Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann, Springer-Verlag, 2002, (295-312)
Research Report No. 419, 2000, *Dept. Theoret. Statist. Aarhus*

Solving the Poisson Disorder Problem

G. PESKIR* *and* A. N. SHIRYAEV*

The Poisson disorder problem seeks to determine a stopping time which is as close as possible to the (unknown) time of 'disorder' when the intensity of an observed Poisson process changes from one value to another. Partial answers to this question are known to date only in some special cases, and the main purpose of the present paper is to describe the structure of the solution in the general case. The method of proof consists of reducing the initial (optimal stopping) problem to a free-boundary differential-difference problem. The key point in the solution is reached by specifying when the principle of smooth fit breaks down and gets superseded by the principle of continuous fit. This can be done in probabilistic terms (by describing the sample path behaviour of the a posteriori probability process) and in analytic terms (via the existence of a singularity point of the free-boundary equation).

1. Introduction

The Poisson disorder problem is less formally stated as follows. Suppose that at time $t = 0$ we begin observing a trajectory of the Poisson process $X = (X_t)_{t \geq 0}$ whose intensity changes from λ_0 to λ_1 at some random (unknown) time θ which is assumed to take value 0 with probability π, and is exponentially distributed with parameter λ given that $\theta > 0$. Based upon the information which is continuously updated through our observation of the trajectory of X, our problem is to terminate the observation (and declare the alarm) at a time τ_* which is as close as possible to θ (measured by a cost function with parameter $c > 0$ specified below).

The problem above was first studied in [2] where a solution has been found in the case when $\lambda + c \geq \lambda_1 > \lambda_0$. This result has been extended in [1] to the case when $\lambda + c \geq \lambda_1 - \lambda_0 > 0$. Many other authors have also studied the problem from a different standpoint (see e.g. [5]). The main purpose of the present paper is to describe the structure of the solution in the general case.

The Wiener process version of the disorder problem (where the drift changes) appeared earlier (see [7]) and is now well-understood (we refer to [8, page 208] for historical comments and references). The method of proof consists of reducing the initial (optimal stopping) problem to a free-boundary differential problem which can be solved explicitly. The principle of smooth fit plays a key role in this context.

In this paper we adopt the same methodology as in the Wiener process case. A discontinuous character of the observed (Poisson) process in the present case, however, forces us to deal with a differential-difference equation forming a free-boundary problem which is more delicate. This in turn leads to a new effect of the breakdown of the smooth fit principle (and its replacement by the principle of continuous fit), and the key issue in the solution is to understand and specify when

---

*Centre for Mathematical Physics and Stochastics, supported by the Danish National Research Foundation.
Mathematics Subject Classification 2000. Primary 62M20, 60G40, 34K10. Secondary 62L15, 62C10, 60J75.
Key words and phrases: Disorder (quickest detection, change-point, disruption, disharmony) problem, Poisson process, optimal stopping, a free-boundary differential-difference problem, the principles of continuous and smooth fit, point (counting) (Cox) process, the innovation process, measure of jumps and its compensator, Itô's formula.

exactly this happens. This can be done, on one hand, in terms of the a posteriori probability process (i.e. its jump structure and sample path behaviour), and on the other hand, in terms of a singularity point of the equation from the free-boundary problem. Moreover, it turns out that the existence of such a singularity point makes explicit computations feasible.

The facts on the principles of continuous and smooth fit found here complement and further extend our findings in [6]. Problems of detecting the arrival of 'disorder' are of central importance in quality control and have also found notable industrial and other applications.

2. The Poisson disorder problem

1. The Poisson disorder problem can be formally stated as follows. Let $N^{\lambda_0} = (N_t^{\lambda_0})_{t \ge 0}$, $N^{\lambda_1} = (N_t^{\lambda_1})_{t \ge 0}$ and $L = (L_t)_{t \ge 0}$ be three independent stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P_\pi)$ with $\pi \in [0,1]$ such that:

(2.1) N^{λ_0} is a Poisson process with intensity $\lambda_0 > 0$;

(2.2) N^{λ_1} is a Poisson process with intensity $\lambda_1 > 0$;

(2.3) L is a continuous Markov chain with two states λ_0 and λ_1, initial distribution $[1-\pi\,;\,\pi]$, and transition-probability matrix $[\,e^{-\lambda t}\,,\,1-e^{-\lambda t}\,;\,0\,,\,1\,]$ for $t > 0$ where $\lambda > 0$.

Thus $P_\pi(L_0 = \lambda_1) = 1 - P_\pi(L_0 = \lambda_0) = \pi$, and given that $L_0 = \lambda_0$, there is a single passage of L from λ_0 to λ_1 at a random time $\theta > 0$ satisfying $P_\pi(\theta > t) = e^{-\lambda t}$ for all $t > 0$.

The process $X = (X_t)_{t \ge 0}$ observed is given by

$$X_t = \int_0^t I(L_{s-} = \lambda_0)\, dN_s^{\lambda_0} + \int_0^t I(L_{s-} = \lambda_1)\, dN_s^{\lambda_1} \tag{2.4}$$

and we set $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$ for $t \ge 0$. Denoting $\theta = \inf\{\, t \ge 0 \,|\, L_t = \lambda_1 \}$ we see that $P_\pi(\theta = 0) = \pi$ and $P_\pi(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ for all $t > 0$. It is assumed that *the time θ of 'disorder'* is unknown (i.e. it cannot be observed directly).

The *Poisson disorder problem* seeks to find a stopping time τ_* of X that is 'as close as possible' to θ as a solution of the following optimal stopping problem:

$$V(\pi) = \inf_\tau \Big(P_\pi(\tau < \theta) + c E_\pi(\tau - \theta)^+ \Big) \tag{2.5}$$

where $P_\pi(\tau < \theta)$ is interpreted as the probability of a '*false alarm*', $E_\pi(\tau - \theta)^+$ is interpreted as the '*average delay*' in detecting the occurrence of 'disorder' correctly, $c > 0$ is a given constant, and the infimum in (2.5) is taken over all stopping times τ of X . [A stopping time of X means a stopping time with respect to the natural filtration $(\mathcal{F}_t^X)_{t \ge 0}$ generated by X . The same terminology will be used for other processes in the sequel as well.]

2. Introducing the *a posteriori probability process*

$$\pi_t = P_\pi(\theta \le t \mid \mathcal{F}_t^X) \tag{2.6}$$

for $t \ge 0$, it is easily seen that $P_\pi(\tau < \theta) = E_\pi(1-\pi_\tau)$ and $E_\pi(\tau-\theta)^+ = E_\pi\big(\int_0^\tau \pi_t\, dt\big)$ for all stopping times τ of X , so that (2.5) can be rewritten as follows:

$$V(\pi) = \inf_\tau \; E_\pi\bigg((1-\pi_\tau) + c \int_0^\tau \pi_t \, dt \bigg) \tag{2.7}$$

where the infimum is taken over all stopping times τ of $(\pi_t)_{t \ge 0}$ (as shown following (2.11) below).

