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Optimal Stopping Games for Markov Processes

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Let $X = (X_t)_{t \geq 0}$ be a strong Markov process, and let G_1, G_2 and G_3 be continuous functions satisfying $G_1 \leq G_3 \leq G_2$ and $\mathsf{E}_x \sup_t |G_i(X_t)| < \infty$ for $i = 1, 2, 3$. Consider the optimal stopping game where the sup-player chooses a stopping time τ to maximise, and the inf-player chooses a stopping time σ to minimise, the expected payoff

$$\mathsf{M}_x(\tau, \sigma) = \mathsf{E}_x\big[G_1(X_\tau)\, I(\tau < \sigma) + G_2(X_\sigma)\, I(\sigma < \tau) + G_3(X_\tau)\, I(\tau = \sigma)\big]$$

where $X_0 = x$ under P_x. Define the upper value and the lower value of the game by

$$V^*(x) = \inf_\sigma \sup_\tau \mathsf{M}_x(\tau, \sigma) \quad \& \quad V_*(x) = \sup_\tau \inf_\sigma \mathsf{M}_x(\tau, \sigma)$$

where the horizon T (the upper bound for τ and σ above) may be either finite or infinite (it is assumed that $G_1(X_T) = G_2(X_T)$ if T is finite and $\liminf_{t \to \infty} G_2(X_t) \leq \limsup_{t \to \infty} G_1(X_t)$ if T is infinite). If X is right-continuous, then the *Stackelberg equilibrium* holds, in the sense that $V^*(x) = V_*(x)$ for all x with $V := V^* = V_*$ defining a measurable function. If X is right-continuous and left-continuous over stopping times (quasi-left-continuous), then the *Nash equilibrium* holds, in the sense that there exist stopping times τ_* and σ_* such that

$$\mathsf{M}_x(\tau, \sigma_*) \leq \mathsf{M}_x(\tau_*, \sigma_*) \leq \mathsf{M}_x(\tau_*, \sigma)$$

for all stopping times τ and σ, implying also that $V(x) = \mathsf{M}_x(\tau_*, \sigma_*)$ for all x. Further properties of the value function V and the optimal stopping times τ_* and σ_* are exhibited in the proof.

1. Introduction

Let $X = (X_t)_{t \geq 0}$ be a strong Markov process, and let G_1, G_2 and G_3 be continuous functions satisfying $G_1 \leq G_3 \leq G_2$ (for further details see Section 2 below). Consider the optimal stopping game where the sup-player chooses a stopping time τ to maximise, and the inf-player chooses a stopping time σ to minimise, the expected payoff

$$\mathsf{M}_x(\tau, \sigma) = \mathsf{E}_x\big[G_1(X_\tau)\, I(\tau < \sigma) + G_2(X_\sigma)\, I(\sigma < \tau) + G_3(X_\tau)\, I(\tau = \sigma)\big] \tag{1.1}$$

where $X_0 = x$ under P_x.

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Define the upper value and the lower value of the game by

$$V^*(x) = \inf_\sigma \sup_\tau \mathsf{M}_x(\tau,\sigma) \quad \& \quad V_*(x) = \sup_\tau \inf_\sigma \mathsf{M}_x(\tau,\sigma) \tag{1.2}$$

where the horizon T (the upper bound for τ and σ above) may be either finite or infinite (it is assumed that $G_1(X_T) = G_2(X_T)$ if T is finite and $\liminf_{t\to\infty} G_2(X_t) \le \limsup_{t\to\infty} G_1(X_t)$ if T is infinite). Note that $V_*(x) \le V^*(x)$ for all x.

In this context one distinguishes: (i) the *Stackelberg equilibrium*, meaning that

$$V^*(x) = V_*(x) \tag{1.3}$$

for all x (in this case $V := V^* = V_*$ unambiguously defines the value of the game); and (ii) the *Nash equilibrium*, meaning that there exist stopping times τ_* and σ_* such that

$$\mathsf{M}_x(\tau,\sigma_*) \le \mathsf{M}_x(\tau_*,\sigma_*) \le \mathsf{M}_x(\tau_*,\sigma) \tag{1.4}$$

for all stopping times τ and σ, and for all x (in other words (τ_*,σ_*) is a saddle point). It is easily seen that the Nash equilibrium implies the Stackelberg equilibrium with $V(x) = \mathsf{M}_x(\tau_*,\sigma_*)$ for all x.

A variant of the problem above was first studied by Dynkin [5] using martingale methods similar to those of Snell [21]. Specific examples of the same problem were studied in [9] and [12] using Markovian methods (see also [13] for martingale methods). In parallel to that Bensoussan and Friedman (cf. [10], [2], [3]) developed an analytic approach (for diffusions) based on variational inequalities. Martingale methods were further advanced in [18] (see also [23]), and Markovian setting was studied in [8] (via Wald-Bellman equations) and [22] (via penalty equations). More recent papers on optimal stopping games include [14], [16], [1], [11], [6], [7] and [15]. These papers study specific problems and often lead to explicit solutions. For optimal stopping games with randomized stopping times see [17] and the references therein. For connections with singular stochastic control (forward/backward SDE) see [4] and the references therein.

The most general martingale result known to date assumes an upper/lower semi-continuity from the left (cf. [18, Theorem 15, page 42]) so that it does not cover the case of Lévy processes for example. The most general Markovian result known to date assumes an asymptotic condition uniformly over initial points (cf. [22, Condition (A3), page 2]) so that it is not always easily verifiable. The present paper aims at closing these gaps.

The main result of the paper (Theorem 2.1) may be summarised as follows. If X is *right-continuous*, then the *Stackelberg equilibrium* holds with a measurable value function. If X is right-continuous and *left-continuous over stopping times* (quasi-left-continuous), then the *Nash equilibrium* holds (see also Example 3.1 and Theorem 3.2). These two sufficient conditions are known to be most general in optimal stopping theory (see e.g. [19] and [20]). Further properties of the value function V and the optimal stopping times τ_* and σ_* are exhibited in the proof.

2. Result and proof

1. Throughout we will consider a strong Markov process $X = (X_t)_{t\ge0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathsf{P}_x)$ and taking values in a measurable space $(E, \mathcal{B})$, where E is a locally compact Hausdorff space with a countable base, and $\mathcal{B}$ is the Borel σ-algebra on E. It will be assumed that the process X starts at x under P_x for $x \in E$ and that the

sample paths of X are (firstly) right-continuous and (then) left-continuous over stopping times. The latter condition is often referred to as quasi-left-continuity and means that $X_{\tau_n} \rightarrow X_\tau$ P_x-a.s. whenever τ_n and τ are stopping times such that $\tau_n \uparrow \tau$ as $n \rightarrow \infty$. (Stopping times are always referred with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ given above.) It is also assumed that the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous (implying that the first entry times to open and closed sets are stopping times) and that $\mathcal{F}_0$ contains all P_x-null sets from $\mathcal{F}_\infty^X = \sigma(X_t : t \geq 0)$ (implying also that the first entry times to Borel sets are stopping times). The main example we have in mind is when $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \cup \mathcal{N})$ where $\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{N} = \{ A \subseteq \Omega : \exists B \in \mathcal{F}_\infty^X, A \subseteq B, \mathsf{P}_x(B) = 0 \}$ for $t \geq 0$ with $\mathcal{F} = \mathcal{F}_\infty$. In addition, it is assumed that the mapping $x \mapsto \mathsf{P}_x(F)$ is (universally) measurable for each $F \in \mathcal{F}$. It follows that the mapping $x \mapsto \mathsf{E}_x(Z)$ is (universally) measurable for each (integrable) random variable Z. Finally, without loss of generality we will assume that Ω equals the canonical space $E^{[0,\infty)}$ with $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \geq 0$, so that the shift operator $\theta_t : \Omega \rightarrow \Omega$ is well defined by $\theta_t(\omega)(s) = \omega(t+s)$ for $\omega \in \Omega$ and $t, s \geq 0$.

2. Given continuous functions $G_1, G_2, G_3 : E \rightarrow I\!R$ satisfying $G_1 \leq G_3 \leq G_2$ and the following integrability condition:

$$\mathsf{E}_x \sup_t |G_i(X_t)| < \infty \quad (i = 1, 2, 3) \tag{2.1}$$

for all $x \in E$, we consider the *optimal stopping game* where the sup-player chooses a stopping time τ to maximise, and the inf-player chooses a stopping time σ to minimise, the expected payoff

$$\mathsf{M}_x(\tau, \sigma) = \mathsf{E}_x\big[G_1(X_\tau) I(\tau < \sigma) + G_2(X_\sigma) I(\sigma < \tau) + G_3(X_\tau) I(\tau = \sigma)\big] \tag{2.2}$$

where $X_0 = x$ under P_x.

Define the upper value and the lower value of the game by

$$V^*(x) = \inf_\sigma \sup_\tau \mathsf{M}_x(\tau, \sigma) \quad \& \quad V_*(x) = \sup_\tau \inf_\sigma \mathsf{M}_x(\tau, \sigma) \tag{2.3}$$

where the horizon T (the upper bound for τ and σ above) may be either finite or infinite. If $T < \infty$ then it will be assumed that $G_1(X_T) = G_2(X_T) = G_3(X_T)$. In this case it is most interesting to assume that X is a time-space process (t, Y_t) for $t \in [0, T]$ so that $G_i = G_i(t, y)$ will be functions of both time and space for $i = 1, 2, 3$. If $T = \infty$ then it will be assumed that $\liminf_{t \rightarrow \infty} G_2(X_t) \leq \limsup_{t \rightarrow \infty} G_1(X_t)$, and the common value for $G_3(X_\infty)$ could formally be assigned as either of the preceding two values (if τ and σ are allowed to take the value ∞) yielding the same results as in Theorem 2.1 below. For simplicity of the exposition, however, we will assume that τ and σ in (2.2) are finite valued.

3. The main result of the paper may now be stated as follows.

Theorem 2.1

Consider the optimal stopping game (2.3). *If X is right-continuous, then the Stackelberg equilibrium* (1.3) *holds with $V := V^* = V_*$ defining a measurable function. If X is right-continuous and left-continuous over stopping times, then the Nash equilibrium* (1.4) *holds with*

$$\tau_* = \inf \{ t : X_t \in D_1 \} \quad \& \quad \sigma_* = \inf \{ t : X_t \in D_2 \} \tag{2.4}$$

where $D_1 = \{ V = G_1 \}$ and $D_2 = \{ V = G_2 \}$.

Proof. Both finite and infinite horizon can be treated by slight modifications of the same method which we will therefore present without referring to horizon.

