

# Embedding Laws in Diffusions by Functions of Time

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We present a constructive probabilistic proof of the fact that if  $B = (B_t)_{t \geq 0}$  is standard Brownian motion started at 0 and  $\mu$  is a given probability measure on  $\mathbb{R}$  such that  $\mu(\{0\}) = 0$  then there exists a unique left-continuous increasing function  $b : (0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\}$  and a unique left-continuous decreasing function  $c : (0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$  such that  $B$  stopped at  $\tau_{b,c} = \inf \{t > 0 \mid B_t \geq b(t) \text{ or } B_t \leq c(t)\}$  has the law  $\mu$ . The method of proof relies upon weak convergence arguments arising from Helly's selection theorem and makes use of the Lévy metric which appears to be novel in the context of embedding theorems. We show that  $\tau_{b,c}$  is minimal in the sense of Monroe so that the stopped process  $B^{\tau_{b,c}} = (B_{t \wedge \tau_{b,c}})_{t \geq 0}$  satisfies natural uniform integrability conditions expressed in terms of  $\mu$ . We also show that  $\tau_{b,c}$  has the smallest truncated expectation among all stopping times that embed  $\mu$  into  $B$ . The main results extend from standard Brownian motion to all recurrent diffusion processes on the real line.

## 1. Introduction

A classic problem in modern probability theory is to find a stopping time  $\tau$  of a standard Brownian motion  $B$  started at zero such that  $B$  stopped at  $\tau$  has a given law  $\mu$ . The existence of a randomised stopping time  $\tau$  for centred laws  $\mu$  was first derived by Skorokhod [22] and the problem is often referred to as the *Skorokhod embedding problem*. A few years later Dubins [8] proved the existence of a non-randomised stopping time  $\tau$  of  $B$  that also holds for more general laws  $\mu$ . Many other solutions have been found in subsequent years and we refer to the survey article by Obłój [17] for a comprehensive discussion (see also [9] for financial applications and [11] for connections to the Cantelli conjecture).

Solutions relevant to the present paper are those found by Root [19] in the setting of  $B$  and Rost [20] in the setting of more general Markov processes and initial laws. Root [19] showed that  $\tau$  can be realised as the first entry time to a *barrier* and Rost [20] showed that  $\tau$  can be characterised in terms of a *filling scheme* dating back to Chacon & Ornstein [4] within ergodic theory (see also [7] for a closely related construction). Subsequently Chacon [3] showed that a stopping time arising from the filling scheme coincides with the first entry time to a *reversed barrier* under some conditions. The proof of Root [19] relies upon a continuous mapping theorem and compactness of barriers in a uniform distance topology. The methods of Rost [20] and Chacon [3] rely on potential theory of general Markov processes. Uniqueness of barriers was studied by Loynes [12]. He described barriers by functions of space. Reversed

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*Mathematics Subject Classification 2010.* Primary 60G40, 60J65. Secondary 60F05, 60J60.

*Key words and phrases:* Skorokhod embedding, Brownian motion, diffusion process, Markov process, Helly's selection theorem, weak convergence, Lévy metric, reversed barrier, minimal stopping time.

barriers can also be described by functions of time. Based on this fact McConnell [13] developed an analytic free-boundary approach relying upon potential theoretic considerations of Rost [20] and Chacon [3]. He proved the existence of functions of time (representing a reversed barrier) when  $\mu$  has a continuous distribution function which is flat around zero. He also showed that these functions are unique under a Tychonov boundedness condition.

In this paper we develop an entirely different approach to the embedding problem and prove the existence and uniqueness of functions of time for general target laws  $\mu$  with no extra conditions imposed. The derivation of  $\tau$  is constructive and the construction itself is purely probabilistic and intuitive. The method of proof relies upon weak convergence arguments for functions of time arising from Helly's selection theorem and makes use of the Lévy metric which appears to be novel in the context of embedding theorems. This enables us to avoid time-reversal arguments (present in previous approaches) and relate the existence arguments directly to the regularity of the sample path with respect to functions of time. The fact that the construction applies to all target laws  $\mu$  with no integrability/regularity assumptions makes the resulting embedding rather canonical and remarkable in the class of known embeddings. Moreover, we show that the resulting stopping time  $\tau$  is minimal in the sense of Monroe [14] so that the stopped process  $B^\tau = (B_{t \wedge \tau})_{t \geq 0}$  satisfies natural uniform integrability conditions which fail to hold for trivial embeddings of any law (see e.g. [18, Exc. 5.7, p. 276]). We also show that the resulting stopping time  $\tau$  has the smallest truncated expectation among all stopping times that embed  $\mu$  into  $B$ . The same result was derived by Chacon [3] for stopping times arising from the filling scheme when their means are finite. A converse result for stopping times arising from barriers was first derived by Rost [21]. The main results extend from standard Brownian motion to all recurrent diffusion processes on the real line. Extending these results to more general Markov processes satisfying specified conditions leads to a research agenda which we leave open for future developments.

When the process is standard Brownian motion then it is possible to check that the sufficient conditions derived by Chacon [3, p. 47] are satisfied so that the filling scheme stopping time used by Rost [20] coincides with the first entry time to a reversed barrier. If  $\mu$  has a continuous distribution function which is flat around zero then the uniqueness result of McConnell [13, pp. 684-690] implies that this reversed barrier is uniquely determined under a Tychonov boundedness condition. When any of these conditions fails however then it becomes unclear whether a reversed barrier is uniquely determined by the filling scheme because in principle there could be many reversed barriers yielding the same law. One consequence of the present paper is that the latter ambiguity gets removed since we show that the filling scheme does indeed determine a reversed barrier uniquely for general target laws  $\mu$  with no extra conditions imposed. Despite this contribution to the theory of filling schemes (see [3] and the references therein) it needs to be noted that the novel methodology of the present paper avoids the filling scheme completely and focuses on constructing the reversed barrier by functions of time directly.

## 2. Existence

In this section we state and prove the main existence result (see also Corollary 8 below).

**Theorem 1 (Existence).** *Let  $B = (B_t)_{t \geq 0}$  be a standard Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $B_0 = 0$ , and let  $\mu$  be a probability measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that  $\mu(\{0\}) = 0$ .*

(I) If  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  then there exists a left-continuous increasing function  $b : (0, \infty) \rightarrow \mathbb{R}$  such that  $B_{\tau_b} \sim \mu$  where  $\tau_b = \inf \{ t > 0 \mid B_t \geq b(t) \}$ .

(II) If  $\text{supp}(\mu) \subseteq \mathbb{R}_-$  then there exists a left-continuous decreasing function  $c : (0, \infty) \rightarrow \mathbb{R}$  such that  $B_{\tau_c} \sim \mu$  where  $\tau_c = \inf \{ t > 0 \mid B_t \leq c(t) \}$ .

(III) If  $\text{supp}(\mu) \cap \mathbb{R}_+ \neq \emptyset$  and  $\text{supp}(\mu) \cap \mathbb{R}_- \neq \emptyset$  then there exist a left-continuous increasing function  $b : (0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\}$  and a left-continuous decreasing function  $c : (0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$  such that  $B_{\tau_{b,c}} \sim \mu$  where  $\tau_{b,c} = \inf \{ t > 0 \mid B_t \geq b(t) \text{ or } B_t \leq c(t) \}$  (see Figure 1 below).

**Proof.** We will first derive (I)+(II) since (III) will then follow by combining and further extending the construction and arguments of (I)+(II). This will enable us to focus more clearly on the subtle technical issues in relation to (a) the competing character of the two boundaries in (III) and (b) the fact that one of them can jump to infinity. Neither of these technical issues is present in (I)+(II) so that the key building block of the construction is best understood by considering this case first.

**(I)+(II) One-sided support:** Clearly it is enough to prove (I) since (II) then follows by symmetry. Let us therefore assume that  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  throughout.

1. *Bounded support.* Assume first that  $\text{supp}(\mu) \subseteq [0, \beta]$  for some  $\beta < \infty$ . Without loss of generality we can assume that  $\beta$  belongs to  $\text{supp}(\mu)$ . Let  $0 = x_0^n < x_1^n < \dots < x_{m_n}^n = \beta$  be a partition of  $[0, \beta]$  such that  $\max_{1 \leq k \leq m_n} (x_k^n - x_{k-1}^n) \rightarrow 0$  as  $n \rightarrow \infty$ . (For example, we could take a dyadic partition defined by  $x_k^n = \frac{k}{2^n} \beta$  for  $k = 0, 1, \dots, 2^n$  but other choices are also possible and will lead to the same result.) Let  $X$  be a random variable (defined on some probability space) having the law equal to  $\mu$  and set

$$(2.1) \quad X_n = \sum_{k=1}^{m_n} x_k^n I(x_{k-1}^n < X \leq x_k^n)$$

for  $n \geq 1$ . Then  $X_n \rightarrow X$  almost surely and hence  $X_n \rightarrow X$  in law as  $n \rightarrow \infty$ . Denoting the law of  $X_n$  by  $\mu_n$  this means that  $\mu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ . We will now construct a left-continuous increasing function  $b_n : (0, \infty) \rightarrow \mathbb{R}$  taking values in  $\{x_1^n, x_2^n, \dots, x_{m_n}^n\}$  such that  $\tau_{b_n} = \inf \{ t > 0 \mid B_t \geq b_n(t) \}$  satisfies  $B_{\tau_{b_n}} \sim \mu_n$  for  $n \geq 1$ .

1.1. *Construction: Discrete case.* For this, set  $p_k^n = \mathbb{P}(x_{k-1}^n < X \leq x_k^n)$  for  $k = 1, 2, \dots, m_n$  with  $n \geq 1$  given and fixed, and let  $k_1$  denote the smallest  $k$  in  $\{1, 2, \dots, m_n\}$  such that  $p_{k_1}^n > 0$ . Consider the sequential movement of two sample paths  $t \mapsto B_t$  and  $t \mapsto x_{k_1}^n$  as  $t$  goes from 0 onwards. From the recurrence of  $B$  it is clear that there exists a unique  $t_1^n > 0$  such that the probability of  $B$  hitting  $x_{k_1}^n$  before  $t_1^n$  equals  $p_{k_1}^n$ . Stop the movement of  $t \mapsto x_{k_1}^n$  at  $t_1^n$  and replace it with  $t \mapsto x_{k_2}^n$  afterwards where  $k_2$  is the smallest  $k$  in  $\{k_1+1, k_1+2, \dots, m_n\}$  such that  $p_k^n > 0$ . Set  $b_n(t) = x_{k_1}^n$  for  $t \in (0, t_1^n]$  and on the event that  $B$  did not hit  $b_n$  on  $(0, t_1^n]$  consider the movement of  $t \mapsto B_t$  and  $t \mapsto x_{k_2}^n$  as  $t$  goes from  $t_1^n$  onwards. From the recurrence of  $B$  it is clear that there exists a unique  $t_2^n > t_1^n$  such that the probability of  $B$  hitting  $x_{k_2}^n$  before  $t_2^n$  equals  $p_{k_2}^n$ . Proceed as before and set  $b_n(t) = x_{k_2}^n$  for  $t \in (t_1^n, t_2^n]$ . Continuing this construction by induction until  $t_i^n = \infty$  for some  $i \leq m_n$  (which clearly has to happen) we obtain  $b_n$  as stated above. Note that  $b_n(t) = x_{k_1}^n$

for  $t \in (0, t_1^n]$  with  $x_{k_1}^n \rightarrow \alpha =: \min \text{supp}(\mu)$  as  $n \rightarrow \infty$  and  $b_n(t) = x_{m_n}^n$  for  $t \in (t_{i-1}^n, \infty)$  since  $x_{m_n}^n = \beta = \max \text{supp}(\mu)$  by assumption.

1.2. *Construction: Passage to limit.* In this way we have obtained a sequence of left-continuous increasing functions  $b_n : (0, \infty) \rightarrow [\alpha, \beta]$  satisfying  $b_n(0+) \rightarrow \alpha$  as  $n \rightarrow \infty$  and  $b_n(+\infty) = \beta$  for  $n \geq 1$ . We can formally extend each  $b_n$  to  $(-\infty, 0]$  by setting  $b_n(t) = b_n(0+)$  for  $t \in (-1, 0]$  and  $b_n(t) = 0$  for  $t \in (-\infty, -1]$  (other definitions are also possible). Then  $\{b_n \mid n \geq 1\}$  is a sequence of left-continuous increasing functions from  $\mathbb{R}$  into  $\mathbb{R}$  such that  $b_n(-\infty) = 0$  and  $b_n(+\infty) = \beta$  for all  $n \geq 1$ . By *Helly's selection theorem* (see e.g. [1, pp. 336-337]) we therefore know that there exists a subsequence  $\{b_{n_k} \mid k \geq 1\}$  and a left-continuous increasing function  $b : \mathbb{R} \rightarrow \mathbb{R}$  such that  $b_{n_k} \rightarrow b$  weakly as  $k \rightarrow \infty$  in the sense that  $b_{n_k}(t) \rightarrow b(t)$  as  $k \rightarrow \infty$  for every  $t \in \mathbb{R}$  at which  $b$  is continuous. (Note that since  $b_n(t) = b_n(0+) \rightarrow \alpha$  as  $n \rightarrow \infty$  for every  $t \in (-1, 0]$  it follows that  $b(0) = \alpha$  by the increase and left-continuity of  $b$ .) Restricting  $b$  to  $(0, \infty)$  and considering the stopping time

$$(2.2) \quad \tau_b = \inf \{ t > 0 \mid B_t \geq b(t) \}$$

we claim that  $B_{\tau_b} \sim \mu$ . This can be seen as follows.

1.3. *Tightness.* We claim that the sequence of generalised distribution functions  $\{b_n \mid n \geq 1\}$  is *tight* (in the sense the mass of the Lebesgue-Stieltjes measure associated with  $b_n$  cannot escape to infinity as  $n \rightarrow \infty$ ). Indeed, if  $\varepsilon > 0$  is given and fixed, then  $\delta_\varepsilon := \mu((\beta - \varepsilon, \beta]) > 0$  since  $\beta$  belongs to  $\text{supp}(\mu)$ . Setting  $\tau_\beta = \inf \{ t > 0 \mid B_t \geq \beta \}$  we see that there exists  $t_\varepsilon > 0$  large enough such that  $\mathbf{P}(\tau_\beta \leq t_\varepsilon) > 1 - \delta_\varepsilon$ . Since  $b_n \leq \beta$  and hence  $\tau_{b_n} \leq \tau_\beta$  this implies that  $\mathbf{P}(\tau_{b_n} \leq t_\varepsilon) > 1 - \delta_\varepsilon$  for all  $n \geq 1$ . From the construction of  $b_n$  the latter inequality implies that  $b_n(t_\varepsilon) > \beta - \varepsilon$  for all  $n \geq 1$ . Recalling the extension of  $b_n$  to  $(-\infty, 0]$  specified above where  $b_n(-1) = 0$  it therefore follows that

$$(2.3) \quad b_n(t_\varepsilon) - b_n(-1) > \beta - \varepsilon$$

for all  $n \geq 1$ . This shows that  $\{b_n \mid n \geq 1\}$  is tight as claimed. From (2.3) we see that  $b(+\infty) = \beta$  and  $b(-\infty) = 0$  so that the Lebesgue-Stieltjes measure associated with  $b$  on  $\mathbb{R}$  has a full mass equal to  $\beta$  like all other  $b_n$  for  $n \geq 1$ . Recalling that  $b(0+) = \alpha$  we see that the Lebesgue-Stieltjes measure associated with  $b$  on  $(0, \infty)$  has a full mass equal to  $\beta - \alpha$ . For our purposes we only need to consider the restriction of  $b$  to  $(0, \infty)$ .

