (X,S) is a two-dimensional continuous semimartingale, where X is of diffusion type while S is an increasing process. The state space of (X,S) is $E = \{(x,s) \mid 0 < x \le s\}$. The function $V_*(x,s)$ defined by (4.4) is twice continuously differentiable in $E \setminus \{\, \big(g_*(s),s\big) \mid s > 0 \,\}$, while on each fixed vertical level $s > 0$ its restriction $x \mapsto V_*(x,s)$ is convex and continuously differentiable on $]0,s[$. These two facts together indicate that Itô's formula can be applied. Since away from the diagonal the process (X,S) moves only horizontally thus $V_*(X_t,S_t)$ can be controlled by the (one-dimensional) Itô-Tanaka formula, while away from the optimal stopping boundary the function $V_*(x,s)$ is C^2 and therefore (two-dimensional) Itô formula can be applied.

In order to formalize this let us choose two continuous (increasing) maps $s \mapsto g_1(s)$ and $s \mapsto g_2(s)$ satisfying $g_*(s) < g_1(s) < g_2(s) < s$ for all $s > 0$. Introduce the stopping times:

$$\sigma_1 = \inf \{ \, t > 0 \mid X_t \le g_1(S_t) \, \} \tag{4.26}$$

$$\sigma_2 = \inf \{ \, t > 0 \mid X_t \ge g_2(S_t) \, \} \tag{4.27}$$

and define $\tau_0 := 0$, $\tau_1 = \sigma_1$, $\tau_2 = \tau_1 + \sigma_2 \circ \theta_{\tau_1}$, $\tau_3 = \tau_2 + \sigma_1 \circ \theta_{\tau_2}$, $\tau_4 = \tau_3 + \sigma_2 \circ \theta_{\tau_3}$ ... Note that $\tau_n \uparrow \infty$. In order to prove (4.15) we only need to show that for each fixed $t > 0$ we have for all $k \ge 0$:

$$\begin{aligned} V_*(X_{t\wedge\tau_k}, S_{t\wedge\tau_k}) &= V_*(x,s) + \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\, 1_{[[0,\tau_k]]}(r)\; dX_r \\ &+ \int_0^t \frac{\partial V_*}{\partial s}(X_r,S_r)\, 1_{[[0,\tau_k]]}(r)\; dS_r + \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\, 1_{[[0,\tau_k]]}(r)\; d\langle X,X\rangle_r \; . \end{aligned} \tag{4.28}$$

We prove this by induction. Fix $t > 0$. For $k = 0$ it is trivial, so suppose it is true for $k-1 \ge 0$ and consider the case k . First assume k is odd, and let $\tilde{V}(x,s)$ be any C^2-function coniciding with $V_*(x,s)$ on $G = \{ (x,s) \in E \mid g_1(s) \le x \}$ (such an extension evidently exists). By applying the ordinary two-dimensional Itô's formula we get:

$$\begin{aligned} V_*(X_{t\wedge\tau_k}, S_{t\wedge\tau_k}) - V_*(X_{t\wedge\tau_{k-1}}, S_{t\wedge\tau_{k-1}}) &= \tilde{V}_*(X_{t\wedge\tau_k}, S_{t\wedge\tau_k}) - \tilde{V}_*(X_{t\wedge\tau_{k-1}}, S_{t\wedge\tau_{k-1}}) \\ &= \int_0^t \frac{\partial \tilde{V}_*}{\partial x}(X_r,S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dX_r + \int_0^t \frac{\partial \tilde{V}_*}{\partial s}(X_r,S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dS_r \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 \tilde{V}_*}{\partial x^2}(X_r,S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; d\langle X,X\rangle_r \\ &= \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dX_r + \int_0^t \frac{\partial V_*}{\partial s}(X_r,S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dS_r \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; d\langle X,X\rangle_r \end{aligned} \tag{4.29}$$

which proves the case k if k is odd. (Observe that $(X_r,S_r) \in G$ if $r \in [\tau_{k-1},\tau_k]$.)