(I) In the first part of the proof we will assume that X is right-continuous, and we will show that this hypothesis implies the Stackelberg equilibrium with $V := V^* = V_*$ defining a measurable function. This will be done in a number of steps as follows.

1. Given $\varepsilon > 0$ set

$$D_1^\varepsilon = \{ V^* \le G_1 + \varepsilon \} \quad \& \quad D_2^\varepsilon = \{ V_* \ge G_2 - \varepsilon \} \tag{2.5}$$

and consider the stopping times

$$\tau_\varepsilon = \inf \{ t : X_t \in D_1^\varepsilon \} \quad \& \quad \sigma_\varepsilon = \inf \{ t : X_t \in D_2^\varepsilon \}. \tag{2.6}$$

The key is to show that

$$\mathsf{M}_x(\tau, \sigma_\varepsilon) - \varepsilon \le V_*(x) \le V^*(x) \le \mathsf{M}_x(\tau_\varepsilon, \sigma) + \varepsilon \tag{2.7}$$

for all τ, σ, x and $\varepsilon > 0$. Indeed, suppose that (2.7) is valid. Then

$$V^*(x) \le \inf_\sigma \mathsf{M}_x(\tau_\varepsilon, \sigma) + \varepsilon \le \sup_\tau \inf_\sigma \mathsf{M}_x(\tau, \sigma) + \varepsilon = V_*(x) + \varepsilon \tag{2.8}$$

for all $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we see that $V^* = V_*$ and the claim follows (up to measurability which will be derived below).

Since the first inequality in (2.7) is analogous to the third one, and since the second inequality holds generally, we focus on establishing the third one which states that

$$V^*(x) \le \mathsf{M}_x(\tau_\varepsilon, \sigma) + \varepsilon \tag{2.9}$$

for all σ, x and $\varepsilon > 0$.

2. To prove (2.9) take any stopping time σ and consider the optimal stopping problem

$$\hat{V}_\sigma^*(x) = \sup_\tau \hat{\mathsf{M}}_x(\tau, \sigma) \tag{2.10}$$

where we set

$$\hat{\mathsf{M}}_x(\tau, \sigma) = \mathsf{E}_x \big[G_1(X_\tau) \, I(\tau < \sigma) + G_2(X_\sigma) \, I(\sigma \le \tau) \big]. \tag{2.11}$$

Note that the gain process G^σ in (2.10) is given by

$$G_t^\sigma = G_1(X_t) \, I(t < \sigma) + G_2(X_\sigma) \, I(\sigma \le t) \tag{2.12}$$

from where we see that G^σ is right-continuous and adapted (satisfying also a sufficient integrability condition which can be derived using (2.1)). Thus general optimal stopping results of the martingale approach (cf. [19]) are applicable to the problem (2.10). In order to make use of these results in the Markovian setting of the present theorem (where P_x forms a family of probability measures when x runs through E) we will first verify a regularity property of the value function $\hat{V}_\sigma^*$.

3. We show that the function $x \mapsto \hat{V}^*_\sigma(x)$ defined in (2.10) is measurable. The basic idea of the proof is to embed the problem (2.10) into a setting of the Wald-Bellman equations (cf. [19]) and then exploit the underlying Markovian structure in this context.

For this, let us first assume that the stopping times τ in (2.10) take values in a finite set, and without loss of generality let us assume that this set equals $\{0, 1, \ldots, N\}$. Introduce the auxiliary optimal stopping problems

$$V_n^N(x) = \sup_{n \le \tau \le N} \mathsf{E}_x G^\sigma_\tau \tag{2.13}$$

for $n = N, \ldots, 1, 0$ and recall that the Wald-Bellman equations in this setting read:

$$S_n^N = G^\sigma_N \quad \text{for } n = N \tag{2.14}$$

$$S_n^N = G^\sigma_n \vee \mathsf{E}_x(S^N_{n+1} \,|\, \mathcal{F}_n) \quad \text{for } n = N-1, \ldots, 1, 0 \tag{2.15}$$

with $V_n^N(x) = \mathsf{E}_x S_n^N$ for $n = N, \ldots, 1, 0$ (see e.g. [19, pp. 3-6]). In particular, since $V_0^N = \hat{V}^*_\sigma$ we see that

$$\hat{V}^*_\sigma(x) = \mathsf{E}_x S_0^N \tag{2.16}$$

for all x. Thus the problem is reduced to showing that $x \mapsto \mathsf{E}_x S_0^N$ is measurable.

If σ is a hitting time, then by the strong Markov property of X it follows using (2.14)-(2.15) inductively that the following identity holds:

$$S_0^N = G_1(x)\, I(0 < \sigma) + F_N(x) + G_2(x)\, I(\sigma = 0) \tag{2.17}$$

under P_x, where $x \mapsto F_N(x)$ is a measurable function obtained by means of the following recursive relations:

$$F_n(x) = \mathsf{E}_x\big[G_1(X_1)\, I(1 < \sigma) \vee F_{n-1}(x)\big] + \mathsf{E}_x\big[G_2(X_\sigma)\, I(0 < \sigma \le 1)\big] \tag{2.18}$$

for $n = 1, 2, \ldots, N$ where $F_0 \equiv -\infty$. Taking E_x in (2.17) and using (2.16) we get

$$\hat{V}^*_\sigma(x) = G_1(x)\, \mathsf{P}_x(0 < \sigma) + F_N(x) + G_2(x)\, \mathsf{P}_x(\sigma = 0) \tag{2.19}$$

for all x. Hence we see that $x \mapsto \hat{V}^*_\sigma(x)$ is measurable as claimed. In the case of a general stopping time σ one can make use of the extended Radon-Nikodym theorem which states that $(x, \omega) \mapsto \mathsf{E}_x(Z_x \,|\, \mathcal{G})(\omega)$ is measurable when $(x, \omega) \mapsto Z_x(\omega)$ is measurable and $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra. Applying this fact inductively in (2.15) and using (2.16) it follows that $x \mapsto \hat{V}^*_\sigma(x)$ is measurable as claimed. [Note that this argument also applies when σ is a hitting time, however, the explicit formula (2.19) is no longer available if σ is a general stopping time.]

Let us now consider the general case when the stopping times τ from (2.10) can take arbitrary values. Setting $\tau_n = k/2^n$ on $\{(k-1)/2^n < \tau \le k/2^n\}$ one knows that each τ_n is a stopping time with values in the set Q_n of dyadic rationals of the form $k/2^n$, and $\tau_n \downarrow \tau$ as $n \to \infty$. Hence by right-continuity of G^σ and Fatou's lemma (using a needed integrability condition which is derived by means of (2.1) above) one gets

$$\mathsf{E}_x G^\sigma_\tau = \mathsf{E}_x\big(\lim_{n \to \infty} G^\sigma_{\tau_n}\big) \le \liminf_{n \to \infty} \mathsf{E}_x G^\sigma_{\tau_n} \le \sup_{n \ge 1} V_n(x) \tag{2.20}$$

where we set

$$V_n(x) = \sup_{\tau \in Q_n} \mathsf{E}_x G^\sigma_\tau . \tag{2.21}$$

Taking supremum in (2.20) over all τ, and using that $V_n \le \hat{V}^*_\sigma$ for all $n \ge 1$, it follows that

$$\hat{V}^*_\sigma(x) = \sup_{n \ge 1} V_n(x) \tag{2.22}$$

for all x. By the first part of the proof above we know that each function $x \mapsto V_n(x)$ is measurable, so it follows from (2.22) that $x \mapsto \hat{V}^*_\sigma(x)$ is measurable as claimed.

4. Since the function $x \mapsto \hat{V}^*_\sigma(x)$ is measurable, it follows that

$$\hat{V}^*_\sigma(X_\rho) = \sup_\tau \hat{\mathsf{M}}_{X_\rho}(\tau, \sigma) \tag{2.23}$$

defines a random variable for any stopping time ρ which is given and fixed. On the other hand, by the strong Markov property we have

$$\begin{aligned} \hat{\mathsf{M}}_{X_\rho}(\tau, \sigma) &= \mathsf{E}_{X_\rho}\big[G_1(X_\tau)\, I(\tau \!<\! \sigma) + G_2(X_\sigma)\, I(\sigma \!\le\! \tau)\big] \\ &= \mathsf{E}_x\big[G_1(X_{\rho+\tau\circ\theta_\rho})\, I(\rho + \tau\circ\theta_\rho < \rho + \sigma\circ\theta_\rho) \\ &\qquad + G_2(X_{\rho+\sigma\circ\theta_\rho})\, I(\rho + \sigma\circ\theta_\rho \le \rho + \tau\circ\theta_\rho) \,|\, \mathcal{F}_\rho\big]. \end{aligned} \tag{2.24}$$

From (2.23) and (2.24) we see that

$$\hat{V}^*_\sigma(X_\rho) = \operatorname*{ess\,sup}_\tau \hat{\mathsf{M}}_x(\rho + \tau\circ\theta_\rho, \rho + \sigma\circ\theta_\rho \,|\, \mathcal{F}_\rho) \tag{2.25}$$

where we set

$$\hat{\mathsf{M}}_x(\tau_\rho, \sigma_\rho \,|\, \mathcal{F}_\rho) = \mathsf{E}_x\big[G_1(X_{\tau_\rho})\, I(\tau_\rho \!<\! \sigma_\rho) + G_2(X_{\sigma_\rho})\, I(\sigma_\rho \!\le\! \tau_\rho) \,|\, \mathcal{F}_\rho\big] \tag{2.26}$$

with $\tau_\rho = \rho + \tau\circ\theta_\rho$ and $\sigma_\rho = \rho + \sigma\circ\theta_\rho$ (being stopping times).