1.4. *Lévy metric and convergence.* If  $b$  and  $c$  are left-continuous increasing functions from  $\mathbb{R}$  into  $\mathbb{R}$  such that  $b(-\infty) = c(-\infty) = 0$  and  $b(+\infty) = c(+\infty) = \beta$ , then the *Lévy metric* is defined by

$$(2.4) \quad d(b, c) = \inf \{ \varepsilon > 0 \mid b(t - \varepsilon) - \varepsilon \leq c(t) \leq b(t + \varepsilon) + \varepsilon \text{ for all } t \in \mathbb{R} \}.$$

It is well known (see e.g. [1, Exc. 14.5]) that  $c_n \rightarrow b$  weakly if and only if  $d(b, c_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Defining functions

$$(2.5) \quad b_\varepsilon(t) := b(t - \varepsilon) - \varepsilon \quad \& \quad b^\varepsilon(t) := b(t + \varepsilon) + \varepsilon$$

for  $t \in \mathbb{R}$  we claim that

$$(2.6) \quad \tau_{b_\varepsilon} \uparrow \tau_b \quad \mathbf{P}\text{-a.s.}$$

$$(2.7) \quad \tau_{b^\varepsilon} \downarrow \tau_b \quad \text{P-a.s.}$$

as  $\varepsilon \downarrow 0$  where in (2.6) we also assume that  $b(0+) > 0$ .

*Proof of (2.6).* Note first that  $b_{\varepsilon'} \leq b_{\varepsilon''} \leq b$  so that  $\tau_{b_{\varepsilon'}} \leq \tau_{b_{\varepsilon''}} \leq \tau_b$  for  $\varepsilon' \geq \varepsilon'' > 0$ . It follows therefore that  $\tau_{b_-} := \lim_{\varepsilon \downarrow 0} \tau_{b_\varepsilon} \leq \tau_b$ . Moreover by definition of  $\tau_{b_\varepsilon}$  we can find a sequence  $\delta_n \downarrow 0$  as  $n \rightarrow \infty$  such that  $B_{\tau_{b_\varepsilon} + \delta_n} \geq b_\varepsilon(\tau_{b_\varepsilon} + \delta_n) = b(\tau_{b_\varepsilon} - \varepsilon + \delta_n) - \varepsilon$  for all  $n \geq 1$  with  $\varepsilon > 0$ . Letting  $n \rightarrow \infty$  it follows that  $B_{\tau_{b_\varepsilon}} \geq b((\tau_{b_\varepsilon} - \varepsilon)_+) - \varepsilon \geq b(\tau_{b_\varepsilon} - \varepsilon) - \varepsilon \geq b(\tau_{b_\varepsilon} - \varepsilon_0) - \varepsilon$  for all  $\varepsilon \in (0, \varepsilon_0)$  with  $\varepsilon_0 > 0$  given and fixed. Since  $b$  is left-continuous and increasing it follows that  $b$  is lower semicontinuous and hence by letting  $\varepsilon \downarrow 0$  in the previous identity we find that  $B_{\tau_{b_-}} \geq \liminf_{\varepsilon \downarrow 0} (b(\tau_{b_\varepsilon} - \varepsilon_0) - \varepsilon) \geq b(\liminf_{\varepsilon \downarrow 0} \tau_{b_\varepsilon} - \varepsilon_0) = b(\tau_{b_-} - \varepsilon_0)$  for all  $\varepsilon_0 > 0$ . Letting  $\varepsilon_0 \downarrow 0$  and using that  $b$  is left-continuous we get  $B_{\tau_{b_-}} \geq b(\tau_{b_-})$ . This implies that  $\tau_b \leq \tau_{b_-}$  and hence  $\tau_{b_-} = \tau_b$  as claimed in (2.6) above.

*Proof of (2.7).* Note first that  $b \leq b^{\varepsilon'} \leq b^{\varepsilon''}$  so that  $\tau_b \leq \tau_{b^{\varepsilon'}} \leq \tau_{b^{\varepsilon''}}$  for  $\varepsilon'' \geq \varepsilon' > 0$ . It follows therefore that  $\tau_b \leq \tau_{b^+} := \lim_{\varepsilon \downarrow 0} \tau_{b^\varepsilon}$ . Moreover setting

$$(2.8) \quad \sigma_b = \inf \{ t > 0 \mid B_t > b(t) \}$$

we claim that

$$(2.9) \quad \tau_b = \sigma_b \quad \text{P-a.s.}$$

so that outside a P-null set we have  $B_{t_n} > b(t_n)$  for some  $t_n \downarrow \tau_b$  with  $t_n > \tau_b$ . Since  $b$  is increasing each  $t_n$  can be chosen as a continuity point of  $b$ , and therefore there exists  $\varepsilon_n > 0$  small enough such that  $B_{t_n} > b^{\varepsilon_n}(t_n) = b(t_n + \varepsilon_n) + \varepsilon_n > b(t_n)$  for all  $n \geq 1$ . This shows that  $\tau_{b^+} \leq t_n$  outside the P-null set for all  $n \geq 1$ . Letting  $n \rightarrow \infty$  we get  $\tau_{b^+} \leq \tau_b$  P-a.s. and hence  $\tau_{b^+} = \tau_b$  P-a.s. as claimed in (2.7) above.

*Proof of (2.9).* Let us first introduce

$$(2.10) \quad \tau_{b+\varepsilon} = \inf \{ t > 0 \mid B_t \geq b(t) + \varepsilon \}$$

and note that  $\tau_{b^+} := \lim_{\varepsilon \downarrow 0} \tau_{b+\varepsilon} = \sigma_b$  as is easily seen from definitions (2.8) and (2.10). Next introduce the truncated versions of (2.2) and (2.10) by setting

$$(2.11) \quad \tau_b^\delta = \inf \{ t > \delta \mid B_t \geq b(t) \}$$

$$(2.12) \quad \tau_{b+\varepsilon}^\delta = \inf \{ t > \delta \mid B_t \geq b(t) + \varepsilon \}$$

with  $\delta > 0$  given and fixed. Note that  $\tau_b^\delta \leq \tau_{b+\varepsilon}^\delta \leq \tau_{b+\varepsilon''}^\delta$  for  $\varepsilon'' \geq \varepsilon' > 0$ . It follows therefore that  $\tau_b^\delta \leq \tau_{b^+}^\delta := \lim_{\varepsilon \downarrow 0} \tau_{b+\varepsilon}^\delta$ . To prove that

$$(2.13) \quad \tau_b^\delta = \tau_{b^+}^\delta \quad \text{P-a.s.}$$

it is enough to establish that

$$(2.14) \quad \mathbb{P}(\tau_{b^+}^\delta > t) \leq \mathbb{P}(\tau_b^\delta > t)$$

for all  $t > 0$ . Indeed, in this case we have  $\mathbf{E}(\tau_{b+}^\delta \wedge N) = \int_0^N \mathbf{P}(\tau_{b+}^\delta > t) dt \leq \int_0^N \mathbf{P}(\tau_b^\delta > t) dt = \mathbf{E}(\tau_b^\delta \wedge N)$  so that  $\tau_{b+}^\delta \wedge N = \tau_b^\delta \wedge N$   $\mathbf{P}$ -a.s. for all  $N \geq 1$ . Letting  $N \rightarrow \infty$  we obtain (2.13) as claimed. Assuming that (2.13) is established note that

$$(2.15) \quad \sigma_b = \tau_{b+} = \lim_{\varepsilon \downarrow 0} \tau_{b+\varepsilon} = \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \tau_{b+\varepsilon}^\delta = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \tau_{b+\varepsilon}^\delta = \lim_{\delta \downarrow 0} \tau_{b+}^\delta = \lim_{\delta \downarrow 0} \tau_b^\delta = \tau_b \quad \mathbf{P}\text{-a.s.}$$

where we use that  $\varepsilon \mapsto \tau_{b+\varepsilon}^\delta$  and  $\delta \mapsto \tau_{b+\varepsilon}^\delta$  are decreasing as  $\varepsilon \downarrow 0$  and  $\delta \downarrow 0$  so that the two limits commute. Hence we see that the proof of (2.9) is reduced to establishing (2.14).

*Proof of (2.14).* Note by Girsanov's theorem that

$$(2.16) \quad \begin{aligned} \mathbf{P}(\tau_{b+}^\delta > t) &= \mathbf{P}\left(\lim_{\varepsilon \downarrow 0} \tau_{b+\varepsilon}^\delta > t\right) \leq \lim_{\varepsilon \downarrow 0} \mathbf{P}(\tau_{b+\varepsilon}^\delta > t) \\ &= \lim_{\varepsilon \downarrow 0} \mathbf{P}(B_s < b(s) + \varepsilon \text{ for all } s \in (\delta, t]) \\ &= \lim_{\varepsilon \downarrow 0} \mathbf{P}\left(B_s - \int_0^s \frac{\varepsilon}{\delta} I(0 \leq r \leq \delta) dr < b(s) \text{ for all } s \in (\delta, t]\right) \\ &= \lim_{\varepsilon \downarrow 0} \mathbf{E}\left[\frac{\mathcal{E}_T^{H^\varepsilon}}{\mathcal{E}_T^{H^\varepsilon}} I\left(B_s - \int_0^s H_r^\varepsilon dr < b(s) \text{ for all } s \in (\delta, t]\right)\right] \\ &= \lim_{\varepsilon \downarrow 0} \tilde{\mathbf{E}}\left[\frac{1}{\mathcal{E}_T^{H^\varepsilon}} I\left(\tilde{B}_s < b(s) \text{ for all } s \in (\delta, t]\right)\right] \end{aligned}$$

where  $H_r^\varepsilon = \frac{\varepsilon}{\delta} I(0 \leq r \leq \delta)$  and  $\mathcal{E}_T^{H^\varepsilon} = \exp(\int_0^T H_r^\varepsilon dB_r - \frac{1}{2} \int_0^T (H_r^\varepsilon)^2 dr)$  so that  $d\tilde{\mathbf{P}} = \mathcal{E}_T^{H^\varepsilon} d\mathbf{P}$  and  $1/\mathcal{E}_T^{H^\varepsilon} = \exp(-\int_0^T H_r^\varepsilon dB_r + \frac{1}{2} \int_0^T (H_r^\varepsilon)^2 dr) = \exp(-\int_0^T H_r^\varepsilon d\tilde{B}_r - \frac{1}{2} \int_0^T (H_r^\varepsilon)^2 dr) = \exp(-\frac{\varepsilon}{\delta} \tilde{B}_\delta - \frac{1}{2} \frac{\varepsilon^2}{\delta})$  with  $\tilde{B}_s = B_s - \int_0^s H_r^\varepsilon dr$  being a standard Brownian motion under  $\tilde{\mathbf{P}}$  for  $s \in [0, T]$ . From (2.16) it therefore follows that

$$(2.17) \quad \begin{aligned} \mathbf{P}(\tau_{b+}^\delta > t) &\leq \lim_{\varepsilon \downarrow 0} \mathbf{E}\left[\exp\left(-\frac{\varepsilon}{\delta} B_\delta - \frac{1}{2} \frac{\varepsilon^2}{\delta}\right) I\left(B_s < b(s) \text{ for all } s \in (\delta, t]\right)\right] \\ &= \mathbf{P}(B_s < b(s) \text{ for all } s \in (\delta, t]) = \mathbf{P}(\tau_b^\delta > t) \end{aligned}$$

using the dominated convergence theorem since  $\mathbf{E}e^{c|B_\delta|} < \infty$  for  $c > 0$ . This completes the verification of (2.14) and thus (2.7) holds as well. (For a different proof of (2.14) in a more general setting see the proof of Corollary 8 below.)

1.5. *Verification.* To prove that  $\tau_b$  from (2.2) satisfies  $B_{\tau_b} \sim \mu$  consider first the case when  $b(0+) > 0$ . Recall that  $b_{n_k} \rightarrow b$  weakly and therefore  $d(b, b_{n_k}) \rightarrow 0$  as  $k \rightarrow \infty$  where  $d$  is the Lévy metric defined in (2.4). To simplify the notation in the sequel let us set  $b_k := b_{n_k}$  for  $k \geq 1$ . This yields the existence of  $\varepsilon_k \downarrow 0$  as  $k \rightarrow \infty$  such that  $b_{\varepsilon_k}(t) \leq b_k(t) \leq b^{\varepsilon_k}(t)$  for all  $t > 0$  and  $k \geq 1$  (recall that  $b_{\varepsilon_k}$  and  $b^{\varepsilon_k}$  are defined by (2.5) above). It follows therefore that  $\tau_{b_{\varepsilon_k}} \leq \tau_{b_k} \leq \tau_{b^{\varepsilon_k}}$  for all  $k \geq 1$ . Letting  $k \rightarrow \infty$  and using (2.6) and (2.7) above we obtain  $\tau_b = \lim_{k \rightarrow \infty} \tau_{b_{\varepsilon_k}} \leq \liminf_{k \rightarrow \infty} \tau_{b_k} \leq \limsup_{k \rightarrow \infty} \tau_{b_k} \leq \lim_{k \rightarrow \infty} \tau_{b^{\varepsilon_k}} = \tau_b$   $\mathbf{P}$ -a.s. This shows that  $\tau_b = \lim_{k \rightarrow \infty} \tau_{b_k}$   $\mathbf{P}$ -a.s. and hence  $B_{\tau_b} = \lim_{k \rightarrow \infty} B_{\tau_{b_k}}$   $\mathbf{P}$ -a.s. Recalling that  $B_{\tau_{b_k}} \sim \mu_k$  for  $k \geq 1$  and that  $\mu_k \rightarrow \mu$  weakly as  $k \rightarrow \infty$  we see that  $B_{\tau_b} \sim \mu$  as claimed.

Consider next the case when  $b(0+) = 0$ . With  $\delta > 0$  given and fixed set  $b^\delta := b \vee \delta$  and  $b_n^\delta := b_n \vee \delta$  for  $n \geq 1$ . Since  $b_k \rightarrow b$  weakly we see that  $b_k^\delta \rightarrow b^\delta$  weakly and hence by the



first part of the proof above (since  $b^\delta(0+) = \delta > 0$ ) we know that  $\tau_{b_k^\delta} \rightarrow \tau_{b^\delta}$   $\mathbb{P}$ -a.s. so that  $B_{\tau_{b_k^\delta}} \rightarrow B_{\tau_{b^\delta}}$   $\mathbb{P}$ -a.s. as  $k \rightarrow \infty$ . Moreover, since  $\tau_{b_k^\delta} \rightarrow \tau_{b_k}$  and  $\tau_{b^\delta} \rightarrow \tau_b$  as  $\delta \downarrow 0$  we see that

$$(2.18) \quad B_{\tau_{b_k^\delta}} \rightarrow B_{\tau_{b_k}} \quad \& \quad B_{\tau_{b^\delta}} \rightarrow B_{\tau_b}$$

as  $\delta \downarrow 0$ . From the fact that the first convergence in  $\mathbb{P}$ -probability is uniform over all  $k \geq 1$  in the sense that we have

$$(2.19) \quad \sup_{k \geq 1} \mathbb{P}(B_{\tau_{b_k^\delta}} \neq B_{\tau_{b_k}}) \leq \sup_{k \geq 1} \mu_k((0, \delta]) \leq \mu((0, \delta]) \rightarrow 0$$

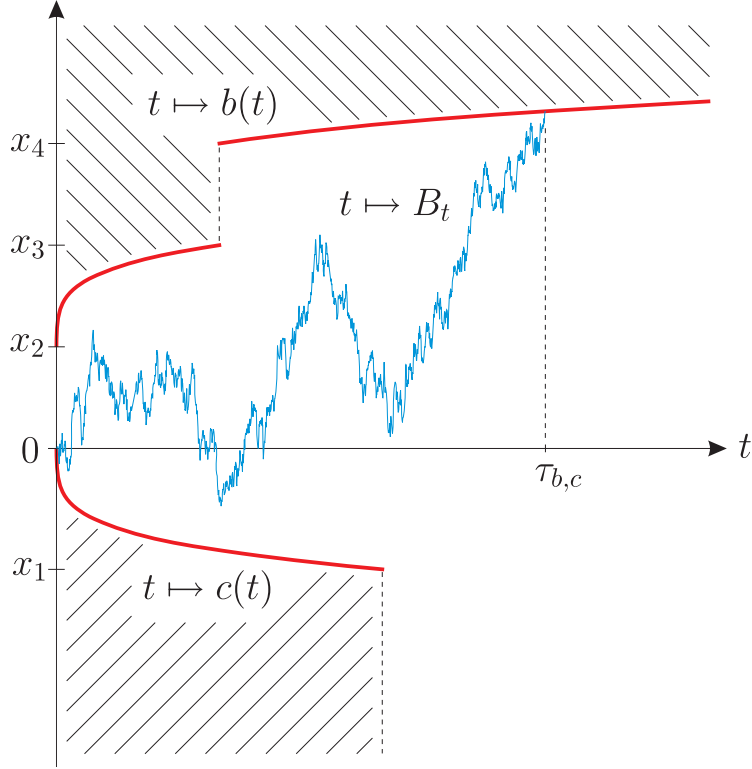
as  $\delta \downarrow 0$ , it follows that the limits in  $\mathbb{P}$ -probability commute so that

$$(2.20) \quad B_{\tau_b} = \lim_{\delta \downarrow 0} B_{\tau_b^\delta} = \lim_{\delta \downarrow 0} \lim_{k \rightarrow \infty} B_{\tau_{b_k}^\delta} = \lim_{k \rightarrow \infty} \lim_{\delta \downarrow 0} B_{\tau_{b_k}^\delta} = \lim_{k \rightarrow \infty} B_{\tau_{b_k}}.$$

Recalling again that  $B_{\tau_{b_k}} \sim \mu_k$  for  $k \geq 1$  and that  $\mu_k \rightarrow \mu$  weakly as  $k \rightarrow \infty$  we see that  $B_{\tau_b} \sim \mu$  in this case as well. Note also that the same arguments show (by dropping the symbol  $B$  from the left-hand side of (2.19) above) that  $\tau_b = \lim_{k \rightarrow \infty} \tau_{b_k}$  in  $\mathbb{P}$ -probability. This will be used in the proof of (III) below.