Now consider the case k even, and let $(S_t^{(n)})_{t\ge 0}$ denote the right continuous increasing process defined by:

$$S_t^{(n)} = \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\, 1_{[(k-1)/2^n, k/2^n[}(S_t) \; + \; n 1_{[n,\infty[}(S_t) \; . \tag{4.30}$$

By applying Itô-Tanaka's formula to the C^1-convex function $x \mapsto V_*(x,s)$ for different $s > 0$ we get ($r \mapsto S_r$ is constant on $[[\tau_{k-1},\tau_k]]$):

$$
\begin{aligned}
(4.31) \qquad & V_*(X_{t\wedge\tau_k}, S_{t\wedge\tau_k}) - V_*(X_{t\wedge\tau_{k-1}}, S_{t\wedge\tau_{k-1}}) = \lim_{n\to\infty} \Big(V_*(X_{t\wedge\tau_k}, S^{(n)}_{t\wedge\tau_k}) - V_*(X_{t\wedge\tau_{k-1}}, S^{(n)}_{t\wedge\tau_{k-1}}) \Big) \\
&= \lim_{n\to\infty} \Big(V_*(X_{t\wedge\tau_k}, S^{(n)}_{\tau_{k-1}}) - V_*(X_{t\wedge\tau_{k-1}}, S^{(n)}_{\tau_{k-1}}) \Big) \\
&= \lim_{n\to\infty} \sum_{j=0}^{n2^n-1} 1_{\{S^{(n)}_{\tau_{k-1}} = j/2^n\}} \Big(V_*(X_{t\wedge\tau_k}, j/2^n) - V_*(X_{t\wedge\tau_{k-1}}, j/2^n) \Big) \\
&= \lim_{n\to\infty} \sum_{j=0}^{n2^n-1} 1_{\{S^{(n)}_{\tau_{k-1}} = j/2^n\}} \bigg(\int_0^t \frac{\partial V_*}{\partial x}(X_r, j/2^n)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dX_r \\
&\qquad + \frac{1}{2} \int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r, j/2^n)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; d\big\langle X, X \big\rangle_r \bigg) \\
&= \lim_{n\to\infty} \bigg(\int_0^t \frac{\partial V_*}{\partial x}(X_r, S^{(n)}_{\tau_{k-1}}))\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dX_r \\
&\qquad + \frac{1}{2} \int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r, S^{(n)}_{\tau_{k-1}})\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; d\big\langle X, X \big\rangle_r \bigg) \\
&= \int_0^t \frac{\partial V_*}{\partial x}(X_r, S_{\tau_{k-1}}))\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dX_r \\
&\qquad + \frac{1}{2} \int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r, S_{\tau_{k-1}})\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; d\big\langle X, X \big\rangle_r \\
&= \int_0^t \frac{\partial V_*}{\partial x}(X_r, S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; dX_r \\
&\qquad + \frac{1}{2} \int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r, S_r)\, 1_{]]\tau_{k-1},\tau_k]]}(r)\; d\big\langle X, X \big\rangle_r
\end{aligned}
$$

The limit identification in the penultimate equality is, for the first integral just an application of the continuity of $\partial V_*/\partial x$ and properties of the stochastic integral with respect to X_t , while for the second integral we use the fact that $\partial^2 V_*/\partial x^2$ is locally bounded and for almost all ω we have $d\big\langle X, X \big\rangle_r(\omega) << dr$ and the Lebesgue measure of $\{\, r > 0 \mid X_r(\omega) = g_*(S_r(\omega)) \,\}$ is zero.

Remark 4.3

It should be observed in the proof of (4.2)+(4.4) above that up to (4.19) we didn't make any use of the specific form of the optimal (maximal) stopping boundary $s \mapsto g_*(s)$ and the corresponding payoff $V_*(x,s)$. In other words, those arguments work out for any candidate for the payoff $V(x,s)$ satisfying (2.10)-(2.13) associated with the stopping boundary $s \mapsto g(s)$ satisfying (2.20). The key point about the optimal (maximal) stopping boundary $s \mapsto g_*(s)$ is to show (4.20). To obtain this, it is clearly seen from (4.22) that we should look for the maximal $s \mapsto g(s)$ satisfying (2.20) which stays below the diagonal. This offers a somewhat loose but indicative analytic argument for the maximality principle to hold.

Remark 4.4

The referee kindly pointed out that the stopping times (4.2), which appear optimal in our problem, have been studied earlier. Due to their significant role in the Skorokhod embedding problem [1], such stopping times are sometimes referred to as *Azéma-Yor stopping times*. We also learned that Jacka [7] makes use of these stopping times in optimal stopping.

5. The maximal inequality

In this section we shall apply the results obtained in the previous section and derive a maximal inequality for geometric Brownian motion. To the best of our knowledge this inequality has not been recorded earlier.