5. By general optimal-stopping results of the martingale approach (cf. [19]) we know that the supermartingale

$$\hat{S}^\sigma_t = \operatorname*{ess\,sup}_{\tau \ge t} \hat{\mathsf{M}}_x(\tau, \sigma \,|\, \mathcal{F}_t) \tag{2.27}$$

admits a right-continuous modification (the Snell envelope) such that

$$\hat{V}^*_\sigma(x) = \mathsf{E}_x \hat{S}^\sigma_\rho \tag{2.28}$$

for every stopping time $\rho \le \tau^\sigma_\varepsilon$ where

$$\tau^\sigma_\varepsilon = \inf\{\, t : \hat{S}^\sigma_t \le G^\sigma_t + \varepsilon \,\}. \tag{2.29}$$

Moreover, by the well-known properties of the Snell envelope (stating that equality between the essential supremum and its right-continuous modification is preserved at stopping times and that the essential supremum is attained over hitting times), we see upon recalling (2.25) above that the following identity holds:

$$\hat{V}^*_\sigma(X_\rho) = \hat{S}^\sigma_\rho \quad \mathsf{P}_x\text{-a.s.} \tag{2.30}$$

for every stopping time $\rho \le \sigma$. The precise meaning of (2.30) is

$$\hat{V}^*_{\sigma^\rho(\omega,\ )}(X_\rho(\omega)) = \hat{S}^\sigma_\rho(\omega) \tag{2.30'}$$

for $\omega \in \Omega \setminus N$ with $\mathsf{P}_x(N) = 0$, where $\sigma(\omega) = \rho(\omega) + \sigma^\rho(\omega, \theta_\rho(\omega))$ for a mapping $(\omega, \omega') \mapsto \sigma^\rho(\omega, \omega')$ which is $\mathcal{F}_\rho \otimes \mathcal{F}_\infty$-measurable and $\omega' \mapsto \sigma^\rho(\omega, \omega')$ is a stopping time for each ω given and fixed. We will simplify the notation in the sequel by dropping ρ and ω from $\sigma^\rho(\omega,\ )$ in (2.30') and simply writing σ instead. This also applies to the expression on the right-hand side of (2.23) above. [Note that if σ is a hitting time then $\sigma^\rho(\omega,\omega') = \sigma(\omega')$ for all ω & ω' and this simplification is exact.] In particular, using (2.30) with $\rho = t$ and the fact that $\hat{S}^\sigma$ and G^σ are right-continuous, we see that τ^σ_ε from (2.29) can be equivalently defined as

$$\tau^\sigma_\varepsilon = \inf\{\, t \in Q : \hat{V}^*_\sigma(X_t) \le G^\sigma_t + \varepsilon \,\} \tag{2.31}$$

where Q is any (given and fixed) countable dense subset of the time set.

6. Setting

$$\hat{V}^*(x) = \inf_\sigma \sup_\tau \hat{\mathsf{M}}_x(\tau, \sigma) \tag{2.32}$$

for $x \in E$, let us assume that the function $x \mapsto \hat{V}^*(x)$ is measurable, and let us consider the stopping time τ_ε from (2.6) above but defined over Q for $\hat{V}^*$, i.e.

$$\tilde{\tau}_\varepsilon = \inf\{\, t \in Q : X_t \in \hat{D}^\varepsilon_1 \,\} \tag{2.33}$$

where $\hat{D}^\varepsilon_1 = \{\hat{V}^* \le G_1 + \varepsilon\}$. Let a stopping time β be given and fixed, and let σ be any stopping time satisfying $\sigma \ge \beta \wedge \tilde{\tau}_\varepsilon$. Then for any $t \in Q$ such that $t < \beta \wedge \tilde{\tau}_\varepsilon$ we have

$$\begin{aligned} \hat{V}^*_\sigma(X_t) &\ge \hat{V}^*(X_t) > G_1(X_t) + \varepsilon \\ &= G_1(X_t)\, I(t<\sigma) + G_2(X_\sigma)\, I(\sigma \le t) + \varepsilon \\ &= G^\sigma_t + \varepsilon \end{aligned} \tag{2.34}$$

since $t < \sigma$ so that $I(\sigma \le t) = 0$. Hence we see by (2.31) that $\beta \wedge \tilde{\tau}_\varepsilon \le \tau^\sigma_\varepsilon$. By (2.28) and (2.30) we can conclude that

$$\hat{V}^*_\sigma(x) = \mathsf{E}_x \hat{V}^*_\sigma(X_{\beta\wedge\tilde{\tau}_\varepsilon}) \tag{2.35}$$

for any $\sigma \ge \beta \wedge \tilde{\tau}_\varepsilon$. Taking the infimum over all such σ we obtain

$$V^*(x) \le \hat{V}^*(x) \le \inf_{\sigma \ge \beta\wedge\tilde{\tau}_\varepsilon} \hat{V}^*_\sigma(x) = \inf_{\sigma \ge \beta\wedge\tilde{\tau}_\varepsilon} \mathsf{E}_x \hat{V}^*_\sigma(X_{\beta\wedge\tilde{\tau}_\varepsilon}) \tag{2.36}$$

for every stopping time β. In the next step we will show that the infimum and the expectation in (2.36) can be interchanged.

7. We show that the family of random variables

$$\Big\{ \sup_\tau \hat{\mathsf{M}}_{X_\rho}(\tau,\sigma) \; : \; \sigma \;\text{ is a stopping time} \Big\} \tag{2.37}$$

is downwards directed. Recall that a family of random variables $\{Z_\sigma : \sigma \in I\}$ is downwards directed if for all $\sigma_1, \sigma_2 \in I$ there exists $\sigma_3 \in I$ such that $Z_{\sigma_3} \le Z_{\sigma_1} \wedge Z_{\sigma_2}$ P_x-a.s. for all x .

To prove the claim, recall that by the strong Markov property we have

$$\hat{\mathsf{M}}_{X_\rho}(\tau,\sigma) = \hat{\mathsf{M}}_x(\rho + \tau \circ \theta_\rho, \rho + \sigma \circ \theta_\rho \,|\, \mathcal{F}_\rho). \tag{2.38}$$

If σ_1 and σ_2 are two stopping times given and fixed, set $\tau_\rho = \rho + \tau \circ \theta_\rho$, $\sigma_1' = \rho + \sigma_1 \circ \theta_\rho$ and $\sigma_2' = \rho + \sigma_2 \circ \theta_\rho$, and define

$$B = \Big\{ \sup_\tau \hat{\mathsf{M}}_x(\tau_\rho, \sigma_1' \,|\, \mathcal{F}_\rho) \le \sup_\tau \hat{\mathsf{M}}_x(\tau_\rho, \sigma_2' \,|\, \mathcal{F}_\rho) \Big\}. \tag{2.39}$$

Then $B \in \mathcal{F}_\rho$ and the random variable

$$\sigma' := \sigma_1' \, I_B + \sigma_2' \, I_{B^c} \tag{2.40}$$

is a stopping time. For this, note that $\{\sigma' \le t\} = (\{\sigma_1' \le t\} \cap B) \cup (\{\sigma_2' \le t\} \cap B^c) = (\{\sigma_1' \le t\} \cap B \cap \{\rho \le t\}) \cup (\{\sigma_2' \le t\} \cap B^c \cap \{\rho \le t\}) \in \mathcal{F}_t$ since B and B^c belong to $\mathcal{F}_\rho$, which verifies the claim.

Moreover, the stopping time σ' can be written as

$$\sigma' = \rho + \sigma \circ \theta_\rho \tag{2.41}$$

for some stopping time σ . Indeed, setting

$$A = \Big\{ \sup_\tau \hat{\mathsf{M}}_{X_0}(\tau, \sigma_1) \le \sup_\tau \hat{\mathsf{M}}_{X_0}(\tau, \sigma_2) \Big\} \tag{2.42}$$

we see that $A \in \mathcal{F}_0$ and $B = \theta_\rho^{-1}(A)$ upon recalling (2.38). Hence from (2.40) we get

$$\begin{aligned} \sigma' &= (\rho + \sigma_1 \circ \theta_\rho) \, I_B + (\rho + \sigma_2 \circ \theta_\rho) \, I_{B^c} \\ &= \rho + \big[(\sigma_1 \circ \theta_\rho)(I_A \circ \theta_\rho) + (\sigma_2 \circ \theta_\rho)(I_{A^c} \circ \theta_\rho) \big] \\ &= \rho + (\sigma_1 I_A + \sigma_2 I_{A^c}) \circ \theta_\rho \end{aligned} \tag{2.43}$$

which implies that (2.41) holds with the stopping time $\sigma = \sigma_1 \, I_A + \sigma_2 \, I_{A^c}$. (The latter is a stopping time since $\{\sigma \le t\} = (\{\sigma_1 \le t\} \cap A) \cup (\{\sigma_2 \le t\} \cap A^c) \in \mathcal{F}_t$ due to the fact that $A \in \mathcal{F}_0 \subseteq \mathcal{F}_t$ for all t .)

Finally, we have

$$\begin{aligned} \sup_\tau \hat{\mathsf{M}}_{X_\rho}(\tau,\sigma) &= \sup_\tau \hat{\mathsf{M}}_x(\tau_\rho, \sigma_1' \,|\, \mathcal{F}_\rho) \, I_B + \sup_\tau \hat{\mathsf{M}}_x(\tau_\rho, \sigma_2' \,|\, \mathcal{F}_\rho) \, I_{B^c} \\ &= \sup_\tau \hat{\mathsf{M}}_x(\tau_\rho, \sigma_1' \,|\, \mathcal{F}_\rho) \wedge \sup_\tau \hat{\mathsf{M}}_x(\tau_\rho, \sigma_2' \,|\, \mathcal{F}_\rho) \\ &= \sup_\tau \hat{\mathsf{M}}_{X_\rho}(\tau, \sigma_1) \wedge \sup_\tau \hat{\mathsf{M}}_{X_\rho}(\tau, \sigma_2) \end{aligned} \tag{2.44}$$

which proves that the family (2.37) is downwards directed as claimed.

8. It is well-known (see e.g. [19, pp. 6-7]) that if a family $\{Z_\sigma : \sigma \in I\}$ of random variables is downwards directed, then there exists a countable subset $J = \{\sigma_n : n \geq 1\}$ of I such that

$$\operatorname*{ess\,inf}_{\sigma \in I} Z_\sigma = \lim_{n\to\infty} Z_{\sigma_n} \quad \mathsf{P}_x\text{-a.s.} \tag{2.45}$$

where $Z_{\sigma_1} \geq Z_{\sigma_2} \geq \ldots$ P_x-a.s. In particular, if there exists a random variable Z such that $\mathsf{E}_x Z < \infty$ and $Z_\sigma \leq Z$ for all $\sigma \in I$, then

$$\mathsf{E}_x \operatorname*{ess\,inf}_{\sigma \in I} Z_\sigma = \lim_{n\to\infty} \mathsf{E}_x Z_{\sigma_n} = \inf_{\sigma \in I} \mathsf{E}_x Z_\sigma \tag{2.46}$$

i.e. the order of the infimum and the expectation can be interchanged.

Applying the preceding general fact to the family in (2.37) upon returning to (2.36) we can conclude that

$$V^*(x) \leq \hat{V}^*(x) \leq \inf_{\sigma \geq \beta \wedge \tilde{\tau}_\varepsilon} \mathsf{E}_x \hat{V}^*_\sigma(X_{\beta\wedge\tilde{\tau}_\varepsilon}) = \mathsf{E}_x \hat{V}^*(X_{\beta\wedge\tilde{\tau}_\varepsilon}). \tag{2.47}$$

In the next step we will relate the process $\hat{V}^*(X^{\tilde{\tau}_\varepsilon})$ to yet another right-continuous modification which will play a useful role in the sequel.