2. *Unbounded support.* Consider now the case when  $\sup \text{supp}(\mu) = +\infty$ . Let  $X$  be a random variable (defined on some probability space) having the law equal to  $\mu$  and set  $X_N = X \wedge \beta_N$  for some  $\beta_N \uparrow \infty$  as  $N \rightarrow \infty$  such that  $\mu((\beta_N - \varepsilon, \beta_N]) > 0$  for all  $\varepsilon > 0$  and  $N \geq 1$ . Let  $N \geq 1$  be given and fixed. Denoting the law of  $X_N$  by  $\mu_N$  we see that  $\text{supp}(\mu_N) \subseteq [0, \beta_N]$  with  $\beta_N \in \text{supp}(\mu_N)$ . Hence by the previous part of the proof we know that there exists a left-continuous increasing function  $b_N : (0, \infty) \rightarrow \mathbb{R}$  such that  $B_{\tau_{b_N}} \sim \mu_N$ . Recall that this  $b_N$  is obtained as the weak limit of a subsequence of the sequence of simple functions constructed by partitioning  $(0, \beta_N)$ . Extending the same construction to partitioning  $[\beta_N, \beta_{N+1})$  while keeping the obtained subsequence of functions with values in  $(0, \beta_N)$  we again know by the previous part of the proof that there exists a left-continuous increasing function  $b_{N+1} : (0, \infty) \rightarrow \mathbb{R}$  such that  $B_{\tau_{b_{N+1}}} \sim \mu_{N+1}$ . This  $b_{N+1}$  is obtained as the weak limit of a further subsequence of the previous subsequence of simple functions. Setting  $t_N = \inf \{t > 0 \mid b_N(t) = \beta_N\}$  it is therefore clear that  $b_{N+1}(t) = b_N(t)$  for all  $t \in (0, t_N]$ . Continuing this process by induction and noticing that  $t_N \uparrow t_\infty$  as  $N \rightarrow \infty$  we obtain a function  $b : (0, t_\infty) \rightarrow \mathbb{R}$  such that  $b(t) = b_N(t)$  for all  $t \in (0, t_N]$  and  $N \geq 1$ . Clearly  $b$  is left-continuous and increasing since each  $b_N$  satisfies these properties. Moreover we claim that  $t_\infty$  must be equal to  $+\infty$ . For this, note that  $\mathbb{P}(B_{\tau_b} \leq x) = \mathbb{P}(B_{\tau_{b_N}} \leq x)$  for  $x < \beta_N$  and  $N \geq 1$ . Letting  $N \rightarrow \infty$  and using that  $B_{\tau_{b_N}} \sim \mu_N$  converges weakly to  $\mu$  since  $X_N \rightarrow X$  we see that  $\mathbb{P}(B_{\tau_b} \leq x) = \mathbb{P}(X \leq x)$  for all  $x > 0$  at which the distribution function of  $X$  is continuous. Letting  $x \uparrow \infty$  over such continuity points we get  $\mathbb{P}(B_{\tau_b} < \infty) = 1$ . Since clearly this is not possible if  $t_\infty$  is finite we see that  $t_\infty = +\infty$  as claimed. Noting that  $b_N = b \wedge \beta_N$  on  $(0, \infty)$  for  $N \geq 1$  it follows that  $\tau_{b_N} = \inf \{t > 0 \mid B_t \geq b_N(t)\} = \inf \{t > 0 \mid B_t \geq b(t) \wedge \beta_N\}$  from where we see that  $\tau_{b_N} \rightarrow \tau_b$  and thus  $B_{\tau_{b_N}} \rightarrow B_{\tau_b}$  as  $N \rightarrow \infty$ . Since  $X_N \rightarrow X$  and thus  $\mu_N \rightarrow \mu$  weakly as  $N \rightarrow \infty$  it follows that  $B_{\tau_b} \sim \mu$  as claimed. This completes the proof of (I).

**(III) Two-sided support:** This will be proved by combining and further extending the construction and arguments of (I) and (II). Novel aspects in this process include the competing character of the two boundaries and the fact that one of them can jump to infinite value.



**Figure 1.** An illustration of the reversed-barrier stopping time  $\tau_{b,c}$  from Theorem 1 that embeds  $\mu$  into  $B$  when  $\text{supp}(\mu) = [x_1, 0] \cup [x_2, x_3] \cup [x_4, \infty)$ .

3. *Bounded support.* As in the one-sided case assume first that  $\text{supp}(\mu) \subseteq [\gamma, \beta]$  for some  $\gamma < 0 < \beta$ . Without loss of generality we can assume that  $\beta$  and  $\gamma$  belong to  $\text{supp}(\mu)$ . Let  $0 = x_0^n < x_1^n < \dots < x_{m_n}^n = \beta$  be a partition of  $[0, \beta]$  such that  $\max_{1 \leq k \leq m_n} (x_k^n - x_{k-1}^n) \rightarrow 0$  as  $n \rightarrow \infty$ , and let  $0 = y_0^n > y_1^n > \dots > y_{l_n}^n = \gamma$  be a partition of  $[\gamma, 0]$  such that  $\max_{1 \leq j \leq l_n} (y_{j-1}^n - y_j^n) \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $X$  be a random variable (defined on some probability space) having the law equal to  $\mu$  and set

$$(2.21) \quad X_n^+ = \sum_{k=1}^{m_n} x_k^n I(x_{k-1}^n < X \leq x_k^n) \quad \& \quad X_n^- = \sum_{j=1}^{l_n} y_j^n I(y_j^n \leq X < y_{j-1}^n)$$

for  $n \geq 1$ . Then  $X_n^+ + X_n^- \rightarrow X$  almost surely and hence  $X_n^+ + X_n^- \rightarrow X$  in law as  $n \rightarrow \infty$ . Denoting the law of  $X_n^+ + X_n^-$  by  $\mu_n$  and recalling that  $X$  has the law  $\mu$  this means that  $\mu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ . We will now construct a left-continuous increasing function  $b_n : (0, \infty) \rightarrow \mathbb{R}$  taking values in  $\{x_1^n, x_2^n, \dots, x_{m_n}^n, +\infty\}$  and a left-continuous decreasing function  $c_n : (0, \infty) \rightarrow \mathbb{R}$  taking values in  $\{y_1^n, y_2^n, \dots, y_{l_n}^n, -\infty\}$  with  $b_n(t) < +\infty$  or  $c_n(t) > -\infty$  for all  $t \in (0, \infty)$  such that  $\tau_{b_n, c_n} = \inf \{t > 0 \mid B_t \geq b_n(t) \text{ or } B_t \leq c_n(t)\}$  satisfies  $B_{\tau_{b_n, c_n}} \sim \mu_n$  for  $n \geq 1$ .

3.1. *Construction: Discrete case.* For this, set  $p_k^n = \mathbb{P}(x_{k-1}^n < X \leq x_k^n)$  for  $k = 1, 2, \dots, m_n$  and  $q_j^n = \mathbb{P}(y_j^n \leq X < y_{j-1}^n)$  for  $j = 1, 2, \dots, l_n$  with  $n \geq 1$  given and fixed. Let  $k_1$  denote



the smallest  $k$  in  $\{1, 2, \dots, m_n\}$  such that  $p_k^n > 0$ , and let  $j_1$  denote the smallest  $j$  in  $\{1, 2, \dots, l_n\}$  such that  $q_j^n > 0$ . Consider the sequential movement of three sample paths  $t \mapsto B_t$ ,  $t \mapsto x_{k_1}^n$  and  $t \mapsto y_{j_1}^n$  as  $t$  goes from 0 onwards. From the recurrence of  $B$  it is clear that there exists a unique  $t_1^n > 0$  such that the probability of  $B$  hitting  $x_{k_1}^n$  before  $y_{j_1}^n$  on  $(0, t_1^n]$  equals  $p_{k_1}^n$ , or the probability of  $B$  hitting  $y_{j_1}^n$  before  $x_{k_1}^n$  on  $(0, t_1^n]$  equals  $q_{j_1}^n$ , whichever happens first (including simultaneous happening). In the first case stop the movement of  $t \mapsto x_{k_1}^n$  at  $t_1^n$  and replace it with  $t \mapsto x_{k_2}^n$  afterwards where  $k_2$  is the smallest  $k$  in  $\{k_1+1, k_1+2, \dots, m_n\}$  such that  $p_k^n > 0$  (if there is no such  $k$  then make no further replacement). In the second case stop the movement of  $t \mapsto y_{j_1}^n$  at  $t_1^n$  and replace it with  $t \mapsto y_{j_2}^n$  afterwards where  $j_2$  is the smallest  $j$  in  $\{j_1+1, j_1+2, \dots, l_n\}$  such that  $q_j^n > 0$  (if there is no such  $j$  then make no further replacement). In the third case, when the first and second case happen simultaneously, stop the movement of both  $t \mapsto x_{k_1}^n$  and  $t \mapsto y_{j_1}^n$  at  $t_1^n$  and replace them with  $t \mapsto x_{k_2}^n$  and  $t \mapsto y_{j_2}^n$  respectively (if there is no  $k_2$  or  $j_2$  then make no replacement respectively). In all three cases set  $b_n(t) = x_{k_1}^n$  and  $c_n(t) = y_{j_1}^n$  for  $t \in (0, t_1^n]$ . On the event that  $B$  did not hit  $b_n$  or  $c_n$  on  $(0, t_1^n]$ , in the first case consider the movement of  $t \mapsto B_t$ ,  $t \mapsto x_{k_2}^n$  and  $t \mapsto y_{j_1}^n$ , in the second case consider the movement of  $t \mapsto B_t$ ,  $t \mapsto x_{k_1}^n$  and  $t \mapsto y_{j_2}^n$ , and in the third case consider the movement of  $t \mapsto B_t$ ,  $t \mapsto x_{k_2}^n$  and  $t \mapsto y_{j_2}^n$  as  $t$  goes from  $t_1^n$  onwards. If there is no  $k_2$  or  $j_2$  we can formally set  $x_{k_2}^n = +\infty$  or  $y_{j_2}^n = -\infty$  respectively (note however that either  $k_2$  or  $j_2$  will always be finite). Continuing this construction by induction until  $t_i^n = \infty$  for some  $i \leq m_n \vee l_n$  (which clearly has to happen) we obtain  $b_n$  and  $c_n$  as stated above.

**3.2. Construction: Passage to limit.** For  $n \geq 1$  given and fixed note that  $b_n$  takes value  $\beta$  on some interval and  $c_n$  takes value  $\gamma$  on some interval since both  $\beta$  and  $\gamma$  belong to  $\text{supp}(\mu)$ . The main technical difficulty is that either  $b_n$  can take value  $+\infty$  or  $c_n$  can take value  $-\infty$  from some time  $t_\zeta$  onwards as well (in which case the corresponding interval is bounded). In effect this means that the corresponding function is not defined on  $(t_\zeta, \infty)$  with values in  $\mathbb{R}$ . To overcome this difficulty we will set  $\bar{b}_n(t) = \beta$  and  $\bar{c}_n(t) = \gamma$  for  $t > t_\zeta$ . Setting further  $\bar{b}_n = b_n$  and  $\bar{c}_n = c_n$  on  $(0, t_\zeta]$  we see that  $\bar{b}_n$  and  $\bar{c}_n$  are generalised distribution functions on  $(0, \infty)$ . Note that we always have either  $\bar{b}_n = b_n$  or  $\bar{c}_n = c_n$  (and often both). Note also that  $\bar{b}_n \neq b_n$  if and only if  $b_n$  takes value  $+\infty$  and  $\bar{c}_n \neq c_n$  if and only if  $c_n$  takes value  $-\infty$ . Note finally that  $\bar{b}_n(+\infty) = \beta$  and  $\bar{c}_n(+\infty) = \gamma$ . Applying the same arguments as in Part 1.2 above (upon extending  $\bar{b}_n$  and  $\bar{c}_n$  to  $\mathbb{R}$  first) we know that there exist subsequences  $\{\bar{b}_{n_k} \mid k \geq 1\}$  and  $\{\bar{c}_{n_k} \mid k \geq 1\}$  such that  $\bar{b}_{n_k} \rightarrow \bar{b}$  and  $\bar{c}_{n_k} \rightarrow \bar{c}$  weakly as  $k \rightarrow \infty$  for some increasing left-continuous function  $\bar{b}$  and some decreasing left-continuous function  $\bar{c}$ .

**3.3. Tightness.** We claim that the sequences of generalised distribution functions  $\{\bar{b}_n \mid n \geq 1\}$  and  $\{\bar{c}_n \mid n \geq 1\}$  are *tight*. Indeed, if  $\varepsilon > 0$  is given and fixed, then  $\delta'_\varepsilon := \mu((\beta - \varepsilon, \beta]) > 0$  and  $\delta''_\varepsilon := \mu([\gamma, \gamma + \varepsilon)) > 0$  since  $\beta$  and  $\gamma$  belong to  $\text{supp}(\mu)$ . Setting  $\delta_\varepsilon := \delta'_\varepsilon \wedge \delta''_\varepsilon$  and considering  $\tau_\beta = \inf\{t > 0 \mid B_t \geq \beta\}$  and  $\tau_\gamma = \inf\{t > 0 \mid B_t \leq \gamma\}$  we see that there exists  $t_\varepsilon > 0$  large enough such that  $\mathbb{P}(\tau_\beta \vee \tau_\gamma \leq t_\varepsilon) > 1 - \delta_\varepsilon$ . Since  $\tau_{b_n, c_n} \leq \tau_\beta \vee \tau_\gamma$  this implies that  $\mathbb{P}(\tau_{b_n, c_n} \leq t_\varepsilon) > 1 - \delta_\varepsilon$  for all  $n \geq 1$ . From the construction of  $b_n$  and  $c_n$  the latter inequality implies that  $b_n(t_\varepsilon) > \beta - \varepsilon$  and  $c_n(t_\varepsilon) < \gamma + \varepsilon$  for all  $n \geq 1$  (note that in all these arguments we can indeed use unbarred functions). The tightness claim then follows using the same arguments as in Part 1.3 above.