Defining the *likelihood ratio process*

$$\varphi_t = \frac{\pi_t}{1-\pi_t} \tag{2.8}$$

it is possible to verify by standard means that the following explicit expression is valid:

$$\varphi_t = e^{\lambda t} e^{X_t \log(\lambda_1/\lambda_0) - (\lambda_1-\lambda_0)t} \bigg(\varphi_0 + \lambda \int_0^t e^{-\lambda s} e^{-X_s \log(\lambda_1/\lambda_0) + (\lambda_1-\lambda_0)s} \, ds \bigg) \tag{2.9}$$

for $t \ge 0$. Hence by Itô's formula (see e.g. [3]) one finds that the processes $(\varphi_t)_{t \ge 0}$ and $(\pi_t)_{t \ge 0}$ solve the following stochastic equations respectively:

$$d\varphi_t = \lambda(1+\varphi_t)\, dt + \bigg(\frac{\lambda_1}{\lambda_0} - 1 \bigg) \varphi_{t-} \, d\big(X_t - \lambda_0 t\big) \tag{2.10}$$

$$d\pi_t = \lambda(1-\pi_t)\, dt + \frac{(\lambda_1-\lambda_0)\, \pi_{t-}(1-\pi_{t-})}{\lambda_1 \pi_{t-} + \lambda_0\, (1-\pi_{t-})} \bigg(dX_t - \Big(\lambda_1 \pi_{t-} + \lambda_0\, (1-\pi_{t-}) \Big) dt \bigg) \tag{2.11}$$

(cf. [2, page 713] or [4, page 307]). It follows that $(\varphi_t)_{t \ge 0}$ and $(\pi_t)_{t \ge 0}$ are time-homogeneous (strong) Markov processes under P_π with respect to the natural filtrations which clearly coincide with $(\mathcal{F}_t^X)_{t \ge 0}$ respectively. Thus, the infimum in (2.7) may indeed be viewed as taken over all stopping times τ of $(\pi_t)_{t \ge 0}$, and the optimal stopping problem (2.7) falls into the class of optimal stopping problems for Markov processes. We thus proceed by finding the infinitesimal operator of the Markov process $(\pi_t)_{t \ge 0}$.

By Itô's formula, upon making use of the fact easily verified (see (2.14) below) that the *innovation process* $\widehat{X}_t = X_t - \int_0^t E_\pi(L_s \mid \mathcal{F}_{s-}^X)\, ds = X_t - \int_0^t (\lambda_1 \pi_{s-} + \lambda_0 (1-\pi_{s-}))\, ds$ is a martingale under P_π with respect to $(\mathcal{F}_t^X)_{t \ge 0}$, it follows from (2.11) that the infinitesimal operator of $(\pi_t)_{t \ge 0}$ acts on $f \in C^1[0,1]$ according to the following rule:

$$\begin{aligned}(\mathbb{L} f)(\pi) = & \Big(\lambda - (\lambda_1 - \lambda_0)\, \pi \Big)\, (1-\pi)\, f'(\pi) \\ & + \Big(\lambda_1\, \pi + \lambda_0\, (1-\pi) \Big) \bigg(f\Big(\frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0\, (1-\pi)} \Big) - f(\pi) \bigg) .\end{aligned} \tag{2.12}$$

It may be noted that the equations (2.10)-(2.12) for $\lambda = 0$ reduce to the analogous equations in [6].

3. We may assume that for each $r \ge 0$ a probability measure Q_r is defined on $(\Omega, \mathcal{F})$ such that $Q_r(\theta = r) = 1$. Thus, under Q_r the observed process $X = (X_t)_{t \ge 0}$ is given by

$$X_t = \int_0^t I(s \le r)\, dN_s^{\lambda_0} + \int_0^t I(s > r)\, dN_s^{\lambda_1} \tag{2.13}$$

for all $t \geq 0$ where $r \geq 0$. It follows that P_π admits the following decomposition:

$$P_\pi = \pi\, Q_0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda r} Q_r \, dr \tag{2.14}$$

which appears to be an elegant tool, for instance, to check that the innovation process $(\widehat{X}_t)_{t \geq 0}$ defined above is a martingale under P_π .

Moreover, using (2.14) it is straightforwardly verified that the following facts are valid:

(2.15) The map $\pi \mapsto V(\pi)$ is concave (continuous) and decreasing on $[0, 1]$;

(2.16) The stopping time $\tau_* = \inf \{ \, t \geq 0 \mid \pi_t \geq B_* \, \}$ is optimal in the problem (2.5+2.7), where B_* is the smallest π from $[0, 1]$ satisfying $V(\pi) = 1 - \pi$.

Thus $V(\pi) < 1 - \pi$ for all $\pi \in [0, B_* \rangle$ and $V(\pi) = 1 - \pi$ for all $\pi \in [B_*, 1]$. It should be noted in (2.16) that $\pi_t = \varphi_t / (1 + \varphi_t)$, and hence by (2.9) we see that π_t is a (path-dependent) functional of the process X observed up to time t . Thus, by observing a trajectory of X it is possible to decide when to stop in accordance with the rule τ_* given in (2.16).

The question arises, however, to determine the optimal threshold B_* in terms of the four parameters $\lambda_0, \lambda_1, \lambda, c$ as well as to compute the value $V(\pi)$ for $\pi \in [0, B_* \rangle$ (especially for $\pi = 0$). We tackle these questions by forming a free-boundary problem.

3. A free-boundary problem

1. Being aided by the general (optimal stopping) theory of Markov processes (see e.g. [8]), and making use of the preceding facts, we are naturally led to formulate the following *free-boundary problem* for $\pi \mapsto V(\pi)$ and B_* defined above:

$$(\mathbb{L}V)(\pi) = -c\pi \quad (0 < \pi < B_*) \tag{3.1}$$

$$V(\pi) = 1 - \pi \quad (B_* \leq \pi \leq 1) \tag{3.2}$$

$$V(B_*-) = 1 - B_* \quad \text{(continuous fit)}. \tag{3.3}$$

In some cases (specified below) the following condition will be satisfied as well:

$$V'(B_*) = -1 \quad \text{(smooth fit)}. \tag{3.4}$$

However, we will also see below that this condition may fail.

Finally, it is easily verified by passing to the limit for $\pi \downarrow 0$ that each continuous solution $\pi \mapsto V(\pi)$ of the system (3.1+3.2) must necessarily satisfy:

$$V'(0+) = 0 \quad \text{(normal entrance)} \tag{3.5}$$

whenever $V(0+)$ is finite. This condition proves useful in the case when $\lambda_1 < \lambda_0$.

For a similar free-boundary differential-difference problem corresponding to the case $\lambda = 0$ above we refer to [6].