9. We show that the process

$$\hat{S}_{t\wedge\tilde{\tau}_\varepsilon} = \operatorname*{ess\,inf}_{\sigma \geq t\wedge\tilde{\tau}_\varepsilon} \hat{S}^\sigma_{t\wedge\tilde{\tau}_\varepsilon} \tag{2.48}$$

admits a right-continuous modification. For this, simplify the notation by setting

$$\hat{S}^\varepsilon_t = \hat{S}_{t\wedge\tilde{\tau}_\varepsilon} \quad \& \quad M^\sigma_t = \hat{S}^\sigma_{t\wedge\tilde{\tau}_\varepsilon} \tag{2.49}$$

and note that the (stopped) process M^σ is a martingale. Indeed, recalling the conclusion in relation to (2.34) above that if $\sigma \geq t\wedge\tilde{\tau}_\varepsilon$ then $t\wedge\tilde{\tau}_\varepsilon \leq \tau^\sigma_\varepsilon$, we see that the martingale property follows by (2.28) above (this is a well-known property of the Snell envelope).

Moreover, since by (2.30) we have

$$\hat{S}^\sigma_{t\wedge\tilde{\tau}_\varepsilon} = \sup_\tau \hat{\mathsf{M}}_{X_{t\wedge\tilde{\tau}_\varepsilon}}(\tau, \sigma) \tag{2.50}$$

when $\sigma \geq t \wedge \tilde{\tau}_\varepsilon$, it follows by (2.37) and (2.48) that there exists a sequence of stopping times $\{\sigma_n : n \geq 1\}$ satisfying $\sigma_n \geq t \wedge \tilde{\tau}_\varepsilon$ such that

$$\hat{S}^\varepsilon_t = \lim_{n\to\infty} M^{\sigma_n}_t \quad \mathsf{P}_x\text{-a.s.} \tag{2.51}$$

where $M^{\sigma_1}_t \geq M^{\sigma_2}_t \geq \ldots$ P_x-a.s. Hence by the conditional monotone convergence theorem (using the integrability condition (2.1) above) we find for $s < t$ that

$$\mathsf{E}_x\big(\hat{S}^\varepsilon_t \,|\, \mathcal{F}_s\big) = \lim_{n\to\infty} \mathsf{E}_x\big(M^{\sigma_n}_t \,|\, \mathcal{F}_s\big) = \lim_{n\to\infty} M^{\sigma_n}_s \geq \hat{S}^\varepsilon_s \tag{2.52}$$

where the martingale property of M^{σ_n} and the definition of $\hat{S}^\varepsilon_s$ are used. This shows that $\hat{S}^\varepsilon$ is a submartingale.

A well-known result in martingale theory states that the submartingale $\hat{S}^\varepsilon$ admits a right-continuous modification if and only if

$$t \mapsto \mathsf{E}_x \hat{S}^\varepsilon_t \text{ is right-continuous.} \tag{2.53}$$

To verify (2.53) note that by the submartingale property of $\hat{S}^\varepsilon$ we have $\mathsf{E}_x \hat{S}^\varepsilon_t \le \ldots \le \mathsf{E}_x \hat{S}^\varepsilon_{t_2} \le \mathsf{E}_x \hat{S}^\varepsilon_{t_1}$ so that $L := \lim_{n\to\infty} \mathsf{E}_x \hat{S}^\varepsilon_{t_n}$ exists and $\mathsf{E}_x \hat{S}^\varepsilon_t \le L$ whenever $t_n \downarrow t$ as $n \to \infty$ is given and fixed. To prove the reverse inequality, fix $N \ge 1$ and by means of (2.51) and the monotone convergence theorem choose $\sigma \ge t \wedge \tilde{\tau}_\varepsilon$ such that

$$\mathsf{E}_x M^\sigma_t \le \mathsf{E}_x \hat{S}^\varepsilon_t + 1/N. \tag{2.54}$$

Fix $\delta > 0$ and note that there is no restriction to assume that $t_n \in [t, t+\delta]$ for all $n \ge 1$. Define a stopping time σ_n by setting

$$\sigma_n = \begin{cases} \sigma & \text{if } \sigma > t_n \wedge \tilde{\tau}_\varepsilon \\ t \wedge \tilde{\tau}_\varepsilon + \delta & \text{if } \sigma \le t_n \wedge \tilde{\tau}_\varepsilon \end{cases} \tag{2.55}$$

for $n \ge 1$. Then for all $n \ge 1$ we have

$$\mathsf{E}_x M^{\sigma_n}_t = \mathsf{E}_x M^{\sigma_n}_{t_n} \ge \mathsf{E}_x \hat{S}^\varepsilon_{t_n} \tag{2.56}$$

by the martingale property of M^{σ_n} and the definition of $\hat{S}^\varepsilon_{t_n}$ using that $\sigma_n \ge t_n \wedge \tilde{\tau}_\varepsilon \ge t \wedge \tilde{\tau}_\varepsilon$.

Since $\{\sigma > t_n \wedge \tilde{\tau}_\varepsilon\}$ and $\{\sigma \le t_n \wedge \tilde{\tau}_\varepsilon\}$ belong to $\mathcal{F}_{t_n \wedge \tilde{\tau}_\varepsilon}$ it is easily verified using (2.30) above that $M^{\sigma_n}_{t_n} I(\sigma > t_n \wedge \tilde{\tau}_\varepsilon) = M^\sigma_{t_n} I(\sigma > t_n \wedge \tilde{\tau}_\varepsilon)$ and $M^{\sigma_n}_{t_n} I(\sigma \le t_n \wedge \tilde{\tau}_\varepsilon) = M^{t\wedge\tilde{\tau}_\varepsilon+\delta}_{t_n} I(\sigma \le t_n \wedge \tilde{\tau}_\varepsilon)$ for all $n \ge 1$. Hence

$$\mathsf{E}_x M^{\sigma_n}_{t_n} = \mathsf{E}_x \big[M^\sigma_{t_n} I(\sigma > t_n \wedge \tilde{\tau}_\varepsilon) + M^{t\wedge\tilde{\tau}_\varepsilon+\delta}_{t_n} I(\sigma \le t_n \wedge \tilde{\tau}_\varepsilon) \big] \tag{2.57}$$

for all $n \ge 1$. Letting $n \to \infty$ in (2.56) and using (2.57) we get

$$\mathsf{E}_x \big[M^\sigma_t I(\sigma > t \wedge \tilde{\tau}_\varepsilon) + M^{t\wedge\tilde{\tau}_\varepsilon+\delta}_t I(\sigma \le t \wedge \tilde{\tau}_\varepsilon) \big] \ge L \tag{2.58}$$

for all $\delta > 0$.

By (2.30) (recall also (2.30')) we have

$$\begin{aligned} M^{t\wedge\tilde{\tau}_\varepsilon+\delta}_t &= \hat{S}^{t\wedge\tilde{\tau}_\varepsilon+\delta}_{t\wedge\tilde{\tau}_\varepsilon} = \hat{V}^*_{t\wedge\tilde{\tau}_\varepsilon+\delta}(X_{t\wedge\tilde{\tau}_\varepsilon}) = \sup_\tau \mathsf{M}_{X_{t\wedge\tilde{\tau}_\varepsilon}}(\tau, \delta) \\ &= \sup_\tau \mathsf{E}_{X_{t\wedge\tilde{\tau}_\varepsilon}} \big[G_1(X_\tau) I(\tau < \delta) + G_2(X_\delta) I(\delta \le \tau) \big] \\ &\le \sup_\tau \mathsf{E}_{X_{t\wedge\tilde{\tau}_\varepsilon}} \big[G_2(X_{\tau\wedge\delta}) \big] \to G_2(X_{t\wedge\tilde{\tau}_\varepsilon}) = M^{t\wedge\tilde{\tau}_\varepsilon}_t \end{aligned} \tag{2.59}$$

where the convergence relation follows by

$$\big| \sup_\tau \mathsf{E}_x G_2(X_{\tau\wedge\delta}) - G_2(x) \big| \le \mathsf{E}_x \sup_{0\le t\le\delta} |G_2(X_t) - G_2(x)| \to 0 \tag{2.60}$$

as $\delta \downarrow 0$ upon using that X is right-continuous (at zero) and that the integrability condition (2.1) holds. Inserting (2.59) in (2.58) and using that $\sigma \ge t \wedge \tilde{\tau}_\varepsilon$ it follows that

$$\mathsf{E}_x M^\sigma_t \ge L\,. \tag{2.61}$$

Combining this with (2.54) we see that $L \le \mathsf{E}_x \hat{S}^\varepsilon_t$ and thus $L = \mathsf{E}_x \hat{S}^\varepsilon_t$. This establishes (2.53) and hence $\hat{S}^\varepsilon$ admits a right-continuous modification (denoted by the same symbol) as claimed.

Moreover, from (2.48) and (2.50) upon using (2.37) it is easily verified that equality between the process in (2.48) and its right-continuous modification extends from deterministic times to all stopping times (via discrete stopping times upon using that each stopping time is the limit of a decreasing sequence of discrete stopping times). Hence by (2.30)+(2.32) and (2.48)+(2.49) we find that

$$\hat{V}^*(X_{\beta\wedge\tilde{\tau}_\varepsilon}) = \hat{S}^\varepsilon_\beta \quad \mathsf{P}_x\text{-a.s.} \tag{2.62}$$

for every stopping time β .

10. We claim that

$$\hat{S}_{\tilde{\tau}_\varepsilon} \le G_1(X_{\tilde{\tau}_\varepsilon}) + \varepsilon \quad \mathsf{P}_x\text{-a.s.} \tag{2.63}$$

To verify this note first that $\tilde{\tau}_{\varepsilon_2} \le \tilde{\tau}_{\varepsilon_1}$ for $\varepsilon_1 < \varepsilon_2$ so that the right-continuous modification of (2.48) extends by letting $\varepsilon \downarrow 0$ to become a right-continuous modification of the process

$$\hat{S}_{t\wedge\tilde{\tau}_{0-}} = \operatorname*{ess\,inf}_{\sigma\ge t\wedge\tilde{\tau}_{0-}} \hat{S}^\sigma_{t\wedge\tilde{\tau}_{0-}} \tag{2.64}$$

where $\tilde{\tau}_{0-} = \lim_{\varepsilon\downarrow 0} \tilde{\tau}_\varepsilon$ is a stopping time. But then by right-continuity of $\hat{S}$ and $G_1(X)$ on $[0, \tau_{0-})$ it follows that on $\{\tilde{\tau}_\varepsilon < \tilde{\tau}_{0-}\}$ we have the inequality (2.63) satisfied. Note that $\tilde{\tau}_{0-} \le \tilde{\tau}_0$ where $\tilde{\tau}_0$ is defined as in (2.33) with $\varepsilon = 0$.