3.4. *Verification.* Applying the same arguments as in Part 1.4 above we know from Part 1.5 above that setting  $\bar{b}_k := \bar{b}_{n_k}$  for  $k \geq 1$  we have  $\tau_{b_k} \rightarrow \tau_{\bar{b}}$  and  $\tau_{c_k} \rightarrow \tau_{\bar{c}}$  in  $\mathbb{P}$ -probability as  $k \rightarrow \infty$ . Setting  $t_k^b = \sup \{ t > 0 \mid b_k(t) = \beta \}$  and  $t_k^c = \sup \{ t > 0 \mid c_k(t) = \gamma \}$  by the construction above we know that either  $t_k^b = \infty$  or  $t_k^c = \infty$  for all  $k \geq 1$ . If there exists  $k_0 \geq 1$  such that both  $t_k^b = \infty$  and  $t_k^c = \infty$  for all  $k \geq k_0$  then  $b_k = \bar{b}_k$  and  $c_k = \bar{c}_k$  for all  $k \geq k_0$  so that  $\tau_{b_k, c_k} = \tau_{b_k} \wedge \tau_{c_k} = \tau_{\bar{b}_k} \wedge \tau_{\bar{c}_k} \rightarrow \tau_{\bar{b}} \wedge \tau_{\bar{c}} = \tau_{\bar{b}, \bar{c}} = \tau_{b, c}$  in  $\mathbb{P}$ -probability as  $k \rightarrow \infty$  where we set  $b := \bar{b}$  and  $c := \bar{c}$ . This implies that  $B_{\tau_{b_k, c_k}} \rightarrow B_{\tau_{b, c}}$  in  $\mathbb{P}$ -probability and thus in law as well while  $B_{\tau_{b_k, c_k}} \sim \mu_k$  with  $\mu_k \rightarrow \mu$  weakly as  $k \rightarrow \infty$  then shows that  $B_{\tau_b, \tau_c} \sim \mu$  as required. Suppose therefore that there is no such  $k_0 \geq 1$ . This means that we have infinitely many  $t_k^b < \infty$  or infinitely many  $t_k^c < \infty$  for  $k \geq 1$ . Without loss of generality assume that the former holds. Then we can pass to a further subsequence such that  $t_{k_l}^b < \infty$  for all  $l \geq 1$  and  $t_{k_l}^b \rightarrow t_\infty^b \in (0, \infty]$  as  $l \rightarrow \infty$ . Set  $b(t) = \bar{b}(t)$  for  $t \in (0, t_\infty^b]$  and  $b(t) = \infty$  for  $t \in (t_\infty^b, \infty)$ . Set also  $c(t) = \bar{c}(t)$  for  $t > 0$  and note that  $c_{k_l} = \bar{c}_{k_l}$  for all  $l \geq 1$ . To simplify the notation set further  $b_l := b_{k_l}$ ,  $\bar{b}_l := \bar{b}_{k_l}$ ,  $c_l := c_{k_l}$  and  $\bar{c}_l := \bar{c}_{k_l}$  for  $l \geq 1$ . Then  $\tau_{\bar{b}_l} \rightarrow \tau_{\bar{b}}$  in  $\mathbb{P}$ -probability and hence  $\tau_{b_l} I(\tau_{\bar{b}} < t_\infty^b) \rightarrow \tau_{\bar{b}} I(\tau_{\bar{b}} < t_\infty^b)$  in  $\mathbb{P}$ -probability as  $l \rightarrow \infty$ . Using definitions of barred functions and the fact that  $t_{k_l}^b \rightarrow t_\infty^b$  one can easily verify that the previous relation implies that  $\tau_{b_l} I(\tau_b < t_\infty^b) \rightarrow \tau_b I(\tau_b < t_\infty^b)$  in  $\mathbb{P}$ -probability as  $l \rightarrow \infty$ . Since  $\mathbb{P}(\tau_b < t_\infty^b) = 1$  it follows that  $\tau_{b_l} \wedge \tau_{c_l} \rightarrow \tau_b \wedge \tau_c$  in  $\mathbb{P}$ -probability as  $l \rightarrow \infty$ . This implies that  $B_{\tau_{b_l, c_l}} \rightarrow B_{\tau_{b, c}}$  in  $\mathbb{P}$ -probability as  $l \rightarrow \infty$  and hence  $B_{\tau_b, c} \sim \mu$  using the same argument as above. This completes the proof in the case when  $\text{supp}(\mu)$  is bounded.

4. *Half bounded support.* Consider now the case when  $\sup \text{supp}(\mu) = +\infty$  and  $\inf \text{supp}(\mu) =: \gamma \in (-\infty, 0)$  (see Figure 1 above). Let  $X$  be a random variable (defined on some probability space) having the law equal to  $\mu$  and set  $X_N = X \wedge \beta_N$  for some  $\beta_N \uparrow \infty$  as  $N \rightarrow \infty$  such that  $\mu((\beta_N - \varepsilon, \beta_N]) > 0$  for all  $\varepsilon > 0$  and  $N \geq 1$ . Let  $N \geq 1$  be given and fixed. Denoting the law of  $X_N$  by  $\mu_N$  we see that  $\text{supp}(\mu_N) \subseteq [\gamma, \beta_N]$  with  $\beta_N$  and  $\gamma$  belonging to  $\text{supp}(\mu_N)$ . Hence by Parts 3.1-3.4 above we know that there exist a left-continuous increasing function  $b_N : (0, \infty) \rightarrow (0, \beta_N] \cup \{+\infty\}$  and a left-continuous decreasing function  $c_N : (0, \infty) \rightarrow [\gamma, 0) \cup \{-\infty\}$  such that  $B_{\tau_{b_N, c_N}} \sim \mu_N$ .

4.1. *Construction.* Recall that these  $b_N$  and  $c_N$  are obtained as the weak limits of subsequences of the sequences of simple functions constructed by partitioning  $(\gamma, 0)$  and  $(0, \beta_N)$ . Extending the same construction to partitioning  $(\gamma, 0)$  and  $[\beta_N, \beta_{N+1})$  while keeping the obtained subsequence of functions with values strictly smaller than  $\beta_N$  we again know by Parts 3.1-3.4 above that there exist a left-continuous increasing function  $b_{N+1} : (0, \infty) \rightarrow (0, \beta_{N+1}] \cup \{+\infty\}$  and a left-continuous decreasing function  $c_{N+1} : (0, \infty) \rightarrow [\gamma, 0) \cup \{-\infty\}$  such that  $B_{\tau_{b_{N+1}, c_{N+1}}} \sim \mu_{N+1}$ . These  $b_{N+1}$  and  $c_{N+1}$  are obtained as the weak limits of further subsequences of the previous subsequences of simple functions. Setting  $t_N = \inf \{ t > 0 \mid b_N(t) = \beta_N \}$  it is therefore clear that  $b_{N+1}(t) = b_N(t)$  and  $c_{N+1}(t) = c_N(t)$  for all  $t \in (0, t_N]$ . Continuing this process by induction and noticing that  $t_N \uparrow t_\infty$  as  $N \rightarrow \infty$  we obtain a left-continuous increasing function  $b : (0, t_\infty) \rightarrow \mathbb{R}$  and a left-continuous decreasing  $c : (0, t_\infty) \rightarrow \mathbb{R} \cup \{-\infty\}$  such that  $b(t) = b_N(t)$  and  $c(t) = c_N(t)$  for all  $t \in (0, t_N]$  and  $N \geq 1$ . Note that  $b$  is finite valued on  $(0, t_\infty)$  with  $b(t_\infty-) = +\infty$ .

4.2. *Verification.* To verify that  $b$  and  $c$  are the required functions consider first the case when  $t_\infty = \infty$ . If  $c$  is finite valued then  $\tau_{b, c} < \infty$   $\mathbb{P}$ -a.s. and hence  $\tau_{b_N, c_N} \rightarrow \tau_{b, c}$   $\mathbb{P}$ -a.s.

as  $N \rightarrow \infty$ . If  $c$  is not finite valued then  $c = c_N$  and hence  $\mathbb{P}(B_{\tau_{b,c}} < \beta_N) = \mathbb{P}(B_{\tau_{b_N, c_N}} < \beta_N) = 1 - \mu([\beta_N, \infty))$  for all  $N \geq N_0$  with some  $N_0 \geq 1$ . Letting  $N \rightarrow \infty$  and using that  $\mu([\beta_N, \infty)) \rightarrow 0$  we find that  $\mathbb{P}(\tau_{b,c} < \infty) = 1$  and hence  $\tau_{b_N, c_N} \rightarrow \tau_{b,c}$  P-a.s. Thus the latter relation always holds and hence  $B_{\tau_{b_N, c_N}} \rightarrow B_{\tau_{b,c}}$  P-a.s. as  $N \rightarrow \infty$ . Since  $B_{\tau_{b_N, c_N}} \sim \mu_N$  and  $X_N \rightarrow X$  so that  $\mu_N \rightarrow \mu$  weakly as  $N \rightarrow \infty$  it follows that  $B_{\tau_{b,c}} \sim \mu$  as required.

Consider next the case when  $t_\infty < \infty$ . To extend the function  $c$  to  $[t_\infty, \infty)$  when  $c(t_\infty -) > \gamma$  (note that when  $c(t_\infty -) = \gamma$  then clearly  $c$  must remain equal to  $\gamma$  on  $[t_\infty, \infty)$  as well) set  $t_N^c = \sup \{t > 0 \mid c_N(t) = \gamma\}$  and define  $\bar{c}_N(t) = c_N(t)$  for  $t \in (0, t_N^c]$  and  $\bar{c}_N(t) = \gamma$  for  $t \in (t_N^c, \infty)$  whenever  $t_N^c < \infty$  for  $N \geq 1$ . Applying the same arguments as in Parts 1.2 and 1.3 above we know that there exists a subsequence  $\{\bar{c}_{N_k} \mid k \geq 1\}$  and a left-continuous function  $\bar{c}$  such that  $\bar{c}_{N_k} \rightarrow \bar{c}$  weakly as  $k \rightarrow \infty$ . Applying the same arguments as in Part 1.4 above we know from Part 1.5 above that setting  $\bar{c}_k := \bar{c}_{N_k}$  for  $k \geq 1$  we have  $\tau_{\bar{c}_k} \rightarrow \tau_{\bar{c}}$  in P-probability as  $k \rightarrow \infty$ . Moreover, we claim that  $t_N^c \rightarrow \infty$  as  $N \rightarrow \infty$ . For this, suppose that  $t_{N_l}^c \leq T < \infty$  for  $l \geq 1$ . Fix  $\varepsilon > 0$  small and set  $c_\varepsilon(t) = c(t)$  for  $t \in (0, t_\infty - \varepsilon)$  and  $c_\varepsilon(t) = c(t_\infty - \varepsilon)$  for  $t \in [t_\infty - \varepsilon, T]$ . Setting  $b_l := b_{N_l}$  and  $c_l := c_{N_l}$  we then have  $\mu([\gamma, \beta_{N_l})) = \mathbb{P}(B_{\tau_{b_l, c_l}} \in [\gamma, \beta_{N_l})) \leq \mathbb{P}(\tau_{b, c_\varepsilon} \leq T)$  for all  $l \geq 1$ . Letting  $l \rightarrow \infty$  and using that  $\mu([\gamma, \beta_{N_l})) \rightarrow 1$  we see that  $\mathbb{P}(\tau_{b, c_\varepsilon} \leq T) = 1$  which clearly is impossible since  $b$  is not defined beyond  $t_\infty$ . Thus  $t_N^c \rightarrow \infty$  as  $N \rightarrow \infty$  and hence  $t_{N_k}^c \rightarrow \infty$  as  $k \rightarrow \infty$ . Setting  $c := \bar{c}$  and  $c_k := c_{N_k}$  for  $k \geq 1$  and using the same arguments as in Part 3.4 above we can therefore conclude that  $\tau_{c_k} I(\tau_c < \infty) \rightarrow \tau_c I(\tau_c < \infty)$  in P-probability as  $k \rightarrow \infty$ . Since  $\mathbb{P}(\tau_c < \infty) = 1$  this shows that  $\tau_{c_k} \rightarrow \tau_c$  in P-probability as  $k \rightarrow \infty$ . Setting  $b_k := b_{N_k}$  and noting that  $\tau_{b_k} \rightarrow \tau_b$  on  $\{\tau_b < \infty\}$  we see that  $\tau_{b_k, c_k} \rightarrow \tau_{b,c}$  in P-probability as  $k \rightarrow \infty$  and hence  $B_{\tau_{b,c}} \sim \mu$  using the same argument as above. The case when  $\sup \text{supp}(\mu) \in (0, +\infty)$  and  $\inf \text{supp}(\mu) = -\infty$  follows in exactly the same way by symmetry.

5. *Fully unbounded support.* Consider finally the case when both  $\sup \text{supp}(\mu) = +\infty$  and  $\inf \text{supp}(\mu) = -\infty$ . Let  $X$  be a random variable (defined on some probability space) having the law equal to  $\mu$  and set  $X_N = \gamma_N \vee X \wedge \beta_N$  for some  $\beta_N \uparrow \infty$  and  $\gamma_N \downarrow -\infty$  as  $N \rightarrow \infty$  such that  $\mu((\beta_N - \varepsilon, \beta_N]) > 0$  and  $\mu([\gamma_N, \gamma_N + \varepsilon)) > 0$  for all  $\varepsilon > 0$  and  $N \geq 1$ . Let  $N \geq 1$  be given and fixed. Denoting the law of  $X_N$  by  $\mu_N$  we see that  $\text{supp}(\mu_N) \subseteq [\gamma_N, \beta_N]$  with  $\beta_N$  and  $\gamma_N$  belonging to  $\text{supp}(\mu_N)$ . Hence by Parts 3.1-3.4 above we know that there exist a left-continuous increasing function  $b_N : (0, \infty) \rightarrow (0, \beta_N] \cup \{+\infty\}$  and a left-continuous decreasing function  $c_N : (0, \infty) \rightarrow [\gamma_N, 0) \cup \{-\infty\}$  such that  $B_{\tau_{b_N, c_N}} \sim \mu_N$ .

5.1. *Construction.* Recall that these  $b_N$  and  $c_N$  are obtained as the weak limits of subsequences of the sequences of simple functions constructed by partitioning  $(\gamma_N, 0)$  and  $(0, \beta_N)$ . Extending the same construction to partitioning  $(\gamma_{N+1}, \gamma_N]$  and  $[\beta_N, \beta_{N+1})$  while keeping the obtained subsequence of functions with values strictly smaller than  $\beta_N$  and strictly larger than  $\gamma_N$  we again know by Parts 3.1-3.4 above that there exist a left-continuous increasing function  $b_{N+1} : (0, \infty) \rightarrow (0, \beta_{N+1}] \cup \{+\infty\}$  and a left-continuous decreasing function  $c_{N+1} : (0, \infty) \rightarrow [\gamma_{N+1}, 0) \cup \{-\infty\}$  such that  $B_{\tau_{b_{N+1}, c_{N+1}}} \sim \mu_{N+1}$ . These  $b_{N+1}$  and  $c_{N+1}$  are obtained as the weak limits of further subsequences of the previous subsequences of simple functions. Setting  $t_N^b = \inf \{t > 0 \mid b_N(t) = \beta_N\}$  and  $t_N^c = \inf \{t > 0 \mid c_N(t) = \gamma_N\}$  it is therefore clear that  $b_{N+1}(t) = b_N(t)$  and  $c_{N+1}(t) = c_N(t)$  for all  $t \in (0, t_N]$  where we set  $t_N := t_N^b \wedge t_N^c$  for  $N \geq 1$ . Continuing this process by induction and noticing that  $t_N \uparrow t_\infty$  as  $N \rightarrow \infty$  we obtain a left-continuous increasing function  $b : (0, t_\infty) \rightarrow \mathbb{R}$  and a left-continuous

decreasing  $c : (0, t_\infty) \rightarrow \mathbb{R}$  such that  $b(t) = b_N(t)$  and  $c(t) = c_N(t)$  for all  $t \in (0, t_N]$  and  $N \geq 1$ .