2. *Solving the free-boundary problem (3.1).* It turns out that the case $\lambda_1 < \lambda_0$ is much different from the case $\lambda_1 > \lambda_0$. Thus assume first that $\lambda_1 > \lambda_0$ and consider the equation (3.1) on $\langle 0, B]$ for some $0 < B < 1$ given and fixed. Introduce the 'step' function

$$S(\pi) = \frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0 (1-\pi)} \tag{3.6}$$

for $\pi \leq B$. Observe that $S(\pi) > \pi$ for all $0 < \pi < 1$ and find points $\ldots < B_2 < B_1 < B_0 := B$ such that $S(B_n) = B_{n-1}$ for $n \geq 1$. It is easily verified that

$$B_n = \frac{(\lambda_0)^n B}{(\lambda_0)^n B + (\lambda_1)^n (1-B)} \quad (n = 0, 1, \ldots) \ . \tag{3.7}$$

Denote $I_n = \langle B_n, B_{n-1}]$ for $n \geq 1$, and introduce the 'distance' function

$$d(\pi, B) = 1 + \left[\log\left(\frac{B}{1-B} \, \frac{1-\pi}{\pi} \right) \Big/ \log\left(\frac{\lambda_1}{\lambda_0} \right) \right] \tag{3.8}$$

for $\pi \leq B$, where $[x]$ denotes the integer part of x . Observe that d is defined to satisfy:

$$\pi \in I_n \iff d(\pi, B) = n \tag{3.9}$$

for all $0 < \pi \leq B$.

Now consider the equation (3.1) first on I_1 upon setting $V(\pi) = 1-\pi$ for $\pi \in \langle B, S(B)]$. This is then a first-order linear differential equation which can be solved explicitly. Imposing a continuity condition at B (which is in agreement with (3.3) above) we obtain a *unique* solution $\pi \mapsto V(\pi; B)$ on I_1 . It is possible to verify that the following formula holds:

$$V(\pi; B) = c_1(B) \, V_g(\pi) + V_{p,1}(\pi; B) \quad (\pi \in I_1) \tag{3.10}$$

where $\pi \mapsto V_{p,1}(\pi; B)$ is a (bounded) *particular* solution of the *non-homogeneous* equation in (3.1):

$$V_{p,1}(\pi; B) = -\frac{\lambda_0(\lambda_1 - c)}{\lambda_1(\lambda_0 + \lambda)} \, \pi + \frac{\lambda_0 \lambda_1 + \lambda c}{\lambda_1(\lambda_0 + \lambda)} \tag{3.11}$$

and $\pi \mapsto V_g(\pi)$ is a *general* solution of the *homogeneous* equation in (3.1):

$$\begin{aligned} V_g(\pi) &= \frac{(1-\pi)^{\gamma_1}}{|\, \lambda - (\lambda_1 - \lambda_0) \, \pi \,|^{\gamma_0}} \quad , \text{ if } \lambda \neq \lambda_1 - \lambda_0 \\ &= (1-\pi) \exp\left(\frac{\lambda_1}{(\lambda_1 - \lambda_0)(1-\pi)} \right) \quad , \text{ if } \lambda = \lambda_1 - \lambda_0 \end{aligned} \tag{3.12}$$

where $\gamma_1 = \lambda_1 / (\lambda_1 - \lambda_0 - \lambda)$ and $\gamma_0 = (\lambda_0 + \lambda)/(\lambda_1 - \lambda_0 - \lambda)$, and the constant $c_1(B)$ is determined by the continuity condition $V(B-; B) = 1 - B$ leading to

$$c_1(B) = -\frac{1}{V_g(B)} \left(\frac{\lambda_1 \lambda + \lambda_0 c}{\lambda_1(\lambda_0 + \lambda)} \, B - \frac{\lambda(\lambda_1 - c)}{\lambda_1(\lambda_0 + \lambda)} \right) \tag{3.13}$$

where $V_g(B)$ is obtained by replacing π in (3.12) by B. [We see from (3.11)-(3.13) however that the continuity condition at B cannot be met when B equals $\widehat{B}$ from (3.16) below unless $\widehat{B}$ equals $\lambda(\lambda_1 - c)/(\lambda\lambda_1 + c\lambda_0)$ from (4.5) below (the latter is equivalent to $c = \lambda_1 - \lambda_0 - \lambda$). Thus, if $B = \widehat{B} \neq \lambda(\lambda_1 - c)/(\lambda\lambda_1 + c\lambda_0)$ then there is no solution $\pi \mapsto V(\pi; B)$ on I_1 that satisfies $V(\pi; B) = 1 - \pi$ for $\pi \in \langle B, S(B)]$ and is continuous at B. It turns out, however, that this analytic fact has no significant implication for the solution of (2.5+2.7).]

Next consider the equation (3.1) on I_2 upon using the solution found on I_1 and setting $V(\pi) = c_1(B)\, V_g(\pi) + V_{p,1}(\pi; B)$ for $\pi \in \langle B_1, S(B_1)]$. This is then again a first-order linear differential equation which can be solved explicitly. Imposing a continuity condition over $I_2 \cup I_1$ at B_1 (which is in agreement with (2.15) above) we obtain a unique solution $\pi \mapsto V(\pi; B)$ on I_2. It turns out, however, that the general solution of this equation cannot be expressed in terms of elementary functions (unless $\lambda = 0$ as shown in [6]) but one needs, for instance, the Gauss hypergeometric function. As these expressions are increasingly complex to record, we omit the explicit formulas in the sequel.

Continuing the preceding procedure by induction as long as possible (considering the equation (3.1) on I_n upon using the solution found on I_{n-1} and imposing a continuity condition over $I_n \cup I_{n-1}$ at B_{n-1}) we obtain a unique solution $\pi \mapsto V(\pi; B)$ on I_n given as

$$V(\pi; B) = c_n(B)\, V_g(\pi) + V_{p,n}(\pi; B) \quad (\pi \in I_n) \tag{3.14}$$

where $\pi \mapsto V_{p,n}(\pi; B)$ is a (bounded) particular solution, $\pi \mapsto V_g(\pi)$ is a general solution given by (3.12), and $B \mapsto c_n(B)$ is a function of B (and the four parameters). [We will see however in Theorem 4.1 below that in the case $B > \widehat{B} > 0$ with $\widehat{B}$ from (3.16) below the solution (3.14) exists for $\pi \in \langle \widehat{B}, B]$ but explodes at $\widehat{B}$ unless $B = B_*$.]

The key difference in the case $\lambda_1 < \lambda_0$ is that $S(\pi) < \pi$ for all $0 < \pi < 1$ so that we need to deal with points $B := B_0 < B_1 < B_2 < \ldots$ such that $S(B_n) = B_{n-1}$ for $n \geq 1$. Then the facts (3.7)-(3.9) remain preserved provided that we set $I_n = [B_{n-1}, B_n\rangle$ for $n \geq 1$. In order to prescribe the initial condition when considering the equation (3.1) on I_1, we can take $B = \varepsilon > 0$ small and make use of (3.5) upon setting $V(\pi) = v$ for all $\pi \in [S(B), B\rangle$ where $v \in \langle 0, 1\rangle$ is a given number satisfying $V(B) = v$. Proceeding by induction as earlier (considering the equation (3.1) on I_n upon using the solution found on I_{n-1} and imposing a continuity condition over $I_{n-1} \cup I_n$ at B_{n-1}) we obtain a unique solution $\pi \mapsto V(\pi; \varepsilon, v)$ on I_n given as

$$V(\pi; \varepsilon, v) = c_n(\varepsilon)\, V_g(\pi) + V_{p,n}(\pi; \varepsilon, v) \quad (\pi \in I_n) \tag{3.15}$$

where $\pi \mapsto V_{p,n}(\pi; \varepsilon, v)$ is a particular solution, $\pi \mapsto V_g(\pi)$ is a general solution given by (3.12), and $\varepsilon \mapsto c_n(\varepsilon)$ is a function of ε (and the four parameters). We shall see in Theorem 4.1 below how these solutions can be used to determine the optimal $\pi \mapsto V(\pi)$ and B_*.