To see what happens on $\{\tilde{\tau}_\varepsilon = \tilde{\tau}_{0-}\}$, let us consider the process

$$\hat{S}_t = \operatorname*{ess\,inf}_{\sigma\ge t} \hat{S}^\sigma_t . \tag{2.65}$$

We then claim that if ρ_n and ρ are stopping times such that $\rho_n \downarrow \rho$ as $n \to \infty$ then

$$\mathsf{E}_x \hat{S}_\rho \le \liminf_{n\to\infty} \mathsf{E}_x \hat{S}_{\rho_n} . \tag{2.66}$$

Indeed, for this note first (since the families are downwards and upwards directed) that

$$\mathsf{E}_x \hat{S}_\rho = \inf_{\sigma\ge\rho} \sup_{\tau\ge\rho} \hat{\mathsf{M}}_x(\tau,\sigma) \le \inf_{\sigma>\rho_n} \sup_{\tau\ge\rho} \hat{\mathsf{M}}_x(\tau,\sigma). \tag{2.67}$$

Taking $\sigma > \rho_n$ we find that

$$\begin{aligned} \hat{\mathsf{M}}_x(\tau,\sigma) &= \mathsf{E}_x\big[\big(G_1(X_\tau)\, I(\tau<\sigma) + G_2(X_\sigma)\, I(\sigma\le\tau)\big)\, I(\tau<\rho_n)\big] \\ &\quad + \mathsf{E}_x\big[(G_1(X_{\tau\vee\rho_n})\, I(\tau\vee\rho_n < \sigma) + G_2(X_\sigma)\, I(\sigma\le\tau\vee\rho_n))\, I(\tau\ge\rho_n)\big] \\ &= \mathsf{E}_x\big[G_1(X_\tau)\, I(\tau<\rho_n)\big] \\ &\quad + \mathsf{E}_x\big[G_1(X_{\tau\vee\rho_n})\, I(\tau\vee\rho_n<\sigma) + G_2(X_\sigma)\, I(\sigma\le\tau\vee\rho_n)\big] \\ &\quad - \mathsf{E}_x\big[(G_1(X_{\tau\vee\rho_n})\, I(\tau\vee\rho_n<\sigma) + G_2(X_\sigma)\, I(\sigma\le\tau\vee\rho_n))\, I(\tau<\rho_n)\big] \\ &= \mathsf{E}_x\big[G_1(X_\tau)\, I(\tau<\rho_n) - G_1(X_{\rho_n})\, I(\tau<\rho_n)\big] \end{aligned} \tag{2.68}$$

$$
\begin{aligned}
&+\mathsf{E}_x\big[G_1(X_{\tau\vee\rho_n})\,I(\tau\vee\rho_n<\sigma)+G_2(X_\sigma)\,I(\sigma\le\tau\vee\rho_n)\big]\\
&=\mathsf{E}_x\big[G_1(X_{\tau\wedge\rho_n})-G_1(X_{\rho_n})\big]+\hat{\mathsf{M}}_x(\tau\vee\rho_n,\sigma)\,.
\end{aligned}
$$

From (2.67) and (2.68) we get

$$
\begin{aligned}
\mathsf{E}_x\hat{S}_\rho &\le \mathsf{E}_x\sup_{\rho\le t\le\rho_n}|G_1(X_t)-G_1(X_{\rho_n})|+\inf_{\sigma>\rho_n}\sup_{\tau\ge\rho_n}\hat{\mathsf{M}}_x(\tau,\sigma)\\
&=\mathsf{E}_x\sup_{\rho\le t\le\rho_n}|G_1(X_t)-G_1(X_{\rho_n})|+\inf_{\sigma\ge\rho_n}\sup_{\tau\ge\rho_n}\hat{\mathsf{M}}_x(\tau,\sigma)\\
&=\mathsf{E}_x\sup_{\rho\le t\le\rho_n}|G_1(X_t)-G_1(X_{\rho_n})|+\mathsf{E}_x\hat{S}_{\rho_n}
\end{aligned}
\tag{2.69}
$$

where the first equality can easily be justified by using that each σ is the limit of a strictly decreasing sequence of discrete stopping times σ_m as $m\to\infty$ yielding

$$
\sup_{\tau\ge\rho_n}\hat{\mathsf{M}}_x(\tau,\sigma)\ge\limsup_{m\to\infty}\,\sup_{\tau\ge\rho_n}\hat{\mathsf{M}}_x(\tau,\sigma_m)
\tag{2.70}
$$

which is obtained directly from (2.93) below. Letting $n\to\infty$ in (2.69) and using that the second last expectation tends to zero since $G_1(X)$ is right-continuous and the integrability condition (2.1) holds, we get (2.66) as claimed.

Returning to the question of $\{\tilde{\tau}_\varepsilon=\tilde{\tau}_{0-}\}$, consider the Borel set $\hat{D}_1^0=\{\hat{V}^*=G_1\}$ and choose compact sets $K_1\subseteq K_2\subseteq\ldots\subseteq\hat{D}_1^0$ such that $\tau_n:=\inf\{\,t: X_t\in K_n\,\}$ satisfy $\tau_n\downarrow\tilde{\tau}_0$ P_x-a.s. as $n\to\infty$. (The latter is a well-known consequence of the fact that each probability measure on E is tight.) Since each K_n is closed we have $\hat{S}_{\tau_n}=\hat{V}^*(X_{\tau_n})=G_1(X_{\tau_n})$ by right-continuity of X for all $n\ge1$. Hence by (2.66) we find

$$
\mathsf{E}_x\hat{S}_{\tilde{\tau}_0}\le\liminf_{n\to\infty}\mathsf{E}_x\hat{S}_{\tau_n}=\liminf_{n\to\infty}\mathsf{E}_xG_1(X_{\tau_n})=\mathsf{E}_xG_1(X_{\tilde{\tau}_0})
\tag{2.71}
$$

by right-continuity of $G_1(X)$ using also the integrability condition (2.1) above. Since $\hat{S}_{\tilde{\tau}_0}\ge G_1(X_{\tilde{\tau}_0})$ P_x-a.s. by definition, we see from (2.71) that $\hat{S}_{\tilde{\tau}_0}=G_1(X_{\tilde{\tau}_0})$ P_x-a.s. Moreover, if we consider the Borel set $\hat{D}_1^\varepsilon=\{\hat{V}^*\le G_1+\varepsilon\}$ and likewise choose stopping times τ_n^ε satisfying $\tau_n^\varepsilon\downarrow\tilde{\tau}_\varepsilon$ P_x-a.s. then the same arguments as in (2.71) show that

$$
\begin{aligned}
\mathsf{E}_xG_1(X_{\tilde{\tau}_{0-}})&=\mathsf{E}_x\lim_{\varepsilon\downarrow0}G_1(X_{\tilde{\tau}_\varepsilon})\le\mathsf{E}_x\liminf_{\varepsilon\downarrow0}\hat{S}_{\tilde{\tau}_\varepsilon}\le\liminf_{\varepsilon\downarrow0}\mathsf{E}_x\hat{S}_{\tilde{\tau}_\varepsilon}\\
&\le\liminf_{\varepsilon\downarrow0}\big(\liminf_{n\to\infty}\mathsf{E}_x\hat{S}_{\tau_n^\varepsilon}\big)\le\liminf_{\varepsilon\downarrow0}\big(\liminf_{n\to\infty}\mathsf{E}_xG_1(X_{\tau_n^\varepsilon})+\varepsilon\big)\\
&=\mathsf{E}_xG_1(X_{\tilde{\tau}_{0-}})
\end{aligned}
\tag{2.72}
$$

upon using that $G_1(X_{\tilde{\tau}_\varepsilon})\le\hat{S}_{\tilde{\tau}_\varepsilon}$ P_x-a.s. and applying Fatou's lemma. Hence all the inequalities in (2.72) are equalities and thus

$$
G_1(X_{\tilde{\tau}_{0-}})=\liminf_{\varepsilon\downarrow0}\hat{S}_{\tilde{\tau}_\varepsilon}\quad\mathsf{P}_x\text{-a.s.}
\tag{2.73}
$$

Since $\tilde{\tau}_\varepsilon\uparrow\tilde{\tau}_{0-}$ as $\varepsilon\downarrow0$, we see from (2.73) that $G_1(X_{\tilde{\tau}_{0-}})=\hat{S}_{\tilde{\tau}_{0-}}$ P_x-a.s. on $\{\tilde{\tau}_\varepsilon=\tilde{\tau}_{0-}\}$. This implies that $\tilde{\tau}_0\le\tilde{\tau}_{0-}$ and thus $\tilde{\tau}_0=\tilde{\tau}_{0-}$ both P_x-a.s. on $\{\tilde{\tau}_\varepsilon=\tilde{\tau}_{0-}\}$. Recalling also that $\hat{S}_{\tilde{\tau}_0}=G_1(X_{\tilde{\tau}_0})$ P_x-a.s. we finally see that on $\{\tilde{\tau}_\varepsilon=\tilde{\tau}_{0-}\}$ one has $\hat{S}_{\tilde{\tau}_\varepsilon}=\hat{S}_{\tilde{\tau}_0}=G_1(X_{\tilde{\tau}_0})=$

$G_1(X_{\tilde{\tau}_\varepsilon}) \le G_1(X_{\tilde{\tau}_\varepsilon}) + \varepsilon$ P_x-a.s. so that (2.63) holds as claimed. [Note that (2.63) can also be obtained by showing that $\hat{S}$ defined in (2.65) admits a right-continuous modification. This proof can be used instead of parts 9 and 10 above which focused on exploiting the submartingale characterisation (2.53) above.]

11. Inserting (2.62) into (2.47) and using (2.63) we get

$$\begin{aligned} (2.74)\qquad V^*(x) &\le \hat{V}^*(x) \le \mathsf{E}_x \hat{S}^\varepsilon_\beta = \mathsf{E}_x\big[\hat{S}_{\tilde{\tau}_\varepsilon} I(\tilde{\tau}_\varepsilon \le \beta) + \hat{S}_\beta I(\beta < \tilde{\tau}_\varepsilon)\big] \\ &\le \mathsf{E}_x\big[\big(G_1(X_{\tilde{\tau}_\varepsilon}) + \varepsilon\big) I(\tilde{\tau}_\varepsilon < \beta) + G_2(X_\beta) I(\beta < \tilde{\tau}_\varepsilon) + \big(G_3(X_{\tilde{\tau}_\varepsilon}) + \varepsilon\big) I(\tilde{\tau}_\varepsilon = \beta)\big] \\ &\le \mathsf{M}_x(\tilde{\tau}_\varepsilon, \beta) + \varepsilon \end{aligned}$$

for every stopping time β. Proceeding as in (2.8) above we find that $\hat{V}^* = V^* = V_*$ and thus (2.63) yields (2.9) with $\tilde{\tau}_\varepsilon$ in place of τ_ε.