Theorem 5.1

Let $B = (B_t)_{t \ge 0}$ *be standard Brownian motion, and let* $\mu < 0$ *and* $\sigma > 0$ *be given and fixed. Then the inequality is valid:*

$$E\left(\max_{0 \le t \le \tau} \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2} \right) t \right) \right) \le 1 - \frac{\sigma^2}{2\mu} + \frac{\sigma^2}{2\mu} \exp\left(- \frac{(\sigma^2 - 2\mu)^2}{2\sigma^2} E(\tau) - 1 \right) \tag{5.1}$$

whenever τ *is a stopping time for* B *.*

Proof. Recalling (2.1)+(2.4) and (4.1), we see that:

$$E_{x,x}\big(S_\tau\big) \le cE(\tau) + V(x,x) \tag{5.2}$$

whenever τ is a stopping time for B (with finite expectation) being given and fixed. By (4.4) and (4.7) we obtain the estimate:

$$\begin{aligned} V(x,x) &= \frac{2c}{\Delta^2 \sigma^2} \left(\left(\frac{x}{g_*(x)} \right)^{\Delta} - \log \left(\frac{x}{g_*(x)} \right)^{\Delta} - 1 \right) + x \\ &\le \frac{\Delta}{\Delta - 1} x + \frac{2c}{\Delta^2 \sigma^2} \log \left(\frac{2c\,(\Delta - 1)}{\Delta^2 \sigma^2} \frac{1}{x} \right) \end{aligned} \tag{5.3}$$

for all $x > 2c(\Delta - 1)/\Delta^2 \sigma^2$. Introduce the function:

$$F(c) = cE(\tau) + \frac{\Delta}{\Delta - 1} x + \frac{2c}{\Delta^2 \sigma^2} \log \left(\frac{2c\,(\Delta - 1)}{\Delta^2 \sigma^2} \frac{1}{x} \right) . \tag{5.4}$$

Then from (5.2) and (5.3) we find:

$$E_{x,x}\big(S_\tau\big) \le \inf_{c > 0} F(c) . \tag{5.5}$$

Solving $F'(c) = 0$ we find that F attains its minimal value on $]0, \infty[$ at the point:

$$c_* = x \, \frac{\Delta^2 \sigma^2}{2(\Delta - 1)} \exp\left(- \frac{\Delta^2 \sigma^2}{2} E(\tau) - 1 \right) . \tag{5.6}$$

(Note that $x > 2c_*(\Delta - 1)/\Delta^2\sigma^2$ as needed.) Moreover, it is easily computed that:

$$F(c_*) = x \left(\frac{\Delta}{\Delta - 1} - \frac{1}{\Delta - 1} \exp\left(- \frac{\Delta^2\sigma^2}{2} E(\tau) - 1 \right) \right) . \tag{5.7}$$

Recalling that $\Delta = 1 - 2\mu/\sigma^2$ and inserting (5.7) into (5.5) gives:

$$E_{x,x}\big(S_\tau\big) \le x \left(1 - \frac{\sigma^2}{2\mu} + \frac{\sigma^2}{2\mu} \exp\left(- \frac{(\sigma^2 - 2\mu)^2}{2\sigma^2} E(\tau) - 1 \right) \right) . \tag{5.8}$$

By the representation (2.1) this reduces to (5.1), and the proof is complete. □

Remark 5.2

The inequality (5.1) might be viewed as a refinement and extension of the equality (2.7) obtained by Doob's theorem (2.6).

Remark 5.3

We find it interesting to record the following consequence of Theorem 4.1 (the payoff is finite if and only if the drift is strictly negative):

$$\sup_\tau E\left(\max_{0 \le t \le \tau} \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2} \right) t \right) - c\tau \right) = +\infty \tag{5.9}$$

which is to be compared with the well-known fact:

$$\sup_\tau E\left(\exp\left(\sigma B_\tau + \left(\mu - \frac{\sigma^2}{2} \right) \tau \right) - c\tau \right) = 1 . \tag{5.10}$$

(The supremums in (5.9) and (5.10) are taken over all stopping times for B .)

From the proof of Theorem 4.1 it is clear what sequence of stopping times τ_n for B that the supremum in (5.9) over equals $+\infty$. If $s \mapsto g_{*,\mu}(s)$ denotes the optimal stopping boundary for (4.1) with drift μ , and $\tau_{*,\mu}$ denotes the corresponding optimal stopping strategy, then it is good enough to take τ_n equal to:

$$\tau_{*,-1/n} = \inf \left\{ t > 0 \mid X_t \le g_{*,-1/n}(S_t) \right\} \tag{5.11}$$

for all $n \ge 1$. (Here X_t and S_t are given by (2.1) and (2.4) with any $s \ge x > 0$.) We were not able to find a simpler sequence of stopping times over which the supremum in (5.9) would be $+\infty$.

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