3. *Two key facts about the solution.* Both of these facts hold only in the case when $\lambda_1 > \lambda_0$. The first fact to be observed is that

$$\widehat{B} = \frac{\lambda}{\lambda_1 - \lambda_0} \tag{3.16}$$

is a *singularity point* of the equation (3.1) whenever $\lambda < \lambda_1 - \lambda_0$. This is clearly seen from (3.12) where $V_g(\pi) \to \infty$ for $\pi \to \widehat{B}$. The second fact of interest is that

$$\widetilde{B} = \frac{\lambda}{\lambda + c} \tag{3.17}$$

is a *smooth-fit point* of the system (3.1)-(3.3) whenever $\lambda_1 > \lambda_0$ and $c \neq \lambda_1 - \lambda_0 - \lambda$, i.e. $V'(\widetilde{B}-;\widetilde{B}) = -1$ in the notation of (3.14) above. This can be verified by (3.10) using (3.11)-(3.13). It means that $\widetilde{B}$ is the unique point which in addition to (3.1)-(3.3) has the power of satisfying the smooth-fit condition (3.4).

It may also be noted in the verification above that the equation $V'(B-;B) = -1$ has no solution when $c = \lambda_1 - \lambda_0 - \lambda$ as the only candidate $\bar{B} := \widetilde{B} = \widehat{B}$ satisfies:

$$V'(\bar{B}-;\bar{B}) = -\frac{\lambda_0}{\lambda_1} . \tag{3.18}$$

This identity follows readily from (3.10)-(3.13) upon noticing that $c_1(\bar{B}) = 0$. Thus, when c runs from $+\infty$ to $\lambda_1 - \lambda_0 - \lambda$, the smooth-fit point $\widetilde{B}$ runs from 0 to the singularity point $\widehat{B}$, and once $\widetilde{B}$ has reached $\widehat{B}$ for $c = \lambda_1 - \lambda_0 - \lambda$, the smooth-fit condition (3.4) breaks down and gets replaced by the condition (3.18) above. We will soon attest below that in all these cases the smooth-fit point $\widetilde{B}$ is actually equal to the optimal-stopping point B_* from (2.16) above.

Observe that the equation (3.1) has no singularity points when $\lambda_1 < \lambda_0$. This analytic fact reveals a key difference between the two cases.

4. Conclusions

In parallel to the two analytic properties displayed above we begin this section by stating the relevant probabilistic properties of the a posteriori probability process.

1. *Sample-path properties of* $(\pi_t)_{t \geq 0}$. First consider the case $\lambda_1 > \lambda_0$. Then from (2.11) we see that $(\pi_t)_{t \geq 0}$ can only jump towards 1 (at times of the jumps of the process X). Moreover, the sign of the drift term $\lambda(1-\pi) - (\lambda_1 - \lambda_0)\pi(1-\pi) = (\lambda_1 - \lambda_0)(\widehat{B} - \pi)(1-\pi)$ is determined by the sign of $\widehat{B} - \pi$. Hence we see that $(\pi_t)_{t \geq 0}$ has a positive drift in $[0, \widehat{B}\rangle$, a negative drift in $\langle \widehat{B}, 1]$, and a zero drift at $\widehat{B}$. Thus, if $(\pi_t)_{t \geq 0}$ starts or ends up at $\widehat{B}$, it is trapped there until the first jump of the process X occurs. At that time $(\pi_t)_{t \geq 0}$ finally leaves $\widehat{B}$ by jumping towards 1 . This also shows that after once $(\pi_t)_{t \geq 0}$ leaves $[0, \widehat{B}\rangle$ it never comes back. The sample-path behaviour of $(\pi_t)_{t \geq 0}$ when $\lambda_1 > \lambda_0$ is depicted in *Figure* 1 (Part i) below.

Next consider the case $\lambda_1 < \lambda_0$. Then from (2.11) we see that $(\pi_t)_{t \geq 0}$ can only jump towards 0 (at times of the jumps of the process X). Moreover, the sign of the drift term $\lambda(1-\pi) - (\lambda_1 - \lambda_0)\pi(1-\pi) = (\lambda + (\lambda_0 - \lambda_1)\pi)(1-\pi)$ is always positive. Thus $(\pi_t)_{t \geq 0}$ always moves continuously towards 1 and can only jump towards 0 . The sample-path behaviour of $(\pi_t)_{t \geq 0}$ when $\lambda_1 < \lambda_0$ is depicted in *Figure* 1 (Part ii) below.

2. *Sample-path behaviour and the principles of smooth and continuous fit.* With a view to (2.16), and taking $0 < B < 1$ given and fixed, we shall now examine the manner in which the process $(\pi_t)_{t \geq 0}$ enters $[B, 1]$ if starting at $B - d\pi$ where $d\pi$ is infinitesimally small. Our previous analysis then shows the following (see *Figure* 1 below).

If $\lambda_1 > \lambda_0$ and $B < \widehat{B}$, or $\lambda_1 < \lambda_0$, then $(\pi_t)_{t \geq 0}$ enters $[B, 1]$ by passing through B continuously. If, however, $\lambda_1 > \lambda_0$ and $B > \widehat{B}$ then the only way for $(\pi_t)_{t \geq 0}$ to enter $[B, 1]$ is by jumping over B . (Jumping exactly at B happens with probability zero.)

The case $\lambda_1 > \lambda_0$ and $B = \widehat{B}$ is special. If starting outside $[B, 1]$ then $(\pi_t)_{t \geq 0}$ travels towards $\widehat{B}$ by either moving continuously or by jumping. However, the closer $(\pi_t)_{t \geq 0}$ gets to $\widehat{B}$ the smaller the drift to the right becomes, and if there were no jump over $\widehat{B}$ eventually, the process $(\pi_t)_{t \geq 0}$ would never reach $\widehat{B}$ as the drift to the right tends to zero together with the distance of $(\pi_t)_{t \geq 0}$ to $\widehat{B}$. This fact can be formally verified by analysing the explicit representation of $(\varphi_t)_{t \geq 0}$ in (2.9) and using that $\pi_t = \varphi_t/(1+\varphi_t)$ for $t \geq 0$. Thus, in this case as well, the only way for $(\pi_t)_{t \geq 0}$ to enter $[\widehat{B}, 1]$ after starting at $B - d\pi$ is by jumping over to $\langle \widehat{B}, 1]$

We will demonstrate below that the sample-path behaviour of the process $(\pi_t)_{t \geq 0}$ during the entrance of $[B_*, 1]$ has a precise analytic counterpart in terms of the free-boundary problem (3.1). If the process $(\pi_t)_{t \geq 0}$ may enter $[B_*, 1]$ by passing through B_* continuously, then the smooth-fit condition (3.4) holds at B_* ; if, however, the process $(\pi_t)_{t \geq 0}$ enters $[B_*, 1]$ exclusively by jumping over B_* , then the smooth-fit condition (3.4) breaks down. In this case the continuous-fit condition (3.3) still holds at B_* , and the existence of a singularity point $\widehat{B}$ can be used to determine the optimal B_* as shown below.