12. To derive (2.9) with τ_ε from (2.6), first note that $\tau_\varepsilon \le \tilde{\tau}_\varepsilon$ and recall from (2.47) that

$$(2.75)\qquad V^*(x) \le \mathsf{E}_x V^*(X_{\beta \wedge \tilde{\tau}_\varepsilon})$$

for every stopping time β. From general theory of Markov processes (upon using that $t \mapsto X_{t \wedge \tilde{\tau}_\varepsilon}$ is right-continuous and adapted) it is known that (2.75) implies that V^* is finely lower semi-continuous up to $\tilde{\tau}_\varepsilon$ in the sense that

$$(2.76)\qquad V^*(x) \le \liminf_{t \downarrow 0} V^*(X_{t \wedge \tilde{\tau}_\varepsilon}) \quad \mathsf{P}_x\text{-a.s.}$$

This in particular implies (since $X^{\tilde{\tau}_\varepsilon}$ is a strong Markov process) that

$$(2.77)\qquad V^*(X_\tau) \le \liminf_{t \downarrow 0} V^*(X_{\tau + t}) \quad \mathsf{P}_x\text{-a.s. on } \{\tau < \tilde{\tau}_\varepsilon\}$$

for every stopping time τ. Indeed, setting $Y_t = X_{t \wedge \tilde{\tau}_\varepsilon}$ and

$$(2.78)\qquad A = \Big\{ V^*(x) \le \liminf_{t \downarrow 0} V^*(Y_t) \Big\} \quad \& \quad B = \Big\{ V^*(Y_\tau) \le \liminf_{t \downarrow 0} V^*(Y_{\tau + t}) \Big\}$$

we see that $B = \theta_\tau^{-1}(A)$ and $B^c = \theta_\tau^{-1}(A^c)$ so that the strong Markov property of Y implies

$$(2.79)\qquad \mathsf{P}_x(B^c) = \mathsf{P}_x(\theta_\tau^{-1}(A^c)) = \mathsf{E}_x\big[\mathsf{E}_x(I_{A^c} \circ \theta_\tau \,|\, \mathcal{F}_\tau)\big] = \mathsf{E}_x \mathsf{E}_{X_\tau}(I_{A^c}) = \mathsf{E}_x \mathsf{P}_{X_\tau}(A^c) = 0$$

since $\mathsf{P}_y(A^c) = 0$ for all y. Hence (2.77) holds as claimed. In particular, if (2.77) is applied to τ_ε, we get

$$(2.80)\qquad V^*(X_{\tau_\varepsilon}) \le G_1(X_{\tau_\varepsilon}) + \varepsilon \quad \mathsf{P}_x\text{-a.s. on } \{\tau_\varepsilon < \tilde{\tau}_\varepsilon\}.$$

With this new information we can now revisit (2.74) via (2.75) upon using (2.63) and (2.80). This gives

$$\begin{aligned} (2.81)\qquad V^*(x) &\le \mathsf{E}_x V^*(X_{\beta \wedge \tau_\varepsilon}) = \mathsf{E}_x\big[V^*(X_{\tau_\varepsilon}) I(\tau_\varepsilon \le \beta) + V^*(X_\beta) I(\beta < \tau_\varepsilon)\big] \\ &= \mathsf{E}_x\big[V^*(X_{\tau_\varepsilon}) I(\tau_\varepsilon \le \beta, \tau_\varepsilon < \tilde{\tau}_\varepsilon) + \hat{S}_{\tilde{\tau}_\varepsilon} I(\tau_\varepsilon \le \beta, \tau_\varepsilon = \tilde{\tau}_\varepsilon) + V^*(X_\beta) I(\beta < \tau_\varepsilon)\big] \\ &\le \mathsf{E}_x\big[\big(G_1(X_{\tau_\varepsilon}) + \varepsilon\big) I(\tau_\varepsilon \le \beta, \tau_\varepsilon < \tilde{\tau}_\varepsilon) + \big(G_1(X_{\tilde{\tau}_\varepsilon}) + \varepsilon\big) I(\tau_\varepsilon \le \beta, \tau_\varepsilon = \tilde{\tau}_\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&\qquad + G_2(X_\beta)\, I(\beta < \tau_\varepsilon)\big]\\
&\le \mathsf{E}_x\big[\big(G_1(X_{\tau_\varepsilon}) + \varepsilon\big)\, I(\tau_\varepsilon < \beta) + G_2(X_\beta)\, I(\beta < \tau_\varepsilon) + \big(G_3(X_{\tau_\varepsilon}) + \varepsilon\big)\, I(\tau_\varepsilon = \beta)\big]\\
&\le \mathsf{M}_x(\tau_\varepsilon, \beta) + \varepsilon
\end{aligned}
$$

for every stopping time β. This completes the proof of (2.9) when the function $x \mapsto \hat{V}^*(x)$ from (2.32) is assumed to be measurable.

13. If $x \mapsto \hat{V}^*(x)$ is not assumed to be measurable, then the proof above can be repeated with reference only to $\hat{S}^\sigma$ and $\hat{S}$ under P_x with x given and fixed. In exactly the same way as above this gives the identity $\hat{V}^*(x) = V^*(x) = V_*(x)$ for this particular and thus all x. But then the measurability follows from the following general fact: If $V^* = V_*$ then $V := V^* = V_*$ defines a measurable function.

To derive this fact consider the optimal stopping game (2.2)+(2.3) when X is a discrete-time Markov chain, so that τ and σ (without loss of generality) take values in $\{0, 1, 2, \dots\}$. The horizon N (the upper bound for τ and σ in (2.2)+(2.3) above) can be either finite or infinite. When N is finite the most interesting case is when $G_i = G_i(x, n)$ for $i = 1, 2, 3$ with $G_1(x, N) = G_2(x, N) = G_3(x, N)$ for all x. When N is infinite then

$$
\liminf_{n\to\infty} G_2(X_n) \le \limsup_{n\to\infty} G_1(X_n) \tag{2.82}
$$

as stipulated following (2.3) above, and the common value for $G_3(X_\infty)$ could formally be assigned as either of the two values in (2.82) (if τ and σ are allowed to take the value ∞).

Then the following Wald-Bellman equations are valid:

$$
V_n(x) = G_1(x) \vee TV_{n-1}(x) \wedge G_2(x) \tag{2.83}
$$

for $n = 1, 2, \dots$ where V_0 is set to be either G_1 or G_2. This yields $V_N = V^* = V_*$ with $V_\infty = \lim_{n\to\infty} V_n$ if $N = \infty$ (see [8] for details).

Recalling that T denotes the transition operator defined by

$$
TF(x) = \mathsf{E}_x F(X_1) \tag{2.84}
$$

one sees that $x \mapsto TF(x)$ is measurable whenever F is so (and $\mathsf{E}_x F(X_1)$ is well defined for all x). Applying this argument inductively in (2.83) we see that $x \mapsto V_N(x)$ is a measurable function. Thus, optimal stopping games for discrete-time Markov chains always lead to measurable value functions.

To treat the case of general X, let Q_n denote the set of all dyadic rationals $k/2^n$ in the time set, and for a given stopping time τ let τ_n be defined by setting $\tau_n = k/2^n$ on $\{\,(k-1)/2^n < \tau \le k/2^n\,\}$. Then each τ_n is a stopping time taking values in Q_n and the following inequality is valid:

$$
\mathsf{M}_x(\tau, \sigma) \le \mathsf{M}_x(\tau_n, \sigma) + \mathsf{E}_x|G_1(X_\tau) - G_1(X_{\tau_n})| \tag{2.85}
$$

for every stopping time $\sigma \in Q_n$ (meaning that σ takes values in Q_n). Indeed, this can be derived as follows:

$$
\mathsf{M}_x(\tau, \sigma) - \mathsf{M}_x(\tau_n, \sigma) \tag{2.86}
$$

$$
\begin{aligned}
&= \mathsf{E}_x\big[G_1(X_\tau)\, I(\tau<\sigma) + G_2(X_\sigma)\, I(\sigma<\tau) + G_3(X_\tau)\, I(\tau=\sigma, \tau\neq\tau_n) \\
&\qquad - G_1(X_{\tau_n})\, I(\tau_n<\sigma) - G_2(X_\sigma)\, I(\sigma<\tau_n) - G_3(X_{\tau_n})\, I(\tau_n=\sigma, \tau_n\neq\tau)\big] \\
&\le \mathsf{E}_x\big[\big(G_1(X_\tau) - G_1(X_{\tau_n})\big)\, I(\tau<\sigma) \\
&\qquad + G_1(X_{\tau_n})\, \big(I(\tau<\sigma) - I(\tau_n<\sigma) - I(\tau_n=\sigma, \tau_n\neq\tau)\big) \\
&\qquad + G_2(X_\sigma)\, \big(I(\sigma<\tau) + I(\tau=\sigma, \tau\neq\tau_n) - I(\sigma<\tau_n)\big)\big] \\
&= \mathsf{E}_x\big[\big(G_1(X_\tau) - G_1(X_{\tau_n})\big)\, I(\tau<\sigma) + \big(G_1(X_{\tau_n}) - G_2(X_\sigma)\big)\, I(\tau<\sigma<\tau_n)\big]
\end{aligned}
$$

being true for any stopping times τ, σ and τ_n such that $\tau \le \tau_n$. In particular, if $\sigma \in Q_n$ (and τ_n is defined as above), then $\{\tau<\sigma<\tau_n\} = \emptyset$ so that (2.86) becomes

$$
\begin{aligned}
\mathsf{M}_x(\tau, \sigma) &\le \mathsf{M}_x(\tau_n, \sigma) + \mathsf{E}_x\big[\big(G_1(X_\tau) - G_1(X_{\tau_n})\big) I(\tau<\sigma)\big] \\
&\le \mathsf{M}_x(\tau_n, \sigma) + \mathsf{E}_x|G_1(X_\tau) - G_1(X_{\tau_n})|
\end{aligned} \tag{2.87}
$$

as claimed in (2.85) above.

Let τ_n^* and σ_n^* denote the optimal stopping times (in the Nash sense) for the optimal stopping game (2.2)+(2.3) with the time set Q_n, and let $V_n(x)$ denote the corresponding value of the game, i.e.

$$
V_n(x) = \mathsf{M}_x(\tau_n^*, \sigma_n^*) \tag{2.88}
$$

for all x. (From (2.83) one sees that such optimal stopping times always exist in the discrete-time setting.) By the first part above (applied to the Markov chain $(X_t)_{t \in Q_n}$) we know that $x \mapsto V_n(x)$ is measurable.