**5.2. Verification.** To verify that  $b$  and  $c$  are the required functions consider first the case when  $t_\infty = \infty$ . Then since  $b(t_N) \leq \beta_N$  and  $c(t_N) \geq \gamma_N$  for any  $A \in \mathcal{B}(\mathbb{R})$  we have  $\mathbb{P}(B_{\tau_{b,c}} \in A \cap (c(t_N), b(t_N))) = \mathbb{P}(B_{\tau_{b_N, c_N}} \in A \cap (c(t_N), b(t_N))) = \mu(A \cap (c(t_N), b(t_N)))$  for all  $N \geq 1$ . Letting  $N \rightarrow \infty$  and using that  $b(t_N) \uparrow \infty$  and  $c(t_N) \downarrow -\infty$  we see that  $\mathbb{P}(B_{\tau_{b,c}} \in A) = \mu(A)$  and this shows that  $B_{\tau_{b,c}} \sim \mu$  as required.

Consider next the case when  $t_\infty < \infty$  and assume first that either  $\{t_N^b \mid N \geq 1\}$  or  $\{t_N^c \mid N \geq 1\}$  is not bounded (we will see below that this is always true). Without loss of generality we can assume (by passing to a subsequence if needed) that  $t_N^c \rightarrow \infty$  so that  $t_N^b \uparrow t_\infty < \infty$  as  $N \rightarrow \infty$ . To extend the function  $c$  to  $[t_\infty, \infty)$  we can now connect to the final paragraph of Part 4 above. Choosing  $M \geq 1$  large enough so that  $\gamma_M < c(t_\infty -)$  we see that we are in the setting of that paragraph with  $\gamma = \gamma_M$  and hence there exists a left-continuous decreasing function  $c_M : (0, \infty) \rightarrow [\gamma_M, 0)$  such that  $B_{\tau_{b,c_M}} \sim X \vee \gamma_M$ . Recall that this  $c_M$  is obtained as the weak limit of a subsequence of the sequence of functions embedding  $B$  into  $[\gamma_M, \beta_N]$  for  $N \geq 1$  and note that  $c_M$  coincides with  $c$  on  $(0, t_\infty)$ . Extending the same construction to embedding  $B$  into  $[\gamma_{M+1}, \beta_N]$  for  $N \geq 1$  while keeping the subsequence of functions obtained previously we again know by the final paragraph of Part 4 above that there exists a left-continuous decreasing function  $c_{M+1} : (0, \infty) \rightarrow [\gamma_{M+1}, 0)$  such that  $B_{\tau_{b,c_{M+1}}} \sim X \vee \gamma_{M+1}$ . This  $c_{M+1}$  is obtained as the weak limit of a further subsequence of the previous sequence of functions. Setting  $t_M^c = \inf\{t > 0 \mid c_M(t) = \gamma_M\}$  it is therefore clear that  $c_{M+1}(t) = c_M(t)$  for  $t \in (0, t_M^c)$ . Continuing this process by induction we obtain a left-continuous decreasing function  $c : (0, \infty) \rightarrow \mathbb{R}$  that coincides with the initial function  $c$  on  $(0, t_\infty)$ . Setting  $t_M^c = \inf\{t > 0 \mid c(t) = \gamma_M\}$  we see that  $c(t_M^c) = \gamma_M \downarrow -\infty$  as  $M \rightarrow \infty$ . Hence for any  $A \in \mathcal{B}(\mathbb{R})$  we see that  $\mathbb{P}(B_{\tau_{b,c}} \in A \cap (c(t_M^c), \infty)) = \mathbb{P}(B_{\tau_{b,c_M}} \in A \cap (c(t_M^c), \infty)) = \mu(A \cap (c(t_M^c), \infty)) \rightarrow \mu(A)$  as  $M \rightarrow \infty$  from where it follows that  $\mathbb{P}(B_{\tau_{b,c}} \in A) = \mu(A)$ . This shows that  $B_{\tau_{b,c}} \sim \mu$  as required. Moreover we claim that this is the only case we need to consider since if both  $\{t_N^b \mid N \geq 1\}$  and  $\{t_N^c \mid N \geq 1\}$  are bounded then without loss of generality we can assume (by passing to a subsequence if needed) that  $t_N^c \rightarrow t_\infty^c < \infty$  with  $t_\infty^c > t_\infty$  first so that  $t_N^b \uparrow t_\infty$  as  $N \rightarrow \infty$ . In this case we can repeat the preceding construction and extend  $c$  to  $[t_\infty, t_\infty^c)$  so that we again have  $B_{\tau_{b,c}} \sim \mu$  by the same argument. If  $t_\infty^c = t_\infty$  however then the same argument as in the case of  $t_\infty = \infty$  above shows that the latter relation also holds. Thus in both cases we have  $t_N^b \leq T$  and  $t_N^c \leq T$  for all  $N \geq 1$  with  $T := t_\infty^c$  so that  $\mu((\gamma_N, \beta_N)) = \mathbb{P}(B_{\tau_{b,c}} \in (\gamma_N, \beta_N)) = \mathbb{P}(B_{\tau_{b,c}} \in (c(t_N^c), b(t_N^b))) \leq \mathbb{P}(\tau_{b,c} \leq T)$  for all  $N \geq 1$ . Letting  $N \rightarrow \infty$  and using that  $\mu((\gamma_N, \beta_N)) \rightarrow 1$  we get  $\mathbb{P}(\tau_{b,c} \leq T) = 1$  which clearly is impossible since  $T < \infty$ . It follows therefore that  $B_{\tau_{b,c}} \sim \mu$  in all possible cases and the proof is complete.  $\square$

**Remark 2.** Note that  $b$  from (I) and  $c$  from (II) are always finite valued since otherwise  $\mu(\mathbb{R}_+) < 1$  or  $\mu(\mathbb{R}_-) < 1$  respectively. Note also that either  $b$  or  $c$  from (III) can formally take value  $+\infty$  or  $-\infty$  respectively from some time onwards, however, when this happens to either function then the other function must remain finite valued (note that (I) and (II) can be seen as special cases of (III) in this sense too). Note finally that the result and proof of Theorem 1 including the same remarks remain valid if  $B_0 \sim \nu$  where  $\nu$  is a probability

measure on  $\mathbb{R}$  such that  $\text{supp}(\nu) \subseteq [-p, q]$  with  $\mu([-p, q]) = 0$  for some  $p > 0$  and  $q > 0$ .

**Remark 3.** Since the arguments in the proof of Theorem 1 can be repeated over any subsequence of  $\{b_n \mid n \geq 1\}$  or  $\{c_n \mid n \geq 1\}$  (when constructed with no upper or lower bound on the partitions of  $\text{supp}(\mu)$  as well) it follows that  $B_{\tau_{b_n, c_n}}$  not only converges to  $B_{\tau_{b, c}}$  over a subsequence P-a.s. but this convergence also holds for the entire sequence in P-probability. Indeed, if this would not be the case then for some subsequence no further subsequence would converge P-a.s. The initial argument of this remark combined with the uniqueness result of Theorem 10 below would then yield a contradiction. The fact that  $B_{\tau_{b_n, c_n}}$  always converges to  $B_{\tau_{b, c}}$  in P-probability as  $n \rightarrow \infty$  makes the derivation fully constructive and amenable to algorithmic calculations described next.

**Remark 4.** The construction presented in the proof above yields a simple algorithm for computing  $b_n$  and  $c_n$  which in turn provide numerical approximations of  $b$  and  $c$ . Key elements of the algorithm can be described as follows. Below we let  $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$  and  $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^x e^{-y^2/2} dy$  for  $x \in \mathbb{R}$  denote the standard normal density and distribution function respectively.

In the one-sided case (I) when  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  recall the well-known expressions (cf. [2])

$$(2.22) \quad \mathbb{P}(B_t \in dx, \tau_y > t) = \frac{1}{\sqrt{t}} \left[ \varphi\left(\frac{x}{\sqrt{t}}\right) - \varphi\left(\frac{x-2y}{\sqrt{t}}\right) \right] dx =: f(t, x, y) dx$$

$$(2.23) \quad \mathbb{P}(\tau_y \leq t) = 2 \left[ 1 - \Phi\left(\frac{y}{\sqrt{t}}\right) \right] =: g(t, y)$$

for  $t > 0$  and  $x < y$  with  $y > 0$  where we set  $\tau_y = \inf \{ t > 0 \mid B_t = y \}$ . Using stationary and independent increments of  $B$  (its Markov property) we then read from Part 1.1 of the proof above that the algorithm runs as follows

$$(2.24) \quad g_k(t) := \int_{-\infty}^{x_{k-1}^n} g(t, x_k^n - y) f_{k-1}(y) dy$$

$$(2.25) \quad t_k^n := t_{k-1}^n + \inf \{ t > 0 \mid g_k(t) = p_k^n \}$$

$$(2.26) \quad f_k(x) := \int_{-\infty}^{x_{k-1}^n} f(t_k^n - t_{k-1}^n, x - y, x_k^n - y) f_{k-1}(y) dy$$

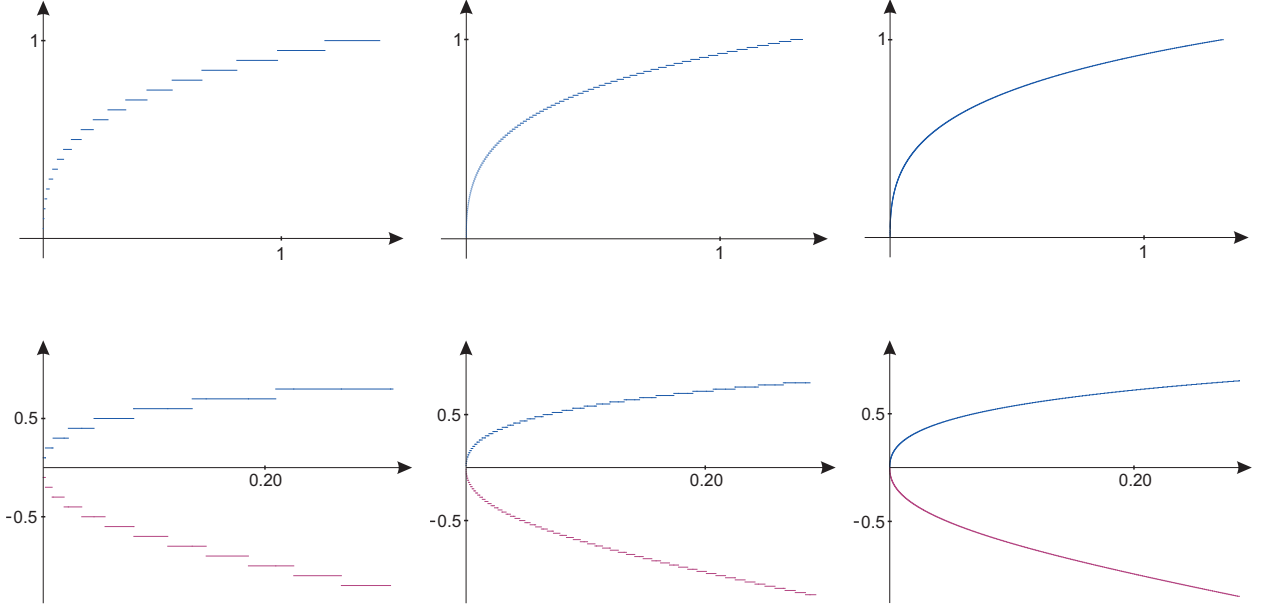
for  $k = 1, 2, \dots, m_n$  where we initially set  $t_0 := 0$ ,  $x_0 := 0$  and  $f_0(x) dx := \delta_0(dx)$ . This yields the time points  $t_1^n, t_2^n, \dots, t_{m_n}^n$  which determine  $b_n$  by the formula

$$(2.27) \quad b_n(t) = \sum_{k=1}^{m_n} x_k^n I(t_{k-1}^n < t \leq t_k^n)$$

for  $t \geq 0$ . The algorithm is stable and completes within a reasonable time frame (see Figure 2 below for the numerical output when the target law  $\mu$  is exponentially distributed with intensity 1).

In the two-sided case (III) when  $\text{supp}(\mu) \subseteq \mathbb{R}$  recall the well-known expressions (cf. [2])

$$(2.28) \quad \mathbb{P}(B_t \in dx, \tau_{y,z} > t) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \left[ \varphi\left(\frac{x+2n(y-z)}{\sqrt{t}}\right) - \varphi\left(\frac{x+2n(y-z)-2y}{\sqrt{t}}\right) \right] dx$$



**Figure 2.** Functions  $b_n$  and  $c_n$  calculated using the algorithm from the proof of Theorem 1 as described in Remark 4. The first row corresponds to the target law  $\mu$  which is exponentially distributed with intensity 1 for  $n = 20, 100, 500$  respectively with equidistant partition of  $\mathbb{R}_+$  having the step size equal to  $1/n$  and the number of time points  $m_n$  equal to  $n$ . The second row corresponds to the target law  $\mu$  which is normally distributed with mean 1 and variance 1 for  $n = 10, 50, 250$  respectively with equidistant partition of  $\mathbb{R}$  having the step size equal to  $1/n$  and the number of time points  $m_n+l_n$  equal to  $2n$ .

$$=: f(t, x, y, z) dx$$

$$(2.29) \quad \mathbb{P}(\tau_y < \tau_z, \tau_{y,z} \leq t) = 2 \sum_{n=0}^{\infty} \left[ \Phi\left(\frac{(2n+1)(y-z)-z}{\sqrt{t}}\right) - \Phi\left(\frac{(2n+1)(y-z)+z}{\sqrt{t}}\right) \right]$$

$$=: g(t, y, z)$$

$$(2.30) \quad \mathbb{P}(\tau_z < \tau_y, \tau_{y,z} \leq t) = 2 \sum_{n=0}^{\infty} \left[ \Phi\left(\frac{(2n+1)(y-z)+y}{\sqrt{t}}\right) - \Phi\left(\frac{(2n+1)(y-z)-y}{\sqrt{t}}\right) \right]$$

$$=: h(t, y, z)$$

for  $t > 0$  and  $z < x < y$  with  $z < 0 < y$  where we set  $\tau_w = \inf\{t > 0 \mid B_t = w\}$  for  $w \in \{y, z\}$  and  $\tau_{y,z} = \tau_y \wedge \tau_z$ . Using stationary and independent increments of  $B$  (its Markov property) we then read from Part 3.1 of the proof above that the algorithm runs as follows

$$(2.31) \quad g_k(t) := \int_{\bar{y}_{k-1}^n}^{\bar{x}_{k-1}^n} g(t, \bar{x}_k^n - z, \bar{y}_k^n - z) f_{k-1}(z) dz$$

$$(2.32) \quad h_k(t) := \int_{\bar{y}_{k-1}^n}^{\bar{x}_{k-1}^n} h(t, \bar{x}_k^n - z, \bar{y}_k^n - z) f_{k-1}(z) dz$$



$$(2.33) \quad t_k^n := t_{k-1}^n + \left( \inf \{ t > 0 \mid g_k(t) = \bar{p}_k^n \} \wedge \inf \{ t > 0 \mid h_k(t) = \bar{q}_k^n \} \right)$$

$$(2.34) \quad f_k(x) := \int_{\bar{y}_{k-1}^n}^{\bar{x}_{k-1}^n} f(t_k^n - t_{k-1}^n, x - z, \bar{x}_k^n - z, \bar{y}_k^n - z) f_{k-1}(z) dz$$

for  $k = 1, 2, \dots, m_n + l_n$  where we initially set  $t_0 := 0$ ,  $\bar{x}_0^n := 0$ ,  $\bar{y}_0^n := 0$ ,  $\bar{x}_1^n := x_1^n$ ,  $\bar{y}_1^n := y_1^n$ ,  $f_0(x) dx := \delta_0(dx)$  and denoting the first infimum in (2.33) by  $I_k^n$  and the second infimum in (2.33) by  $J_k^n$  this is then continued as follows: if  $I_k^n > J_k^n$  then  $\bar{x}_{k+1}^n := \inf \{ x_l \mid x_l > \bar{x}_k^n \}$ ,  $\bar{y}_{k+1}^n := \bar{y}_k^n$ ,  $\bar{p}_{k+1}^n := p(\bar{x}_{k+1}^n)$ ,  $\bar{q}_{k+1}^n := \bar{q}_k^n - h_k(I_k^n)$ ; if  $J_k^n > I_k^n$  then  $\bar{y}_{k+1}^n := \sup \{ y_l \mid y_l < \bar{y}_k^n \}$ ,  $\bar{x}_{k+1}^n := \bar{x}_k^n$ ,  $\bar{q}_{k+1}^n := q(\bar{y}_{k+1}^n)$ ,  $\bar{p}_{k+1}^n := \bar{p}_k^n - g_k(J_k^n)$ ; if  $I_k^n = J_k^n$  then  $\bar{x}_{k+1}^n := \inf \{ x_l \mid x_l > \bar{x}_k^n \}$ ,  $\bar{y}_{k+1}^n := \sup \{ y_l \mid y_l < \bar{y}_k^n \}$ ,  $\bar{p}_{k+1}^n := p(\bar{x}_{k+1}^n)$ ,  $\bar{q}_{k+1}^n := q(\bar{y}_{k+1}^n)$  where we set  $p(x) = p_k^n$  for  $x = x_k^n$  and  $q(y) = q_k^n$  for  $y = y_k^n$ . This yields the time points  $t_1^n, t_2^n, \dots, t_{m_n+l_n}^n$  which determine  $b_n$  and  $c_n$  by the formulae

$$(2.35) \quad b_n(t) = \sum_{k=1}^{m_n+l_n} \bar{x}_k^n I(t_{k-1}^n < t \leq t_k^n) \quad \& \quad c_n(t) = \sum_{k=1}^{m_n+l_n} \bar{y}_k^n I(t_{k-1}^n < t \leq t_k^n)$$

for  $t \geq 0$ . The algorithm is stable and completes within a reasonable time frame (see Figure 2 above for the numerical output when the target law  $\mu$  is normally distributed with mean 1 and variance 1).