3. The preceding considerations may now be summarized as follows.

Theorem 4.1

Consider the Poisson disorder problem (2.5) and the equivalent optimal-stopping problem (2.7) where the process $(\pi_t)_{t \geq 0}$ from (2.6) solves (2.11) and $\lambda_0, \lambda_1, \lambda, c > 0$ are given and fixed.

Then there exists $B_* \in \langle 0, 1 \rangle$ such that the stopping time

$$\tau_* = \inf \{ t \geq 0 \mid \pi_t \geq B_* \} \tag{4.1}$$

is optimal in (2.5) and (2.7). Moreover, the optimal cost function $\pi \mapsto V(\pi)$ from (2.5+2.7) solves the free-boundary problem (3.1)-(3.3), and the optimal threshold B_* is determined as follows.

(i): If $\lambda_1 > \lambda_0$ and $c > \lambda_1 - \lambda_0 - \lambda$, then the smooth-fit condition (3.4) holds at B_* , and the following explicit formula is valid (cf. [2] and [1]):

$$B_* = \frac{\lambda}{\lambda + c} \,. \tag{4.2}$$

In this case $B_* < \widehat{B}$ where $\widehat{B}$ is a singularity point of the free-boundary equation (3.1) given in (3.16) above (see Figure 2 below).

(ii): If $\lambda_1 > \lambda_0$ and $c = \lambda_1 - \lambda_0 - \lambda$, then the smooth-fit condition breaks down at B_* and gets replaced by the condition (3.18) above ($V'(B_*-) = -\lambda_0/\lambda_1$). The optimal threshold B_* is still given by (4.2), and in this case $B_* = \widehat{B}$ (see Figure 3 below).

(iii): If $\lambda_1 > \lambda_0$ and $c < \lambda_1 - \lambda_0 - \lambda$, then the smooth-fit condition does not hold at B_* , and the optimal threshold B_* is determined as a unique solution in $\langle \widehat{B}, 1 \rangle$ of the following equation:

$$c_{d(\widehat{B}, B_*)}(B_*) = 0 \tag{4.3}$$

where the map $B \mapsto d(\widehat{B}, B)$ *is defined in (3.8), and the map* $B \mapsto c_n(B)$ *is defined by (3.13) and (3.14) above (see Figure 4 below). In particular, when* c *satisfies:*

$$\frac{\lambda_1 \lambda_0 (\lambda_1 - \lambda_0 - \lambda)}{\lambda_1 \lambda_0 + (\lambda_1 - \lambda_0)(\lambda + \lambda_0)} \le c < \lambda_1 - \lambda_0 - \lambda \tag{4.4}$$

then the following explicit formula is valid:

$$B_* = \frac{\lambda(\lambda_1 - c)}{\lambda \lambda_1 + c \lambda_0} \tag{4.5}$$

which in the case $c = \lambda_1 - \lambda_0 - \lambda$ *reduces again to (4.2) above.*

In the cases (i)-(iii) the optimal cost function $\pi \mapsto V(\pi)$ *from (2.5+2.7) is given by (3.14) with* B_* *in place of* B *for all* $0 < \pi \le B_*$ *(with* $V(0) = V(0+)$*) and* $V(\pi) = 1 - \pi$ *for* $B_* \le \pi \le 1$ *.*

(iv): If $\lambda_1 < \lambda_0$ *then the smooth-fit condition holds at* B_* *, and the optimal threshold* B_* *can be determined using the normal entrance condition (3.5) as follows (see Figure 5). For* $\varepsilon > 0$ *small let* v_ε *denote a unique number in* $\langle 0, 1 \rangle$ *for which the map* $\pi \mapsto V(\pi; \varepsilon, v_\varepsilon)$ *from (3.15) hits the map* $\pi \mapsto 1 - \pi$ *smoothly at some* B_*^ε *from* $\langle 0, 1 \rangle$ *. Then we have:*

$$B_* = \lim_{\varepsilon \downarrow 0} B_*^\varepsilon \tag{4.6}$$

$$V(\pi) = \lim_{\varepsilon \downarrow 0} V(\pi; \varepsilon, v_\varepsilon) \tag{4.7}$$

for all $0 < \pi \le B_*$ *(with* $V(0) = V(0+)$*) and* $V(\pi) = 1 - \pi$ *for* $B_* \le \pi \le 1$ *.*

Proof. We have already established in (2.16) above that τ_* from (4.1) is optimal in (2.5) and (2.7) for some $B_* \in [0, 1]$ to be found. It thus follows by the strong Markov property of the process $(\pi_t)_{t \ge 0}$ together with (2.15) above that the optimal cost function $\pi \mapsto V(\pi)$ from (2.5+2.7) solves the free-boundary problem (3.1)-(3.3). Some of these facts will also be reproved below.

First consider the case $\lambda_1 > \lambda_0$. In Section 3.2 above it was shown that for each given and fixed $B \in \langle 0, \widehat{B} \rangle$ the problem (3.1)-(3.3) with B in place of B_* has a unique continuous solution given by the formula (3.14). Moreover, this solution is (at least) C^1 everywhere but possibly at B where it is (at least) C^0. As explained following (3.13) above, these facts also hold for $B = \widehat{B}$ when $\widehat{B}$ equals $\lambda(\lambda_1 - c)/(\lambda \lambda_1 + c \lambda_0)$ from (4.5) above. We will now show how the optimal threshold B_* is determined among all these candidates B when $c \ge \lambda_1 - \lambda_0 - \lambda$.

(i)+(ii): Since the innovation process $\widehat{X}_t = X_t - \int_0^t (\lambda_1 \pi_{s-} + \lambda_0 (1 - \pi_{s-}))\, ds$ is a martingale under P_π with respect to $(\mathcal{F}_t^X)_{t \ge 0}$, it follows by (2.11) that

$$\pi_t = \pi + \lambda \int_0^t (1 - \pi_{s-})\, ds + M_t \tag{4.8}$$

where $M = (M_t)_{t \ge 0}$ is a martingale under P_π with respect to $(\mathcal{F}_t^X)_{t \ge 0}$. Hence by the optional sampling theorem we easily find:

$$E_\pi \bigg((1 - \pi_\tau) + c \int_0^\tau \pi_t\, dt \bigg) = (1 - \pi) + (\lambda + c)\, E_\pi \bigg(\int_0^\tau \Big(\pi_t - \frac{\lambda}{\lambda + c} \Big)\, dt \bigg) \tag{4.9}$$