Setting $\varepsilon_n(x, \tau) = \mathsf{E}_x|G_1(X_\tau) - G_1(X_{\tau_n})|$ we see that (2.85) reads

$$
\mathsf{M}_x(\tau, \sigma) \le \mathsf{M}_x(\tau_n, \sigma) + \varepsilon_n(x, \tau) \tag{2.89}
$$

for every τ and every $\sigma \in Q_n$. Hence we find that

$$
\mathsf{M}_x(\tau, \sigma_n^*) \le \mathsf{M}_x(\tau_n, \sigma_n^*) + \varepsilon_n(x, \tau) \le \mathsf{M}_x(\tau_n^*, \sigma_n^*) + \varepsilon_n(x, \tau) = V_n(x) + \varepsilon_n(x, \tau). \tag{2.90}
$$

This implies that

$$
\inf_\sigma \mathsf{M}_x(\tau, \sigma) \le \liminf_{n\to\infty} V_n(x) \tag{2.91}
$$

since $\varepsilon_n(x, \tau) \to 0$ by right-continuity of X and the fact that $\tau_n \downarrow \tau$ as $n \to \infty$ (using also the integrability condition (2.1) above). Taking the supremum over all τ we conclude that

$$
V_*(x) \le \liminf_{n\to\infty} V_n(x) \tag{2.92}
$$

for all x.

On the other hand, similarly to (2.86) one finds that

$$
\begin{aligned}
\mathsf{M}_x(\tau, \sigma) - \mathsf{M}_x(\tau, \sigma_n) \ge \mathsf{E}_x\big[&\big(G_2(X_\sigma) - G_2(X_{\sigma_n})\big)\, I(\sigma<\tau) \\
&+ \big(G_2(X_{\sigma_n}) - G_1(X_\tau)\big)\, I(\sigma<\tau<\sigma_n)\big]
\end{aligned} \tag{2.93}
$$

for any stopping times τ , σ and σ_n such that $\sigma \le \sigma_n$. If σ_n is defined analogously to τ_n above (with σ in place of τ), then (2.93) yields the following analogue of (2.89) above:

$$\mathsf{M}_x(\tau,\sigma) \ge \mathsf{M}_x(\tau,\sigma_n) - \delta_n(x,\sigma) \tag{2.94}$$

where $\delta_n(x,\sigma) = \mathsf{E}_x|G_2(X_\sigma) - G_2(X_{\sigma_n})| \to 0$ as $n \to \infty$ for the same reasons as above. This analogously yields

$$\limsup_{n\to\infty} V_n(x) \le V^*(x) \tag{2.95}$$

for all x . Thus, if $V^* = V_*$ then by (2.92) and (2.95) we see that $V := V^* = V_*$ satisfies

$$V(x) = \lim_{n\to\infty} V_n(x) \tag{2.96}$$

for all x . Since each V_n is measurable we see that V is measurable as claimed. This completes the first part of the proof.

(II) In the second part of the proof we will assume that X is right-continuous and left-continuous over stopping times, and we will show that these hypotheses imply the Nash equilibrium (1.4) with τ_* and σ_* from (2.4).

1. Since X is right-continuous we know by the first part of the proof above that $V^* = V_*$ with $V := V^* = V_*$ defining a measurable function which by (2.7) satisfies

$$\mathsf{M}_x(\tau,\sigma_\varepsilon) - \varepsilon \le V(x) \le \mathsf{M}_x(\tau_\varepsilon,\sigma) + \varepsilon \tag{2.97}$$

for all τ , σ , x and $\varepsilon > 0$. Recalling from (2.5)+(2.6) that

$$\tau_\varepsilon = \inf\{\, t : X_t \in D_1^\varepsilon \,\} \tag{2.98}$$

where $D_1^\varepsilon = \{\, V \le G_1 + \varepsilon \,\}$, we will now show that the second inequality in (2.97) implies

$$V(x) \le \mathsf{M}_x(\tau_0,\sigma) \tag{2.99}$$

for all σ and x , where $\tau_0 = \inf\{\, t : X_t \in D_1^0 \,\}$ with $D_1^0 = \{\, V = G_1 \,\}$. (Note that τ_0 coincides with τ_* in the notation above.)

2. It is clear from the definitions that $\tau_\varepsilon \uparrow \tau_{0-}$ as $\varepsilon \downarrow 0$ where τ_{0-} is a stopping time satisfying $\tau_{0-} \le \tau_0$. We will now show that $\tau_{0-} = \tau_0$ P_x-a.s. For this, let us first establish the following general fact: If ρ_n and ρ are stopping times such that $\rho_n \uparrow \rho$ as $n \to \infty$, then

$$\mathsf{E}_x V(X_\rho) \le \liminf_{n\to\infty} \mathsf{E}_x V(X_{\rho_n}). \tag{2.100}$$

To see this recall from the first part of the proof above that $V(X_\beta) = \hat{V}(X_\beta) = \hat{S}_\beta = \check{V}(X_\beta) = \check{S}_\beta$ for every stopping time β , where $\check{V}$ and $\check{S}$ are defined analogously to $\hat{V}$ and $\hat{S}$ but with $\check{\mathsf{M}}_x(\tau,\sigma) = \mathsf{E}_x\big[G_1(X_\tau)\, I(\tau \le \sigma) + G_2(X_\sigma)\, I(\sigma < \tau)\big]$ in place of $\hat{\mathsf{M}}_x(\tau,\sigma)$ and with the order of the supremum and the infimum being interchanged. Hence we find that

$$\mathsf{E}_x V(X_{\rho_n}) = \mathsf{E}_x \hat{S}_{\rho_n} = \sup_{\tau \ge \rho_n} \inf_{\sigma \ge \rho_n} \hat{\mathsf{M}}_x(\tau,\sigma) \ge \sup_{\tau \ge \rho} \inf_{\sigma \ge \rho_n} \hat{\mathsf{M}}_x(\tau,\sigma). \tag{2.101}$$

Taking $\tau \geq \rho$ we find that

$$
\begin{aligned}
(2.102)\qquad \hat{\mathsf{M}}_x(\tau,\sigma) &= \mathsf{E}_x\big[\big(G_1(X_\tau)\,I(\tau<\sigma) + G_2(X_\sigma)\,I(\sigma\leq\tau)\big)\,I(\sigma\leq\rho)\big] \\
&\quad + \mathsf{E}_x\big[\big(G_1(X_\tau)\,I(\tau<\sigma\vee\rho) + G_2(X_{\sigma\vee\rho})\,I(\sigma\vee\rho\leq\tau)\big)\,I(\sigma>\rho)\big] \\
&= \mathsf{E}_x\big[G_2(X_\sigma)\,I(\sigma\leq\rho)\big] \\
&\quad + \mathsf{E}_x\big[G_1(X_\tau)\,I(\tau<\sigma\vee\rho) + G_2(X_{\sigma\vee\rho})\,I(\sigma\vee\rho\leq\tau)\big] \\
&\quad - \mathsf{E}_x\big[\big(G_1(X_\tau)\,I(\tau<\sigma\vee\rho) + G_2(X_{\sigma\vee\rho})\,I(\sigma\vee\rho\leq\tau)\big)\,I(\sigma\leq\rho)\big] \\
&= \mathsf{E}_x\big[G_2(X_\sigma)\,I(\sigma\leq\rho) - G_2(X_\rho)\,I(\sigma\leq\rho)\big] \\
&\quad + \mathsf{E}_x\big[G_1(X_\tau)\,I(\tau<\sigma\vee\rho) + G_2(X_{\sigma\vee\rho})\,I(\sigma\vee\rho\leq\tau)\big] \\
&= \mathsf{E}_x\big[G_2(X_{\sigma\wedge\rho}) - G_2(X_\rho)\big] + \hat{\mathsf{M}}_x(\tau,\sigma\vee\rho).
\end{aligned}
$$

From (2.101) and (2.102) we get

$$
\begin{aligned}
(2.103)\qquad \mathsf{E}_x V(X_{\rho_n}) &\geq \inf_{\sigma\geq\rho_n} \mathsf{E}_x\big[G_2(X_{\sigma\wedge\rho}) - G_2(X_\rho)\big] + \sup_{\tau\geq\rho}\inf_{\sigma\geq\rho} \hat{\mathsf{M}}_x(\tau,\sigma) \\
&= \mathsf{E}_x V(X_\rho) + \inf_{\rho_n\leq\sigma\leq\rho} \mathsf{E}_x\big[G_2(X_\sigma) - G_2(X_\rho)\big]\,.
\end{aligned}
$$

Letting $n\to\infty$ and using that the final expectation tends to zero since X is left-continuous over stopping times and the integrability condition (2.1) holds, we get (2.100) as claimed.

Applying (2.100) to τ_ε and τ_{0-}, and recalling from the first part of the proof above that $V(X_{\tau_\varepsilon}) \leq G_1(X_{\tau_\varepsilon}) + \varepsilon$ P_x-a.s. it follows that

$$
(2.104)\qquad \mathsf{E}_x V(X_{\tau_{0-}}) \leq \liminf_{\varepsilon\downarrow 0} \mathsf{E}_x V(X_{\tau_\varepsilon}) \leq \liminf_{\varepsilon\downarrow 0} \mathsf{E}_x\big[G_1(X_{\tau_\varepsilon}) + \varepsilon\big] = \mathsf{E}_x G_1(X_{\tau_{0-}})
$$

upon using that $G_1(X)$ is left-continuous over stopping times (as well as the integrability condition (2.1) above). Since on the other hand we have $V(X_{\tau_{0-}}) \geq G_1(X_{\tau_{0-}})$ we see from (2.104) that $V(X_{\tau_{0-}}) = G_1(X_{\tau_{0-}})$ and thus $\tau_0 \leq \tau_{0-}$ P_x-a.s. proving that $\tau_0 = \tau_{0-}$ P_x-a.s. as claimed.