**Remark 5.** Note that  $\tau_b$  from (I) could also be defined by

$$(2.36) \quad \tau_b = \inf \{ t > 0 \mid B_t = b(t) \}$$

and that  $B_{\tau_b} = b(\tau_b)$ . This is easily verified since  $b$  is left-continuous and increasing. The same remark applies to  $\tau_c$  from (II) and  $\tau_{b,c}$  from (III) with  $B_{\tau_{b,c}}$  being equal to  $b(\tau_{b,c})$  or  $c(\tau_{b,c})$ . From (2.8) and (2.9) we also see that these inequalities and equalities in the definitions of the stopping times can be replaced by strict inequalities and that all relations remain valid almost surely in this case. Similarly, in all these definitions we could replace left-continuous functions  $b$  and  $c$  with their right-continuous versions defined by  $b(t) := b(t+)$  and  $c(t) := c(t+)$  for  $t > 0$  respectively. All previous facts in this remark remain valid in this case too.

**Remark 6.** If  $\mu(\{0\}) =: p > 0$  in Theorem 1 then we can generate a random variable  $\zeta$  independently from  $B$  such that  $\zeta$  takes two values 0 and  $\infty$  with probabilities  $p$  and  $1-p$  respectively. Performing the same construction with the stopped sample path  $t \mapsto B_{t \wedge \zeta}$  yields the existence of functions  $b$  and  $c$  as in Theorem 1 with  $B^\zeta = (B_{t \wedge \zeta})_{t \geq 0}$  in place of  $B = (B_{t \geq 0})_{t \geq 0}$ . The resulting stopping time may be viewed as randomised through the initial condition.

**Remark 7.** Two main ingredients in the proof of Theorem 1 above are (i) embedding in discrete laws and (ii) passage to the limit from discrete to general laws. If the standard Brownian motion  $B$  is replaced by a continuous (time-homogeneous) Markov process  $X$  we see from the proof above that (i) can be achieved when

$$(2.37) \quad t \mapsto \mathbf{P}_x(\tau_y < \tau_z, \tau_{y,z} \leq t) \quad \& \quad t \mapsto \mathbf{P}_x(\tau_z < \tau_y, \tau_{y,z} \leq t)$$

are continuous on  $\mathbb{R}_+$  and  $\mathbb{P}_x(\tau_{y,z} > t) \downarrow 0$  as  $t \uparrow \infty$  for all  $-\infty \leq z < x < y \leq \infty$  with  $|z| \wedge |y| < \infty$  and  $\mathbb{P}_x(X_0 = x) = 1$  where we set  $\tau_w = \inf \{t > 0 \mid X_t = w\}$  for  $w \in \{y, z\}$  and  $\tau_{y,z} = \tau_y \wedge \tau_z$ . We also see from the proof above that (ii) can be achieved when

$$(2.38) \quad \tau_b = \sigma_b \quad \mathbb{P}_0\text{-a.s.} \quad \& \quad \tau_c = \sigma_c \quad \mathbb{P}_0\text{-a.s.}$$

where the first equality holds for any left-continuous increasing function  $b$  with  $\tau_b = \inf \{t > 0 \mid X_t \geq b(t)\}$  and  $\sigma_b = \inf \{t > 0 \mid X_t > b(t)\}$ , and the second equality holds for any left-continuous decreasing function  $c$  with  $\tau_c = \inf \{t > 0 \mid X_t \leq c(t)\}$  and  $\sigma_c = \inf \{t > 0 \mid X_t < c(t)\}$ . In particular, by verifying (2.37) and (2.38) in the proof of Corollary 8 below we will establish that the result of Theorem 1 extends to all recurrent diffusion processes  $X$  in the sense of Itô and McKean [10] (see [2, Chapter II] for a review). This extension should also hold for non-recurrent diffusion processes  $X$  and ‘admissible’ target laws  $\mu$  (cf. [15]) as well as for more general standard Markov processes  $X$  satisfying suitable modifications of (2.37) and (2.38) in the admissible setting. We leave precise formulations of these more general statements and proofs as informal conjectures open for future developments.

**Corollary 8.** *The result of Theorem 1 remains valid if the standard Brownian motion  $B$  is replaced by any recurrent diffusion process  $X$ .*

**Proof.** As pointed out above the proof can be carried out in the same way as the proof of Theorem 1 if we show that (2.37) and (2.38) are satisfied. Note that  $\mathbb{P}_x(\tau_{y,z} > t) \downarrow 0$  as  $t \uparrow \infty$  for all  $-\infty \leq z < x < y \leq \infty$  with  $|z| \wedge |y| < \infty$  since  $X$  is recurrent. Recall also that all recurrent diffusions are regular (see [2, Chapter II] for definitions).

1. We first show that the functions in (2.37) are continuous. Clearly by symmetry it is enough to show that the first function is continuous. For this, set  $F(t) = \mathbb{P}_x(\tau_y < \tau_z, \tau_{y,z} \leq t)$  for  $t \geq 0$  where  $-\infty \leq z < x < y < \infty$  are given and fixed. Since  $t \mapsto F(t)$  is increasing and right-continuous we see that it is enough to disprove the existence of  $t_1 > 0$  such that  $F(t_1) - F(t_1-) = \mathbb{P}_x(\tau_y < \tau_z, \tau_y = t_1) > 0$ . Since this implies that  $\mathbb{P}_x(\tau_y = t_1) > 0$  we see that it is enough to show that the distribution function  $t \mapsto \mathbb{P}_x(\tau_y \leq t)$  is continuous for  $x < y$  in  $\mathbb{R}$  given and fixed. For this, let  $p$  denote the transition density of  $X$  with respect to its speed measure  $m$  in the sense that  $\mathbb{P}_x(X_t \in A) = \int_A p(t; x, y) m(dy)$  holds for all  $t > 0$  and all  $A \in \mathcal{B}(\mathbb{R})$ . It is well known (cf. [10, p. 149]) that  $p$  may be chosen to be jointly continuous (in all three variables). Next note that for any  $s > 0$  given and fixed the mapping  $t \mapsto \mathbb{E}_x[\mathbb{P}_{X_s}(\tau_y \leq t)] = \int_{\mathbb{R}} \mathbb{P}_z(\tau_y \leq t) p(s; x, z) m(dz)$  is increasing and right-continuous on  $(0, \infty)$  so that  $G(t, s) := \mathbb{E}_x[\mathbb{P}_{X_s}(\tau_y = t)] = \int_{\mathbb{R}} \mathbb{P}_z(\tau_y = t) p(s; x, z) m(dz) = 0$  for all  $t \in (0, \infty) \setminus C_s$  where the set  $C_s$  is at most countable. Setting  $C := \cup_{s \in \mathbb{Q}_+} C_s$  where  $\mathbb{Q}_+$  denotes the set of rational numbers in  $(0, \infty)$ , we see that the set  $C$  is at most countable and  $G(t, s) = 0$  for all  $t \in (0, \infty) \setminus C$  and all  $s \in \mathbb{Q}_+$ . Since each  $z \mapsto p(s; x, z)$  is a density function integrating to 1 over  $m(dz)$  and  $s \mapsto p(s; x, z)$  is continuous on  $(0, \infty)$ , we see by Scheffé’s theorem (see e.g. [1, p. 215]) that  $G(t, s_n) \rightarrow G(t, s)$  as  $s_n \rightarrow s$  in  $(0, \infty)$  for any  $t > 0$  fixed. Choosing these  $s_n$  from  $\mathbb{Q}_+$  for given  $s > 0$  it follows therefore that  $G(t, s) = 0$  for all  $t \in (0, \infty) \setminus C$  and all  $s > 0$ . By the Markov property we moreover see that  $\mathbb{P}_x(\tau_y = t+s) \leq \mathbb{P}_x(\tau_y \circ \theta_s = t) = G(t, s) = 0$  and hence  $\mathbb{P}_x(\tau_y = t+s) = 0$  for all  $t \in (0, \infty) \setminus C$  and all  $s > 0$ . Since the set  $C$  is at most countable it follows that  $\mathbb{P}_x(\tau_y = t) = 0$  for all  $t > 0$ . This implies that  $F$  is continuous and the proof of (2.37) is complete.

2. We next show that the equalities in (2.38) are satisfied. Clearly by symmetry it is enough to derive the first equality. Note that replacing  $B$  by  $X$  in the proof of (2.9) above and using exactly the same arguments yields the first equality in (2.38) provided that (2.14) is established for  $X$  in place of  $B$ . This shows that the first equality in (2.38) reduces to establishing that

$$(2.39) \quad \mathbb{P}_0(\sigma_b^\delta > t) \leq \mathbb{P}_0(\tau_b^\delta > t)$$

for all  $t > 0$  where  $\sigma_b^\delta = \inf \{ t > \delta \mid X_t > b(t) \}$  and  $\tau_b^\delta = \inf \{ t > \delta \mid X_t \geq b(t) \}$  for  $\delta > 0$  given and fixed. Observe that  $\sigma_b^\delta$  coincides with  $\tau_{b^+}^\delta := \lim_{\varepsilon \downarrow 0} \tau_{b+\varepsilon}^\delta$  where  $\tau_{b+\varepsilon}^\delta = \inf \{ t > \delta \mid X_t \geq b(t)+\varepsilon \}$  as is easily seen from the definitions so that (2.39) is indeed equivalent to (2.14) as stated above.

To establish (2.39) consider first the case when  $b$  is flat on some time interval  $I \subseteq (\delta, \infty)$  and denote the joint value of  $b$  on  $I$  by  $y$  meaning that  $b(t) = y$  for all  $t \in I$ . Consider the stopping times  $\tau := \inf \{ t > \delta \mid X_t = y \}$  and  $\sigma := \inf \{ t > \tau \mid X_t > y \}$ . Since  $X$  is recurrent we know that both  $\tau$  and  $\sigma$  are finite valued under  $\mathbb{P}_0$ . Note that  $\sigma = \tau + \rho \circ \theta_\tau$  where  $\rho := \inf \{ t > 0 \mid X_t > y \}$  is a stopping time. By the strong Markov property of  $X$  applied at  $\tau$  we thus have  $\mathbb{P}_0(\sigma = \tau) = \mathbb{P}_0(\rho \circ \theta_\tau = 0) = \mathbb{P}_{X_\tau}(\rho = 0) = \mathbb{P}_y(\rho = 0) = 1$  where the final equality follows since  $X$  is regular (cf. [2, p. 13]). Hence we see that  $X_{\tau+t} > y$  for infinitely many  $t$  in each  $(0, \varepsilon]$  for  $\varepsilon > 0$  with  $\mathbb{P}_0$ -probability one. In particular, this shows that on the set  $\{\sigma_b^\delta > t\}$  with  $t > 0$  given and fixed the sample path of  $X$  stays strictly below  $b$  on the time interval  $I \setminus \text{sup}(I)$  with  $\mathbb{P}_0$ -probability one for each time interval  $I \subseteq (\delta, t)$  on which  $b$  is flat. Since  $(\delta, t)$  can be written as a countable union of disjoint intervals on each of which  $b$  is either flat or strictly increasing, we see that the previous conclusion implies that

$$(2.40) \quad \begin{aligned} \mathbb{P}_0(\sigma_b^\delta > t) &\leq \mathbb{P}_0(X_s < b(s+h) \text{ for all } s \in (\delta, t)) \\ &\leq \mathbb{P}_0(X_{r-h} < b(r) \text{ for all } r \in (\delta+h, t+h)) \\ &\leq \mathbb{P}_0(X_{r-h} < b(r) \text{ for all } r \in (\delta+h_0, t]) \end{aligned}$$

for any  $h \in (0, h_0)$  where  $h_0 \in (0, \delta/2)$  is given and fixed. By the Markov property and Scheffé's theorem applied as above we find that

$$(2.41) \quad \begin{aligned} \mathbb{P}_0(X_{r-h} < b(r) \text{ for all } r \in (\delta+h_0, t]) &= \mathbb{E}_0[\mathbb{P}_{X_{\delta/2-h}}(X_{r-\delta/2} < b(r) \text{ for all } r \in (\delta+h_0, t])] \\ &= \int_{\mathbb{R}} \mathbb{P}_y(X_{r-\delta/2} < b(r) \text{ for all } r \in (\delta+h_0, t]) p(\delta/2-h; 0, y) m(dy) \\ &\longrightarrow \int_{\mathbb{R}} \mathbb{P}_y(X_{r-\delta/2} < b(r) \text{ for all } r \in (\delta+h_0, t]) p(\delta/2; 0, y) m(dy) \\ &= \mathbb{E}_0[\mathbb{P}_{X_{\delta/2}}(X_{r-\delta/2} < b(r) \text{ for all } r \in (\delta+h_0, t])] \\ &= \mathbb{P}_0(X_r < b(r) \text{ for all } r \in (\delta+h_0, t]) \end{aligned}$$

as  $h \downarrow 0$ . Combining (2.40) and (2.41) we get

$$(2.42) \quad \mathbb{P}_0(\sigma_b^\delta > t) \leq \mathbb{P}_0(X_r < b(r) \text{ for all } r \in (\delta+h_0, t])$$

for all  $h_0 \in (0, \delta/2)$ . Letting  $h_0 \downarrow 0$  in (2.42) we find that

$$(2.43) \quad \mathbb{P}_0(\sigma_b^\delta > t) \leq \mathbb{P}_0(X_r < b(r) \text{ for all } r \in (\delta, t]) = \mathbb{P}_0(\tau_b^\delta > t)$$

for all  $t > 0$ . This establishes (2.39) and hence  $\tau_b = \sigma_b$   $\mathbb{P}_0$ -a.s. as explained above. The proof of (2.38) is therefore complete.  $\square$

Note that the claims of Remarks 2-6 extend to the setting of Corollary 8 with suitable modifications in Remark 4 since the process no longer has stationary and independent increments and some of the expressions may no longer be available in closed form.