for all stopping times τ of $(\pi_t)_{t\ge 0}$. Recalling the sample-path behaviour of $(\pi_t)_{t\ge 0}$ in the case $\lambda_1 > \lambda_0$ as displayed in Section 4.1 above (cf. *Figure* 1 (Part i)), and the definition of $V(\pi)$ in (2.7) together with the fact that $\widetilde{B} = \lambda/(\lambda+c) \le \widehat{B}$ when $c \ge \lambda_1 - \lambda_0 - \lambda$, we clearly see from (4.9) that it is never optimal to stop $(\pi_t)_{t\ge 0}$ in $[0, \widetilde{B}\rangle$, as well as that $(\pi_t)_{t\ge 0}$ must be stopped immediately after entering $[\widetilde{B}, 1]$ as it will never return to the 'favourable' region $[0, \widetilde{B}\rangle$ again. This proves that $\widetilde{B}$ equals the optimal threshold B_* , i.e. that τ_* from (4.1) with B_* from (4.2) is optimal in (2.5) and (2.7). The claim about the breakdown of the smooth-fit condition (3.4) when $c = \lambda_1 - \lambda_0 - \lambda$ has been already established in the paragraph containing (3.18) above (cf. *Figure* 3). The general answer (4.2) has been obtained in [1].

(iii): It was shown in Section 3.2 above that for each given and fixed $B \in \langle \widehat{B}, 1\rangle$ the problem (3.1)-(3.3) with B in place of B_* has a unique continuous solution on $\langle \widehat{B}, 1]$ given by the formula (3.14). We will now show that there exists a unique point $B_* \in \langle \widehat{B}, 1\rangle$ such that $\lim_{\pi \downarrow \widehat{B}} V(\pi; B) = \pm\infty$ if $B \in \langle \widehat{B}, B_*\rangle \cup \langle B_*, 1\rangle$ and $\lim_{\pi \downarrow \widehat{B}} V(\pi; B_*)$ is finite. This point is the optimal threshold, i.e. the stopping time τ_* from (4.1) is optimal in (2.5) and (2.7). Moreover, the point B_* can be characterized as a unique solution of the equation (4.3) in $\langle \widehat{B}, 1\rangle$.

In order to verify the preceding claims we will first state the following observation which proves useful. Setting $g(\pi) = 1 - \pi$ for $0 < \pi < 1$ we have:

$$(\mathbb{L}g)(\pi) \ge -c\pi \iff \pi \ge \widetilde{B} \tag{4.10}$$

where $\widetilde{B}$ is given in (3.17). This is verified straightforwardly using (2.12).

Now since $\widehat{B}$ is a singularity point of the equation (3.1) (recall our discussion in Section 3.3 above), and moreover $\pi \mapsto V(\pi)$ from (2.5+2.7) solves (3.1)-(3.3), we see that the optimal threshold B_* from (2.16) must satisfy (4.3). This is due to the fact that a particular solution $\pi \mapsto V_{p,n}(\pi; B_*)$ for $n = d(\widehat{B}, B_*)$ in (3.14) above is taken bounded. The key remaining fact to be established is that there cannot be two (or more) points in $\langle \widehat{B}, 1\rangle$ satisfying (4.3).

Assume on the contrary that there are two such points B_1 and B_2 . We may however assume that both B_1 and B_2 are larger than $\widetilde{B}$ since for $B \in \langle \widehat{B}, \widetilde{B}\rangle$ the solution $\pi \mapsto V(\pi; B)$ is ruled out by the fact that $V(\pi; B) > 1 - \pi$ for $\pi \in \langle B - \varepsilon, B\rangle$ with $\varepsilon > 0$ small. This fact is verified directly using (3.10)-(3.13). Thus, each map $\pi \mapsto V(\pi; B_i)$ solves (3.1)-(3.3) on $\langle 0, B_i]$ and is continuous (bounded) at $\widehat{B}$ for $i = 1, 2$. Since $S(\pi) > \pi$ for all $0 < \pi < 1$ when $\lambda_1 > \lambda_0$, it follows easily from (2.12) that each solution $\pi \mapsto V(\pi; B_i)$ of (3.1)-(3.3) must also satisfy $-\infty < V(0+; B_i) < +\infty$ for $i = 1, 2$.

In order to make use of the preceding fact we shall set $h_\beta(\pi) = (1 + (\beta - 1)\widehat{B}) - \beta\pi$ for $0 \le \pi \le \widehat{B}$ and $h_\beta(\pi) = 1 - \pi$ for $\widehat{B} \le \pi \le 1$. Since both maps $\pi \mapsto V(\pi; B_i)$ are bounded on $\langle 0, \widehat{B}\rangle$ we can fix $\beta > 0$ large enough so that $V(\pi; B_i) \le h_\beta(\pi)$ for all $0 < \pi \le \widehat{B}$ and $i = 1, 2$. Consider then the auxiliary optimal stopping problem:

$$W(\pi) := \inf_{\tau} E_\pi\bigg(h_\beta(\pi_\tau) + c\int_0^\tau \pi_t\, dt \bigg) \tag{4.11}$$

where the supremum is taken over all stopping times τ of $(\pi_t)_{t\ge 0}$. Extend the map $\pi \mapsto V(\pi; B_i)$ on $[B_i, 1]$ by setting $V(\pi; B_i) = 1 - \pi$ for $B_i \le \pi \le 1$ and denote the resulting (continuous) map on $[0, 1]$ by $\pi \mapsto V_i(\pi)$ for $i = 1, 2$. Then $\pi \mapsto V_i(\pi)$ satisfies (3.1)-(3.3), and since

$B_i \geq \widetilde{B}$, we see by means of (4.10) that the following condition is also satisfied:

$$(\mathbb{L}V_i)(\pi) \geq -c\pi \tag{4.12}$$

for $\pi \in [B_i, 1]$ and $i = 1, 2$. We will now show that the preceding two facts have the power of implying that $V_i(\pi) = W(\pi)$ for all $\pi \in [0, 1]$ with either $i \in \{1, 2\}$ given and fixed.

It follows by Itô formula that

$$V_i(\pi_t) = V_i(\pi) + \int_0^t (\mathbb{L}V_i)(\pi_{s-})\, ds + M_t \tag{4.13}$$

where $M = (M_t)_{t \geq 0}$ is a martingale (under P_π) given by

$$M_t = \int_0^t \Big(V_i\big(\pi_{s-} + \Delta\pi_s\big) - V_i(\pi_{s-}) \Big)\, d\widehat{X}_s \tag{4.14}$$

and $\widehat{X}_t = X_t - \int_0^t (\lambda_1 \pi_{s-} + \lambda_0 (1 - \pi_{s-}))\, ds$ is the innovation process. By the optional sampling theorem it follows from (4.13) using (4.12) and the fact that $V_i(\pi) \leq h_\beta(\pi)$ for all $\pi \in [0, 1]$ that $V_i(\pi) \leq W(\pi)$ for all $\pi \in [0, 1]$. Moreover, defining $\tau_i = \inf\{\, t \geq 0 \mid \pi_t \geq B_i \,\}$ it is easily seen by (4.8) for instance that $E_\pi(\tau_i) < \infty$. Using then that $\pi \mapsto V_i(\pi)$ is bounded on $[0, 1]$, it follows easily by the optional sampling theorem that $E_\pi(M_{\tau_i}) = 0$. Since moreover $V_i(\pi_{\tau_i}) = h_\beta(\pi_{\tau_i})$ and $(\mathbb{L}V_i)(\pi_{s-}) = -c\pi_{s-}$ for all $s \leq \tau_i$, we see from (4.13) that the inequality $V_i(\pi) \leq W(\pi)$ derived above is actually equality for all $\pi \in [0, 1]$. This proves that $V(\pi; B_1) = V(\pi; B_2)$ for all $\pi \in [0, 1]$, or in other words, that there cannot be more than one point B_* in $\big\langle \widehat{B}, 1 \big\rangle$ satisfying (4.3). Thus, there is only one solution $\pi \mapsto V(\pi)$ of (3.1)-(3.3) which is finite at $\widehat{B}$ (see *Figure* 4 below), and the proof of the claim is complete.