3. Motivated by passing to the limit in (2.97) for $\varepsilon\downarrow 0$, we will now establish the following general fact: If τ_n and τ are stopping times such that $\tau_n \uparrow \tau$ then

$$
(2.105)\qquad \limsup_{n\to\infty} \mathsf{M}_x(\tau_n,\sigma) \leq \mathsf{M}_x(\tau,\sigma)
$$

for every stopping time σ given and fixed. To see this, note that

$$
\begin{aligned}
(2.106)\qquad & \mathsf{M}_x(\tau,\sigma) - \mathsf{M}_x(\tau_n,\sigma) \\
&= \mathsf{E}_x\big[G_1(X_\tau)\,I(\tau<\sigma) + G_2(X_\sigma)\,I(\sigma<\tau) + G_3(X_\tau)\,I(\tau=\sigma,\tau\neq\tau_n) \\
&\qquad - G_1(X_{\tau_n})\,I(\tau_n<\sigma) - G_2(X_\sigma)\,I(\sigma<\tau_n) - G_3(X_{\tau_n})\,I(\tau_n=\sigma,\tau_n\neq\tau)\big] \\
&\geq \mathsf{E}_x\big[\big(G_1(X_\tau) - G_1(X_{\tau_n})\big)\,I(\tau<\sigma) \\
&\qquad + G_1(X_{\tau_n})\,\big(I(\tau<\sigma) + I(\tau=\sigma,\tau\neq\tau_n) - I(\tau_n<\sigma)\big) \\
&\qquad + G_2(X_\sigma)\,\big(I(\sigma<\tau) - I(\sigma<\tau_n) - I(\tau_n=\sigma,\tau_n\neq\tau)\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathsf{E}_x\big[\big(G_1(X_\tau)-G_1(X_{\tau_n})\big)\, I(\tau<\sigma) + \big(G_2(X_\sigma)-G_1(X_{\tau_n})\big)\, I(\tau_n<\sigma<\tau)\big] \\
&\geq -\mathsf{E}_x|G_1(X_\tau)-G_1(X_{\tau_n})| - \mathsf{E}_x\big[\big(\sup_t |G_2(X_t)| + \sup_t |G_1(X_t)|\big)\, I(\tau_n<\sigma<\tau)\big].
\end{aligned}
$$

Letting $n \to \infty$ and using the fact that the final two expectations tend to zero since $G_1(X)$ is left-continuous over stopping times and the integrability condition (2.1) holds, we see that (2.105) follows as claimed.

Applying (2.105) to τ_ε and τ_0 upon letting $\varepsilon \downarrow 0$ in (2.97) we get (2.99). The inequality

$$
\mathsf{M}_x(\tau, \sigma_0) \leq V(x) \tag{2.107}
$$

can be established analogously. Combining (2.99) and (2.107) we get (1.4) and the proof is complete. □

3. Concluding remarks

The following example shows that the Nash equilibrium (1.4) may fail when X is right-continuous but not left-continuous over stopping times.

Example 3.1. Let the state space E of the process X be $[-1,1]$. If X starts at $x \in (-1,1)$ let X be a standard Brownian motion until it hits either -1 or 1; at this time let X start afresh from 0 as an independent copy of B until it hits either -1 or 1; and so on. If X starts at $x \in \{-1,1\}$ let X stay at the same x for the rest of time.

It follows that X is a right-continuous strong Markov process which is not left-continuous over stopping times. Indeed, if we consider the first hitting time ρ_ε of X to b_ε under P_x for $x \in (-1,1)$ given and fixed, where b_ε equals either $-1+\varepsilon$ or $1-\varepsilon$ for all $\varepsilon > 0$ sufficiently small, then $\rho_\varepsilon \uparrow \rho$ as $\varepsilon \downarrow 0$ so that ρ is a stopping time, however, the value $X_{\rho_\varepsilon} = b_\varepsilon$ does not converge to $X_\rho = 0$ as $\varepsilon \downarrow 0$, implying the claim.

Let $G_1(x) = x(x+1)-1$ and $G_2(x) = -x(x-1)+1$ for $x \in [-1,1]$, and let G_3 be equal to G_1 on $[-1,1]$. Note that $G_i(-1) = -1$ and $G_i(1) = 1$ for $i = 1,2,3$. To include stopping times τ and σ which are allowed to take the value ∞ below, let us set $G_3(X_\infty) = \limsup_{t\to\infty} G_1(X_t)$. Note that $G_3(X_\infty) \equiv 1$ under P_x when $x \in (-1,1]$ and $G_3(X_\infty) \equiv -1$ under P_x when $x = -1$.

It is then easily seen (using the first part of Theorem 2.1 above) that $V^*(x) = V_*(x) = x$ for all $x \in [-1,1]$ with $\tau_\varepsilon = \inf\{\, t : X_t \leq a_\varepsilon^1 \text{ or } X_t \geq b_\varepsilon^1 \,\}$ (where $a_\varepsilon^1 < b_\varepsilon^1$ satisfy $G_1(a_\varepsilon^1) = a_\varepsilon^1 - \varepsilon$ and $G_1(b_\varepsilon^1) = b_\varepsilon^1 - \varepsilon$) and $\sigma_\varepsilon = \inf\{\, t : X_t \leq a_\varepsilon^2 \text{ or } X_t \geq b_\varepsilon^2 \,\}$ (where $a_\varepsilon^2 < b_\varepsilon^2$ satisfy $G_2(a_\varepsilon^2) = a_\varepsilon^2 + \varepsilon$ and $G_2(b_\varepsilon^2) = b_\varepsilon^2 + \varepsilon$) being approximate stopping times satisfying (2.7) above. (Note that $a_\varepsilon^i \downarrow -1$ and $b_\varepsilon^i \uparrow 1$ as $\varepsilon \downarrow 0$ for $i = 1,2$.)

Thus the Stackelberg equilibrium (1.3) holds with $V(x) = x$ for all $x \in [-1,1]$. It is clear, however, that the Nash equilibrium fails as it is impossible to find stopping times τ_* and σ_* satisfying (1.4) above. [Note that the natural candidates $\tau \equiv \infty$ and $\sigma \equiv \infty$ are ruled out since $\mathsf{M}_x(\infty,\infty) = 1$ for $x \in (-1,1]$ and $\mathsf{M}_x(\infty,\infty) = -1$ for $x = -1$.]

The methodology used in the proof of Theorem 2.1 above (second part) extends from the Markovian approach to the martingale approach for optimal stopping games. For the sake of completeness we will formulate the analogous results of the martingale approach.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathsf{P})$ be a filtered probability space such that $(\mathcal{F}_t)_{t\ge 0}$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets from $\mathcal{F}$. Given adapted stochastic processes G^1, G^2, G^3 on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathsf{P})$ satisfying $G_t^1 \le G_t^3 \le G_t^2$ for all t and the integrability condition

$$\mathsf{E} \sup_t |G_t^i| < \infty \quad (i = 1, 2, 3) \tag{3.1}$$

consider the optimal stopping game where the sup-player chooses a stopping time τ to maximise, and the inf-player chooses a stopping time σ to minimise, the expected payoff

$$\mathsf{M}(\tau, \sigma \,|\, \mathcal{F}_t) = \mathsf{E}\big[G_\tau^1 I(\tau < \sigma) + G_\sigma^2 I(\sigma < \tau) + G_\tau^3 I(\tau = \sigma) \,|\, \mathcal{F}_t\big] \tag{3.2}$$

for each t given and fixed. Note that if $\mathcal{F}_0$ is trivial (in the sense that $\mathsf{P}(F)$ equals either 0 or 1 for all $F \in \mathcal{F}_0$) then $\mathsf{M}(\tau, \sigma \,|\, \mathcal{F}_0)$ equals $\mathsf{E}\big[G_\tau^1 I(\tau < \sigma) + G_\sigma^2 I(\sigma < \tau) + G_\tau^3 I(\tau = \sigma)\big]$ and this expression is then denoted by $\mathsf{M}(\tau, \sigma)$ for all τ and σ.

Define the upper value and the lower value of the game by

$$V_t^* = \operatorname*{ess\,inf}_{\sigma \ge t} \operatorname*{ess\,sup}_{\tau \ge t} \mathsf{M}(\tau, \sigma \,|\, \mathcal{F}_t) \quad \& \quad V_*^t = \operatorname*{ess\,sup}_{\tau \ge t} \operatorname*{ess\,inf}_{\sigma \ge t} \mathsf{M}(\tau, \sigma \,|\, \mathcal{F}_t) \tag{3.3}$$

where the horizon T (the upper bound for τ and σ above) may be either finite or infinite. If $T < \infty$ then it is assumed that $G_T^1 = G_T^2 = G_T^3$. If $T = \infty$ then it is assumed that $\liminf_{t\to\infty} G_t^2 \le \limsup_{t\to\infty} G_t^1$, and the common value for G_∞^3 could formally be assigned as either of the preceding two values (if τ and σ are allowed to take the value ∞).

Theorem 3.2

Consider the optimal stopping game (3.3)*. If G_i is right-continuous for $i = 1, 2, 3$ then the Stackelberg equilibrium holds in the sense that*

$$V_t^* = V_*^t \quad \mathsf{P}\text{-a.s.} \tag{3.4}$$

with $V_t := V_t^* = V_*^t$ defining a right-continuous process (modification) for $t \ge 0$. Moreover, the stopping times

$$\tau_\varepsilon = \inf\{\, t : V_t \le G_t^1 + \varepsilon \,\} \quad \& \quad \sigma_\varepsilon = \inf\{\, t : V_t \ge G_t^2 - \varepsilon \,\} \tag{3.5}$$

satisfy the following inequalities:

$$\mathsf{M}(\tau, \sigma_\varepsilon \,|\, \mathcal{F}_t) - \varepsilon \le \mathsf{M}(\tau_\varepsilon, \sigma_\varepsilon \,|\, \mathcal{F}_t) \le \mathsf{M}(\tau_\varepsilon, \sigma \,|\, \mathcal{F}_t) + \varepsilon \tag{3.6}$$

for each t and every $\varepsilon > 0$. If G_i is right-continuous and left-continuous over stopping times for $i = 1, 2, 3$ then the Nash equilibrium holds in the sense that the stopping times

$$\tau_* = \inf\{\, t : V_t = G_t^1 \,\} \quad \& \quad \sigma_* = \inf\{\, t : V_t = G_t^2 \,\} \tag{3.7}$$

satisfy the following inequalities:

$$\mathsf{M}(\tau, \sigma_* \,|\, \mathcal{F}_t) \le \mathsf{M}(\tau_*, \sigma_* \,|\, \mathcal{F}_t) \le \mathsf{M}(\tau_*, \sigma \,|\, \mathcal{F}_t) \tag{3.8}$$

for each t and all stopping times τ and σ.

Proof. The first part of the theorem (Stackelberg equilibrium) was established in [18] (under slightly more restrictive conditions on integrability and the common value at the end of time but the same method extends to cover the present case without major changes). The second part of the theorem (Nash equilibrium) can be derived using the same arguments as in the second part of the proof of Theorem 2.1 above. □

Note that the second part of Theorem 3.2 (Nash equilibrium) is applicable to all Lévy processes (without additional hypotheses on the jump structure).

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