In the setting of Theorem 1 or Corollary 8 let  $F_\mu$  denote the distribution function of  $\mu$ . The following proposition shows that (i) jumps of  $b$  or  $c$  correspond exactly to flat intervals of  $F_\mu$  (i.e. no mass of  $\mu$ ) and (ii) flat intervals of  $b$  or  $c$  correspond exactly to jumps of  $F_\mu$  (i.e. atoms of  $\mu$ ). In particular, from (i) we see that if  $F_\mu$  is strictly increasing on  $\mathbb{R}_+$  then  $b$  is continuous, and if  $F_\mu$  is strictly increasing on  $\mathbb{R}_-$  then  $c$  is continuous. Similarly, from (ii) we see that if  $F_\mu$  is continuous on  $\mathbb{R}_+$  then  $b$  is strictly increasing, and if  $F_\mu$  is continuous on  $\mathbb{R}_-$  then  $c$  is strictly decreasing.

**Proposition 9 (Continuity).** *In the setting of Theorem 1 or Corollary 8 we have:*

$$(2.44) \quad b(t+) > b(t) \text{ if and only if } \mu((b(t), b(t+))) = 0$$

$$(2.45) \quad b(t) = b(t-\varepsilon) \text{ for some } \varepsilon > 0 \text{ if and only if } \mu(\{b(t)\}) > 0$$

$$(2.46) \quad c(t+) < c(t) \text{ if and only if } \mu((c(t+), c(t))) = 0$$

$$(2.47) \quad c(t) = c(t-\varepsilon) \text{ for some } \varepsilon > 0 \text{ if and only if } \mu(\{c(t)\}) > 0$$

for any  $t > 0$  given and fixed.

**Proof.** All statements follow from the construction and basic properties of  $b$  and  $c$  derived in the proof of Theorem 1.  $\square$

### 3. Uniqueness

In this section we state and prove the main uniqueness result. Note that the result and proof remain valid in the more general case addressed at the end of Remark 2 and the method of proof is also applicable to more general processes (cf. Remark 7).

**Theorem 10 (Uniqueness).** *In the setting of Theorem 1 or Corollary 8 the functions  $b$  and  $c$  are uniquely determined by the law  $\mu$ .*

**Proof.** To simplify the exposition we will derive (I) in full detail. It is clear from the proof below that the same arguments can be used to derive (II) and (III).

1. Let us assume that  $b_1 : (0, \infty) \rightarrow \mathbb{R}_+$  and  $b_2 : (0, \infty) \rightarrow \mathbb{R}_+$  are left-continuous increasing functions such that  $X_{\tau_{b_1}} \sim \mu$  and  $X_{\tau_{b_2}} \sim \mu$  where  $\tau_{b_1} = \inf \{t > 0 \mid X_t \geq b_1(t)\}$  and  $\tau_{b_2} = \inf \{t > 0 \mid X_t \geq b_2(t)\}$ . We then need to show that  $b_1 = b_2$ . For this, we will first

show that  $b := b_1 \wedge b_2$  also solves the embedding problem in the sense that  $X_{\tau_b} \sim \mu$  where  $\tau_b = \inf \{ t > 0 \mid X_t \geq b(t) \}$ . The proof of this fact can be carried out as follows.

2. Let  $A = \{ x \in \text{supp}(\mu) \mid \mu(\{x\}) > 0 \}$  and for any given  $x \in A$  set  $\ell_i(x) = \inf \{ t \in (0, \infty) \mid b_1(t) = x \}$  and  $r_i(x) = \sup \{ t \in (0, \infty) \mid b_1(t) = x \}$  when  $i = 1, 2$ . By (2.45) we know that  $[\ell_i(x), r_i(x)]$  is a non-empty interval. Moreover, note that the functions  $\ell_i$  and  $r_i$  are also well defined on  $\text{supp}(\mu) \setminus A$  (with the convention  $\inf \emptyset = \sup \emptyset = +\infty$ ) in which case we have  $\ell_i = r_i$  for  $i = 1, 2$ . With this notation in mind consider the sets

$$(3.1) \quad G_{1,1} = \{ x \in \text{supp}(\mu) \setminus A \mid \ell_1(x) < \ell_2(x) \}$$

$$(3.2) \quad G_{1,2} = \{ x \in A \mid r_1(x) < r_2(x) \}$$

$$(3.3) \quad G_{1,3} = \{ x \in A \mid \ell_1(x) < \ell_2(x) \ \& \ r_1(x) = r_2(x) \}$$

$$(3.4) \quad G_{2,1} = \{ x \in \text{supp}(\mu) \setminus A \mid \ell_2(x) \leq \ell_1(x) \}$$

$$(3.5) \quad G_{2,2} = \{ x \in A \mid r_2(x) < r_1(x) \}$$

$$(3.6) \quad G_{2,3} = \{ x \in A \mid \ell_2(x) < \ell_1(x) \ \& \ r_1(x) = r_2(x) \}$$

$$(3.7) \quad G_{2,4} = \{ x \in A \mid \ell_1(x) = \ell_2(x) \ \& \ r_1(x) = r_2(x) \}.$$

Set  $G_1 := G_{1,1} \cup G_{1,2} \cup G_{1,3}$  and  $G_2 := G_{2,1} \cup G_{2,2} \cup G_{2,3} \cup G_{2,4}$ . Note that  $G_1$  and  $G_2$  are disjoint and  $\text{supp}(\mu) = G_1 \cup G_2$ . Setting  $\tau_1 := \tau_{b_1}$  and  $\tau_2 := \tau_{b_2}$  we claim that

$$(3.8) \quad \mathbf{P}(X_{\tau_1} \in G_1, X_{\tau_2} \in G_2) = 0.$$

Indeed, if  $X_{\tau_1} \in G_1$  then  $X_{\tau_1} = b_1(\tau_1) \geq b_2(\tau_1)$  so that  $\tau_2 \leq \tau_1$ , while if  $X_{\tau_2} \in G_2$  then  $X_{\tau_2} = b_2(\tau_2) \geq b_1(\tau_2)$  so that  $\tau_1 \leq \tau_2$ . Since  $G_1$  and  $G_2$  are disjoint this shows that the set in (3.8) is empty and thus has  $\mathbf{P}$ -probability zero as claimed. From (3.8) we see that

$$(3.9) \quad \mathbf{P}(X_{\tau_1} \in G_1) = \mathbf{P}(X_{\tau_1} \in G_1, X_{\tau_2} \in G_1).$$

Since  $X_{\tau_1} \sim X_{\tau_2}$  this is further equal to

$$(3.10) \quad \mathbf{P}(X_{\tau_2} \in G_1) = \mathbf{P}(X_{\tau_2} \in G_1, X_{\tau_1} \in G_1) + \mathbf{P}(X_{\tau_2} \in G_1, X_{\tau_1} \in G_2)$$

from where we also see that

$$(3.11) \quad \mathbf{P}(X_{\tau_1} \in G_2, X_{\tau_2} \in G_1) = 0.$$

It follows therefore that

$$(3.12) \quad \mathbf{P}(X_{\tau_1} \in G_2) = \mathbf{P}(X_{\tau_1} \in G_2, X_{\tau_2} \in G_2).$$

From (3.9) and (3.12) we see that the sets  $\Omega_1 = \{ X_{\tau_1} \in G_1, X_{\tau_2} \in G_1 \}$  and  $\Omega_2 = \{ X_{\tau_1} \in G_2, X_{\tau_2} \in G_2 \}$  form a partition of  $\Omega$  with  $\mathbf{P}$ -probability one. Moreover, note that for  $\omega \in \Omega_1$  we have  $X_{\tau_1}(\omega) \in G_1$  so that  $\tau_2(\omega) \leq \tau_1(\omega)$  and hence  $\tau_b(\omega) = \tau_2(\omega)$ , and for  $\omega \in \Omega_2$  we have  $X_{\tau_2}(\omega) \in G_2$  so that  $\tau_1(\omega) \leq \tau_2(\omega)$  and hence  $\tau_b(\omega) = \tau_1(\omega)$ . This implies that for every  $C \in \mathcal{B}(\text{supp}(\mu))$  we have

$$(3.13) \quad \mathbf{P}(X_{\tau_b} \in C) = \mathbf{P}(\{X_{\tau_2} \in C\} \cap \Omega_1) + \mathbf{P}(\{X_{\tau_1} \in C\} \cap \Omega_2)$$

$$\begin{aligned}
&= \mathbb{P}(X_{\tau_1} \in G_1, X_{\tau_2} \in C \cap G_1) + \mathbb{P}(X_{\tau_1} \in C \cap G_2, X_{\tau_2} \in G_2) \\
&= \mathbb{P}(X_{\tau_2} \in C \cap G_1) + \mathbb{P}(X_{\tau_1} \in C \cap G_2) \\
&= \mu(C \cap G_1) + \mu(C \cap G_2) = \mu(C)
\end{aligned}$$

where we also use (3.11) in the third equality. This shows that  $X_{\tau_b} \sim \mu$  as claimed.

3. To conclude the proof we can now proceed as follows. Since  $b \leq b_i$  we see that  $X_{\tau_b} \leq X_{\tau_{b_i}}$  for  $i = 1, 2$ . Moreover, since  $X_{\tau_b} \sim X_{\tau_{b_i}}$  from the latter inequality we see that  $X_{\tau_b} = X_{\tau_{b_i}}$   $\mathbb{P}$ -a.s. for  $i = 1, 2$ . As clearly this is not possible if for some  $t > 0$  we would have  $b_1(t) \neq b_2(t)$  it follows that  $b_1 = b_2$  and the proof is complete.  $\square$

## 4. Minimality

In this section we show that the stopping time from Theorem 1 or Corollary 8 is minimal in the sense of Monroe (see [14, p. 1294]).

**Proposition 11 (Minimality).** *In the setting of Theorem 1 or Corollary 8 let  $\tau = \tau_{b,c}$  with  $c = -\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  and  $b = +\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_-$ . Let  $\sigma$  be any stopping time such that*

$$(4.1) \quad X_\sigma \sim X_\tau$$

$$(4.2) \quad \sigma \leq \tau \text{ } \mathbb{P}\text{-a.s.}$$

Then  $\sigma = \tau$   $\mathbb{P}$ -a.s.

**Proof.** Since  $\int_0^N \mathbb{P}(\sigma \geq t) dt = \mathbb{E}(\sigma \wedge N) \leq \mathbb{E}(\tau \wedge N) = \int_0^N \mathbb{P}(\tau \geq t) dt$  for all  $N \geq 1$  by (4.2) above, we see that it is enough to show that  $\mathbb{P}(\sigma \geq t) \geq \mathbb{P}(\tau \geq t)$  or equivalently

$$(4.3) \quad \mathbb{P}(\sigma < t) \leq \mathbb{P}(\tau < t)$$

for all  $t > 0$ . For this, note that from (4.1) and (4.2) combined with the facts that  $b$  and  $c$  are left-continuous increasing and decreasing functions respectively it follows that

$$\begin{aligned}
(4.4) \quad \mathbb{P}(\sigma < t) &= \mathbb{P}(\sigma < t, X_\sigma \in (c(t), b(t))) + \mathbb{P}(\sigma < t, X_\sigma \notin (c(t), b(t))) \\
&\leq \mathbb{P}(X_\sigma \in (c(t), b(t))) + \mathbb{P}(\sigma < t, \tau \leq \sigma, X_\sigma \notin (c(t), b(t))) \\
&= \mathbb{P}(X_\tau \in (c(t), b(t))) + \mathbb{P}(\sigma < t, \tau = \sigma, X_\sigma \notin (c(t), b(t))) \\
&\leq \mathbb{P}(\tau < t, X_\tau \in (c(t), b(t))) + \mathbb{P}(\tau < t, X_\tau \notin (c(t), b(t))) \\
&= \mathbb{P}(\tau < t)
\end{aligned}$$

for all  $t > 0$  proving the claim.  $\square$

**Corollary 12 (Uniform integrability).** *In the setting of Theorem 1 let  $\tau = \tau_{b,c}$  with  $c = -\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  and  $b = +\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_-$ .*

$$(4.5) \quad \text{If } \int x \mu(dx) = 0 \text{ then } \{B_{t \wedge \tau} \mid t \geq 0\} \text{ is uniformly integrable.}$$



(4.6) If  $0 < \int x \mu(dx) < +\infty$  then  $\{B_{t \wedge \tau}^+ \mid t \geq 0\}$  is uniformly integrable.

(4.7) If  $-\infty < \int x \mu(dx) < 0$  then  $\{B_{t \wedge \tau}^- \mid t \geq 0\}$  is uniformly integrable.

**Proof.** The statement (4.5) follows by combining Proposition 11 above and Theorem 3 in [14, p. 1294]. The statements (4.6) and (4.7) follow by combining Proposition 11 above and Theorem 3 in [5, p. 397]. This completes the proof.  $\square$

**Proposition 13 (Finiteness).** *In the setting of Theorem 1 suppose that  $\text{supp}(\mu) \cap \mathbb{R}_+ \neq \emptyset$  and  $\text{supp}(\mu) \cap \mathbb{R}_- \neq \emptyset$ .*

(4.8) If  $\sup \text{supp}(\mu) < \infty$  then there exists  $T > 0$  such that  $b(t) = +\infty$  for all  $t > T$  if and only if  $-\infty \leq \int x \mu(dx) < 0$ .

(4.9) If  $\inf \text{supp}(\mu) > -\infty$  then there exists  $T > 0$  such that  $c(t) = -\infty$  for all  $t > T$  if and only if  $0 < \int x \mu(dx) \leq +\infty$ .

**Proof.** It is enough to prove (4.8) since (4.9) then follows by symmetry. For this, suppose first that  $b(t) = +\infty$  for all  $t > T$  with some minimal  $T > 0$ . Since  $\sup \text{supp}(\mu) < \infty$  we know that  $b(T) < \infty$ . Set  $b_1(t) = b(t)$  for  $t \in (0, T]$  and  $b_1(t) = b(T)$  for  $t > T$ . Set  $c_1(t) = c(t)$  for  $t \in (0, T]$  and  $c_1(t) = c(T)$  for  $t > T$  (recall that  $c$  must be finite valued). Then  $|B_{t \wedge \tau_{b_1, c_1}}| \leq b(T) \vee (-c(T)) < \infty$  for all  $t \geq 0$  so that  $\{B_{t \wedge \tau_{b_1, c_1}} \mid t \geq 0\}$  is uniformly integrable and hence  $\mathbb{E}B_{\tau_{b_1, c_1}} = 0$ . Note that  $B_{\tau_{b, c}} \leq B_{\tau_{b_1, c_1}}$  and moreover  $B_{\tau_{b, c}} < B_{\tau_{b_1, c_1}}$  on the set of a strictly positive  $\mathbb{P}$ -measure where  $B$  hits  $b_1$  after  $T$  before hitting  $c_1$ . This implies that  $\mathbb{E}B_{\tau_{b, c}} < \mathbb{E}B_{\tau_{b_1, c_1}} = 0$  as claimed in (4.8) above.