(iv): It was shown in Section 3.2 above that the map $\pi \mapsto V(\pi; \varepsilon, v)$ from (3.15) is a unique continuous solution of the equation $(\mathbb{L}V)(\pi) = -c\pi$ for $\varepsilon < \pi < 1$ satisfying $V(\pi) = v$ for all $\pi \in [S(\varepsilon), \varepsilon]$. It can be checked using (3.12) that

$$V_{p,1}(\pi; \varepsilon, v) = \frac{c\lambda_0}{\lambda_1(\lambda_0 + \lambda)}\, \pi + \frac{c\lambda}{\lambda_1(\lambda_0 + \lambda)} + v \tag{4.15}$$

$$c_1(\varepsilon) = -\frac{1}{V_g(\varepsilon)} \left(\frac{c\lambda_0}{\lambda_1(\lambda_0 + \lambda)}\, \varepsilon + \frac{c\lambda}{\lambda_1(\lambda_0 + \lambda)} \right) \tag{4.16}$$

for $\pi \in I_1 = [\varepsilon, \varepsilon_1\big\rangle$ where $S(\varepsilon_1) = \varepsilon$. Moreover, it may be noted directly from (2.12) above that $\mathbb{L}(f + c) = \mathbb{L}(f)$ for every constant c , and thus $V(\pi; \varepsilon, v) = V(\pi; \varepsilon, 0) + v$ for all $\pi \in [S(\varepsilon), 1\big\rangle$. Consequently, the two maps $\pi \mapsto V(\pi; \varepsilon, v')$ and $\pi \mapsto V(\pi; \varepsilon, v'')$ do not intersect in $[S(\varepsilon), 1\big\rangle$ when v' and v'' are different.

Each map $\pi \mapsto V(\pi; \varepsilon, v)$ is concave on $[S(\varepsilon), 1\big\rangle$. This fact can be proved by a probabilistic argument using (2.14) upon considering the auxiliary optimal stopping problem (4.11) where the map $\pi \mapsto h_\beta(\pi)$ is replaced by the concave map $h_v(\pi) = v \wedge (1 - \pi)$. [It is a matter of fact that $\pi \mapsto W(\pi)$ from (4.11) is concave on $[0, 1]$ whenever $\pi \mapsto h_\beta(\pi)$ is so.] Moreover, using (3.12)+(4.15)+(4.16) in (3.15) with $n = 1$ it is possible to see that for v close to 0 we have $V(\pi; \varepsilon, v) < 0$ for some $\pi > \varepsilon$, and for v close to 1 we have $V(\pi; \varepsilon, v) > 1 - \pi$ for some $\pi > \varepsilon$ (see *Figure* 5 below). Thus a simple concavity argument implies the existence of a unique point $B_*^\varepsilon \in \langle 0,1 \rangle$ at which $\pi \mapsto V(\pi;\varepsilon,v_\varepsilon)$ for some $v_\varepsilon \in \langle 0,1 \rangle$ hits $\pi \mapsto 1-\pi$ smoothly. The key non-trivial point in the verification that $V(\pi;\varepsilon,v_\varepsilon)$ equals the value function $W(\pi)$ of the optimal stopping problem (4.11) with $\pi \mapsto h_{v_\varepsilon}(\pi)$ in place of $\pi \mapsto h_\beta(\pi)$ is to establish that $(\mathbb{L}(V(\,\cdot\,;\varepsilon,v_\varepsilon)))(\pi) \geq -c\pi$ for all $\pi \in \langle B_*^\varepsilon, S^{-1}(B_*^\varepsilon) \rangle$. Since B_*^ε is a smooth-fit point, however, this can be done using the same method which we applied in part 3 of the proof of Theorem 2.1 in [6]. Moreover, when $\varepsilon \downarrow 0$ then clearly (4.6) and (4.7) are valid (recall (2.15) and (3.5) above), and the proof of the theorem is complete. □

Concluding the paper we would like to mention that the fixed false-alarm formulation of the Poisson disorder problem (cf. [8, page 205]) raises some new interesting questions not present in the Wiener process version of the same problem.

REFERENCES

[1] DAVIS, M. H. A. (1976). A note on the Poisson disorder problem. *Banach Center Publ.* 1 (65-72).

[2] GAL'CHUK, L. I. *and* ROZOVSKII, B. L. (1971). The "disorder" problem for a Poisson process. *Theory Probab. Appl.* 16 (712-716).

[3] JACOD, J. *and* SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin Heidelberg.

[4] LIPTSER, R. S. *and* SHIRYAYEV, A. N. (1978). *Statistics of Random Processes II*. Springer-Verlag, New York.

[5] MARCELLUS, R. L. (1990). A Markov renewal approach to the Poisson disorder problem. *Comm. Statist. Stochastic Models* 6 (213-228).

[6] PESKIR, G. *and* SHIRYAEV, A. N. (1998). Sequential testing problems for Poisson processes. *Research Report* No. 400, *Dept. Theoret. Statist. Aarhus* (20 pp). *Ann. Statist.* 28, 2000, (837-859).

[7] SHIRYAEV, A. N. (1967). Two problems of sequential analysis. *Cybernetics* 3 (63-69).

[8] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules*. Springer-Verlag, New York.