Conversely, suppose that  $\mathbb{E}B_{\tau_{b, c}} < 0$  and consider first the case when  $c(t) = -\infty$  for  $t > T$  with some  $T > 0$  at which  $c(T) > -\infty$ . Set  $c_1(t) = c(t)$  for  $t \in (0, T]$  and  $c_1(t) = c(T)$  for  $t > T$ . Since  $B_{\tau_{b, c_1}} \leq \sup \text{supp}(\mu) < \infty$  when  $b$  is finite valued we see that  $|B_{t \wedge \tau_{b, c_1}}| \leq \sup \text{supp}(\mu) \vee (-c(T)) < \infty$  for all  $t \geq 0$  so that  $\{B_{t \wedge \tau_{b, c_1}} \mid t \geq 0\}$  is uniformly integrable and hence  $\mathbb{E}B_{\tau_{b, c_1}} = 0$ . Note that  $B_{\tau_{b, c}} \geq B_{\tau_{b, c_1}}$  so that  $\mathbb{E}B_{\tau_{b, c}} \geq 0$  and this contradicts the hypothesis. Next consider the case when  $c(t) > -\infty$  for all  $t \geq 0$ . Set  $c_n(t) = c(t)$  for  $t \in (0, n]$  and  $c_n(t) = -\infty$  for  $t > n$  with  $n \geq 1$ . Set  $d_n(t) = c(t)$  for  $t \in (0, n]$  and  $d_n(t) = c(n)$  for  $t > n$  with  $n \geq 1$ . Then as above  $\mathbb{E}B_{\tau_{b, d_n}} = 0$  and since  $B_{\tau_{b, c_n}} \geq B_{\tau_{b, d_n}}$  it follows that  $\mathbb{E}B_{\tau_{b, c_n}} \geq 0$  for all  $n \geq 1$ . Moreover, since  $B_{\tau_{b, c_n}} \leq \sup \text{supp}(\mu) < \infty$  for all  $n \geq 1$  when  $b$  is finite valued by Fatou's lemma we get

$$(4.10) \quad \mathbb{E}B_{\tau_{b, c}} = \mathbb{E} \lim_{n \rightarrow \infty} B_{\tau_{b, c_n}} \geq \limsup_{n \rightarrow \infty} \mathbb{E}B_{\tau_{b, c_n}} \geq 0$$

and this contradicts the hypothesis. Thus in both cases we see that  $b$  cannot be finite valued and this completes the proof.  $\square$

## 5. Optimality

In this section we show that the stopping time from Theorem 1 has the smallest truncated expectation among all stopping times that embed  $\mu$  into  $B$ . The same optimality result for stopping times arising from the filling scheme when their means are finite was derived by Chacon

[3, p. 34] using a different method of proof. The proof we present below is based on a recent proof of Rost's optimality result [21] given by Cox & Wang [6, Section 5]. The verification technique we employ avoids stochastic calculus and invokes a general martingale/Markovian result to describe the supermartingale structure. This technique applies in the setting of Corollary 8 as well and should also be of interest in other/more general settings of this kind.

**Theorem 14.** *In the setting of Theorem 1 or Corollary 8 let  $\tau = \tau_{b,c}$  with  $c = -\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  and  $b = +\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_-$ . If  $\sigma$  is any stopping time such that  $B_\sigma \sim B_\tau$  then we have*

$$(5.1) \quad \mathbb{E}(\tau \wedge T) \leq \mathbb{E}(\sigma \wedge T)$$

for all  $T > 0$ .

**Proof.** Let  $\mathbb{P}_{t,x}$  denote the probability measure under which  $\mathbb{P}_{t,x}(X_t = x) = 1$  and consider the function  $H$  defined by

$$(5.2) \quad H(t, x) = \mathbb{P}_{t,x}(\tau \leq T)$$

for  $(t, x) \in [0, T] \times \mathbb{R}$  with  $T > 0$  given and fixed. Extend  $H$  outside  $[0, T]$  by setting  $H(t, x) = 0$  for  $t > T$  and  $x \in \mathbb{R}$ . Define the (right) inverse  $\rho$  of  $b$  and  $c$  by setting

$$(5.3) \quad \begin{aligned} \rho(x) &= \inf \{ t > 0 \mid b(t) \geq x \} \quad \text{if } x \geq b(0+) \\ &= \inf \{ t > 0 \mid c(t) \leq x \} \quad \text{if } x \leq c(0+) \end{aligned}$$

Then  $x \mapsto \rho(x)$  is right-continuous and increasing on  $[b(0+), \infty)$  and left-continuous and decreasing on  $(-\infty, c(0+)]$ . Set  $D = (-\infty, c(0+)] \cup [b(0+), \infty)$  to denote the domain of  $\rho$  and note that  $\rho(x) \geq 0$  for all  $x \in D$ .

1. For  $x \in D$  such that  $\rho(x) \leq T$  and  $t \leq \rho(x)$  we have  $H(s, x) = 1$  for all  $s \in [t, \rho(x)]$ . Hence we see that the following identity holds

$$(5.4) \quad \rho(x) - t = \int_t^{\rho(x)} H(s, x) ds$$

whenever  $t \leq \rho(x) \leq T$ . Since  $H \leq 1$  we see that this identity extends as

$$(5.5) \quad \rho(x) - t \leq \int_t^{\rho(x)} H(s, x) ds$$

for  $\rho(x) < t \leq T$ . Since  $\rho(x) - t = (T - t)^+ - (T - \rho(x))^+$  for  $t \vee \rho(x) \leq T$  and  $H(s, x) = 0$  for  $s > T$  it is easily verified using the same arguments as above that (5.4) and (5.5) yield

$$(5.6) \quad (T - t)^+ \leq \int_t^{\rho(x) \wedge T} H(s, x) ds + (T - \rho(x))^+$$

for all  $t \geq 0$  and  $x \in D$ . Let us further rewrite (5.6) as follows

$$(5.7) \quad (T - t)^+ \leq F(t, x) + G(x)$$

where the functions  $F$  and  $G$  are defined by

$$(5.8) \quad F(t, x) = \int_t^T H(s, x) ds$$

$$(5.9) \quad G(x) = (T - \rho(x))^+ - \int_{\rho(x) \wedge T}^T H(s, x) ds$$

for  $t \geq 0$  and  $x \in D$ .

2. It is easily seen from definitions of  $\tau$  and  $\rho$  (using that  $b$  and  $c$  are increasing and decreasing respectively) that  $\rho(X_\tau) \geq \tau$ . Combining this with the fact that  $H(s, x) = 1$  for all  $s \in [t, \rho(x) \wedge T]$  and  $x \in D$  we see that equality in (5.6) is attained at  $(\tau, X_\tau)$ . Since (5.7) is equivalent to (5.6) it follows that

$$(5.10) \quad (T - \tau)^+ = F(\tau, X_\tau) + G(X_\tau).$$

We now turn to examining (5.7) for other stopping times.

3. To understand the structure of the function  $F$  from (5.8) define

$$(5.11) \quad D_t = \{ (s, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x \geq b(t+s) \text{ or } x \leq c(t+s) \}$$

and note by time-homogeneity of  $X$  that

$$(5.12) \quad H(t, x) = \mathbb{P}_{t,x}(\tau \leq T) = \mathbb{P}_x(\tau_t \leq T - t)$$

for  $(t, x) \in [0, T] \times \mathbb{R}$  where we set

$$(5.13) \quad \tau_t = \inf \{ s > 0 \mid X_s \in D_{t+s} \}$$

with respect to the probability measure  $\mathbb{P}_x$  under which  $\mathbb{P}_x(X_0 = x) = 1$ . Hence we see that

$$(5.14) \quad \begin{aligned} F(t, x) &= \int_t^T H(s, x) ds = \int_t^T \mathbb{P}_x(\tau_s \leq T - s) ds \\ &= \int_0^{T-t} \mathbb{P}_x(\tau_{T-s} \leq s) ds = \mathbb{E}_x \int_0^{T-t} Z_s ds \end{aligned}$$

for  $(t, x) \in [0, T] \times \mathbb{R}$  where we set

$$(5.15) \quad Z_s = I(\tau_{T-s} \leq s)$$

for  $s \in [0, T-t]$ . Noting that each  $Z_s$  is  $\mathcal{F}_s$ -measurable where  $\mathcal{F}_s = \sigma(X_r \mid 0 \leq r \leq s)$  we can now invoke a general martingale/Markovian result and conclude that

$$(5.16) \quad M_t := F(t, X_t) + \int_0^t Z_s ds$$

is a martingale with respect to  $\mathcal{F}_t$  for  $t \in [0, T]$ . Indeed, for this note that by the Markov property of  $X$  we have

$$(5.17) \quad \mathbb{E}_x(M_{t+h} \mid \mathcal{F}_t) = \mathbb{E}_x \left( F(t+h, X_{t+h}) + \int_0^{t+h} Z_s ds \mid \mathcal{F}_t \right)$$

$$\begin{aligned}
&= \mathbb{E}_x \left( \mathbb{E}_{X_{t+h}} \left( \int_0^{T-t-h} Z_s ds \right) + \int_0^{t+h} Z_s ds \mid \mathcal{F}_t \right) \\
&= \mathbb{E}_x \left( \mathbb{E}_x \left( \int_0^{T-t-h} Z_s ds \circ \theta_{t+h} \mid \mathcal{F}_{t+h} \right) + \int_0^{t+h} Z_s ds \mid \mathcal{F}_t \right) \\
&= \mathbb{E}_x \left( \int_0^T Z_s ds \mid \mathcal{F}_t \right) = \mathbb{E}_x \left( \int_t^T Z_s ds \mid \mathcal{F}_t \right) + \int_0^t Z_s ds \\
&= \mathbb{E}_x \left( \int_0^{T-t} Z_s ds \circ \theta_t \mid \mathcal{F}_t \right) + \int_0^t Z_s ds \\
&= \mathbb{E}_{X_t} \left( \int_0^{T-t} Z_s ds \right) + \int_0^t Z_s ds \\
&= F(t, X_t) + \int_0^t Z_s ds = M_t
\end{aligned}$$

for all  $0 \leq t \leq t+h \leq T$  showing that (5.16) holds as claimed. Extend the martingale  $M$  to  $(T, \infty)$  by setting  $M_t = M_T$  for  $t > T$ . Since  $F(t, x) = 0$  for  $t > T$  and  $x \in \mathbb{R}$  this is equivalent to setting  $Z_s = 0$  for  $s > T$  in (5.16) above. Since  $Z_s \geq 0$  for all  $s \geq 0$  we see from (5.16) that  $F(t, X_t)$  is a supermartingale with respect to  $\mathcal{F}_t$  for  $t \geq 0$ .

4. We next note that

$$(5.18) \quad \int_0^{t \wedge \tau} Z_s ds = 0$$

for all  $t \geq 0$ . Indeed, this is due to the fact that  $\tau_{T-s} = \inf \{ r > 0 \mid X_r \in D_{T-s+r} \} \geq \inf \{ r > 0 \mid X_r \in D_0 \} = \tau$  for all  $s \in [0, \tau \wedge T)$  since  $b$  is increasing and  $c$  is decreasing. Hence from (5.15) we see that  $Z_s = 0$  for all  $s \in [0, \tau)$  and this implies (5.18) as claimed. Combining (5.16) and (5.18) we see that  $F(t \wedge \tau, X_{t \wedge \tau})$  is a martingale with respect to  $\mathcal{F}_{t \wedge \tau}$  for  $t \geq 0$ .

5. Taking now any stopping time  $\sigma$  such that  $X_\sigma \sim X_\tau$  it follows by (5.10), (5.18), (5.16) and (5.7) using the optional sampling theorem that

$$\begin{aligned}
(5.19) \quad \mathbb{E}(T-\tau)^+ &= \mathbb{E}F(\tau, X_\tau) + \mathbb{E}G(X_\tau) = \mathbb{E}M_\tau + \mathbb{E}G(X_\sigma) \\
&= \mathbb{E}M_\sigma + \mathbb{E}G(X_\sigma) \geq \mathbb{E}F(\sigma, X_\sigma) + \mathbb{E}G(X_\sigma) \geq \mathbb{E}(T-\sigma)^+.
\end{aligned}$$

Noting that  $\mathbb{E}(T-\tau)^+ = T - \mathbb{E}(\tau \wedge T)$  and  $\mathbb{E}(T-\sigma)^+ = T - \mathbb{E}(\sigma \wedge T)$  we see that this is equivalent to (5.1) and the proof is complete.  $\square$

**Remark 15.** In the setting of Theorem 1 if  $\int x^2 \mu(dx) < \infty$  then  $\mathbb{E}B_\tau^2 < \infty$  and hence  $\mathbb{E}\tau < \infty$  since  $\tau$  is minimal (Section 4). If moreover  $\mathbb{E}\sigma < \infty$  then by Itô's formula and the optional sampling theorem we know that  $\mathbb{E}\sigma = \mathbb{E}\tau$ . When  $\int x^2 \mu(dx) = \infty$  however it is not clear a priori whether the 'expected waiting time' for  $\tau$  compares favourably with the 'expected waiting time' for any other stopping time  $\sigma$  that embeds  $\mu$  into  $B$ . The result of Theorem 14 states the remarkable fact that  $\tau$  has the smallest truncated expectation among all stopping times  $\sigma$  that embed  $\mu$  into  $B$  (note that this fact is non-trivial even when  $\mathbb{E}\tau$

and  $\mathbf{E}\sigma$  are finite). It is equally remarkable that this holds for all laws  $\mu$  with no extra conditions imposed.

The optimality result of Theorem 14 extends to more general concave functions using standard techniques.

**Corollary 16 (Optimality).** *In the setting of Theorem 1 or Corollary 8 let  $\tau = \tau_{b,c}$  with  $c = -\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_+$  and  $b = +\infty$  if  $\text{supp}(\mu) \subseteq \mathbb{R}_-$  and let  $F : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be a concave function such that  $\mathbf{E}F(\tau)$  exists. Then we have*

$$(5.20) \quad \mathbf{E}F(\tau) \leq \mathbf{E}F(\sigma)$$

for any stopping time  $\sigma$  such that  $X_\sigma \sim X_\tau$ .

**Proof.** By (5.1) we know that

$$(5.21) \quad \int_0^t \mathbf{P}(\tau > s) ds \leq \int_0^t \mathbf{P}(\sigma > s) ds$$

for all  $t \geq 0$ . It is easy to check using Fubini's theorem that for any non-negative random variable  $\rho$  we have

$$(5.22) \quad \mathbf{E}F(\rho) = F(0) - \int_0^\infty \int_0^t \mathbf{P}(\rho > s) ds F'(dt)$$

whenever  $F$  is a concave function satisfying  $tF'(t) \rightarrow 0$  as  $t \downarrow 0$  and  $F'(t) \rightarrow 0$  as  $t \rightarrow \infty$  where  $F'$  denotes the right derivative of  $F$ . Applying (5.22) to  $\tau$  and  $\sigma$  respectively, recalling that  $F'(dt)$  defines a negative measure, and using (5.21) we get (5.20) for those functions  $F$ . The general case then follows easily by tangent approximation (from the left) and/or truncation (from the right) using monotone convergence.  $\square$

**Remark 17.** In addition to the temporal optimality of  $b$  and  $c$  established in (5.20) there also exists their spatial optimality arising from the optimal stopping problem

$$(5.23) \quad \sup_{0 \leq \tau \leq T} \mathbf{E} \left( |B_\tau| - 2 \int_0^{B_\tau} F_\mu(x) dx \right)$$

where  $F_\mu$  denotes the distribution function of  $\mu$ . Indeed McConnell [13, Section 5] shows that (under his conditions) the optimal stopping time in (5.23) equals

$$(5.24) \quad \tau_* = \inf \{ t \in [0, T] \mid B_t \geq b(T-t) \text{ or } B_t \leq c(T-t) \}$$

where  $b$  and  $c$  are functions from Theorem 1 (compare (5.23) with the optimal stopping problem derived in [16]). This can be checked by Itô-Tanaka's formula and the optional sampling theorem from the local time reformulation of (5.23) that reads

$$(5.25) \quad \sup_{0 \leq \tau \leq T} \mathbf{E} \left( \int_{\mathbb{R}} \ell_\tau^x \nu(dx) - \int_{\mathbb{R}} \ell_\tau^x \mu(dx) \right)$$

where  $\ell$  is the local time of  $B$  and  $\nu$  is a probability measure on  $\mathbb{R}$  such that  $\text{supp}(\nu) \subseteq [-p, q]$  with  $\mu([-p, q]) = 0$  for some  $p > 0$  and  $q > 0$ . Since the existence and uniqueness result of Theorem 1 and Theorem 10 with  $B_0 \sim \nu$  remain valid in this case as well (recall Remark 2 and the beginning of Section 3) we see that McConnell [13, Section 5] implies that (under his conditions) the resulting stopping time (5.24) is optimal in (5.25).

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