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk

Albert N. Shiryaev
Steklov Mathematical Institute
Gubkina str. 8
117966 Moscow
Russia
shiryaev@mi.ras.ru

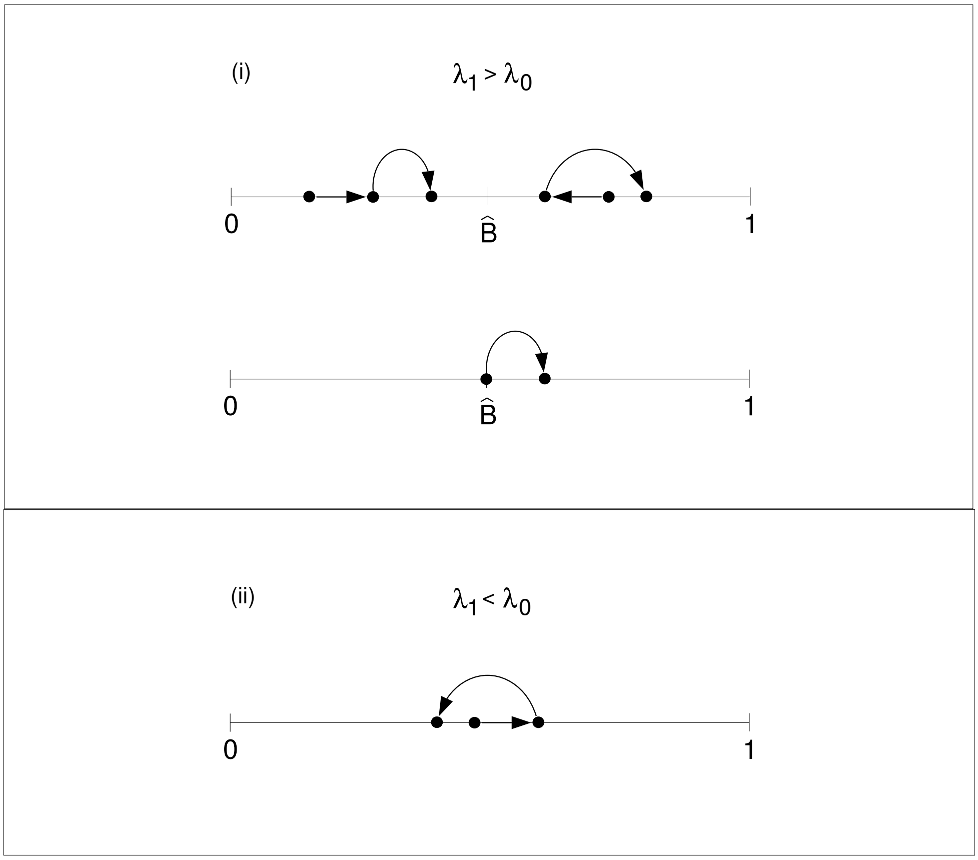

Figure 1. Sample-path properties of the a posteriori probability process $(\pi_t)_{t \geq 0}$ from (2.6+2.11). The point $\widehat{B}$ is a singularity point (3.16) of the free-boundary equation (3.1).

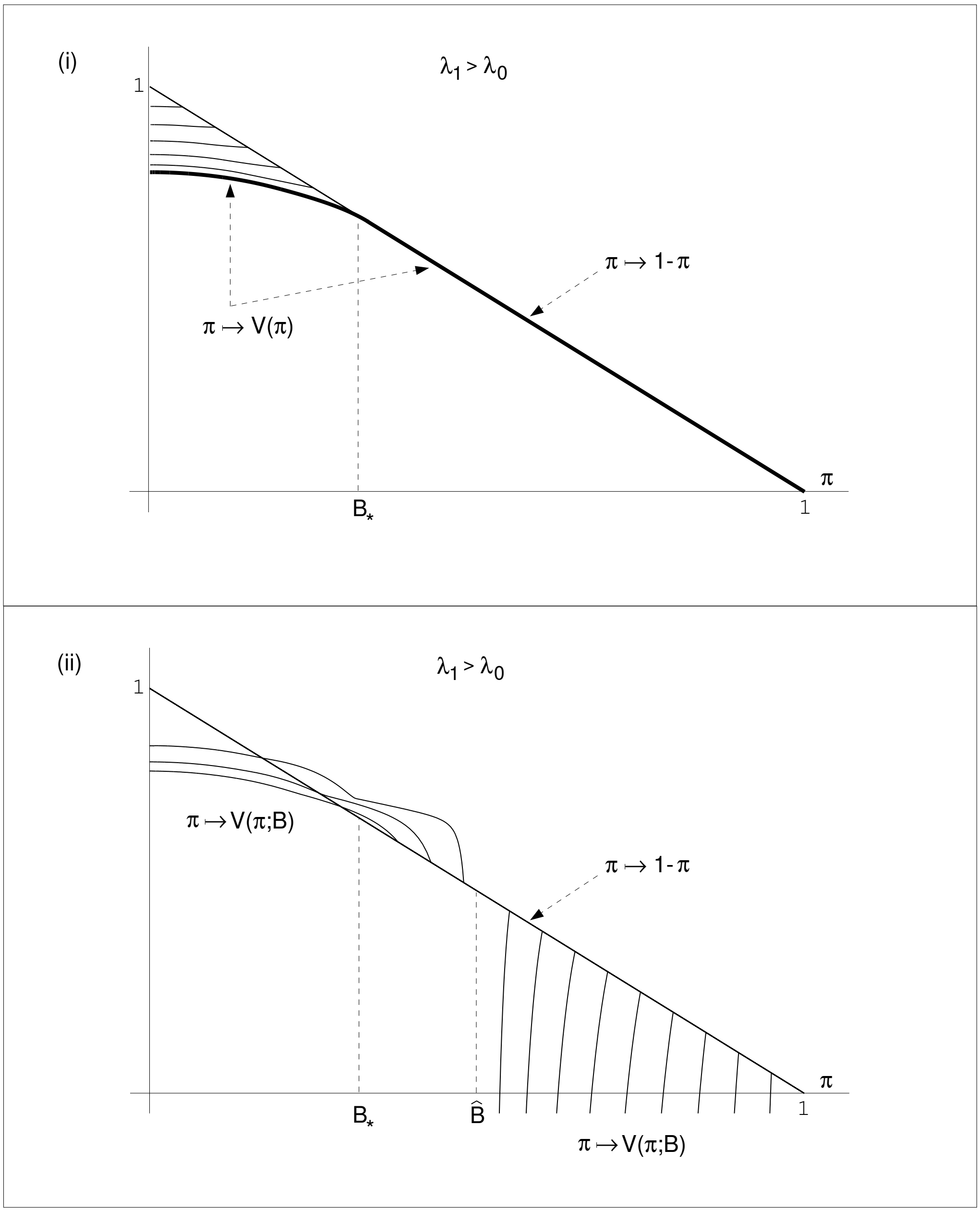

Figure 2. A computer drawing of the maps $\pi \mapsto V(\pi;B)$ from (3.14) for different B from $\langle 0,1\rangle$ in the case $\lambda_1 = 4$, $\lambda_0 = 2$, $\lambda = 1$, $c = 2$. The singularity point $\widehat{B}$ from (3.16) equals $1/2$, and the smooth-fit point $\widetilde{B}$ from (3.17) equals $1/3$. The optimal threshold B_* coincides with the smooth-fit point $\widetilde{B}$. The optimal cost function $\pi \mapsto V(\pi)$ from (2.5+2.7) equals $\pi \mapsto V(\pi;B_*)$ for $0 \le \pi \le B_*$ and $1-\pi$ for $B_* \le \pi \le 1$. (This is presented in part (i) above.) The solutions $\pi \mapsto V(\pi;B)$ for $B > B_*$ are ruled out since they fail to satisfy $0 \le V(\pi) \le 1-\pi$ for all $\pi \in [0,1]$. (This is shown in part (ii) above.) The general case $\lambda_1 > \lambda_0$ with $c > \lambda_1 - \lambda_0 - \lambda$ looks very much the same.

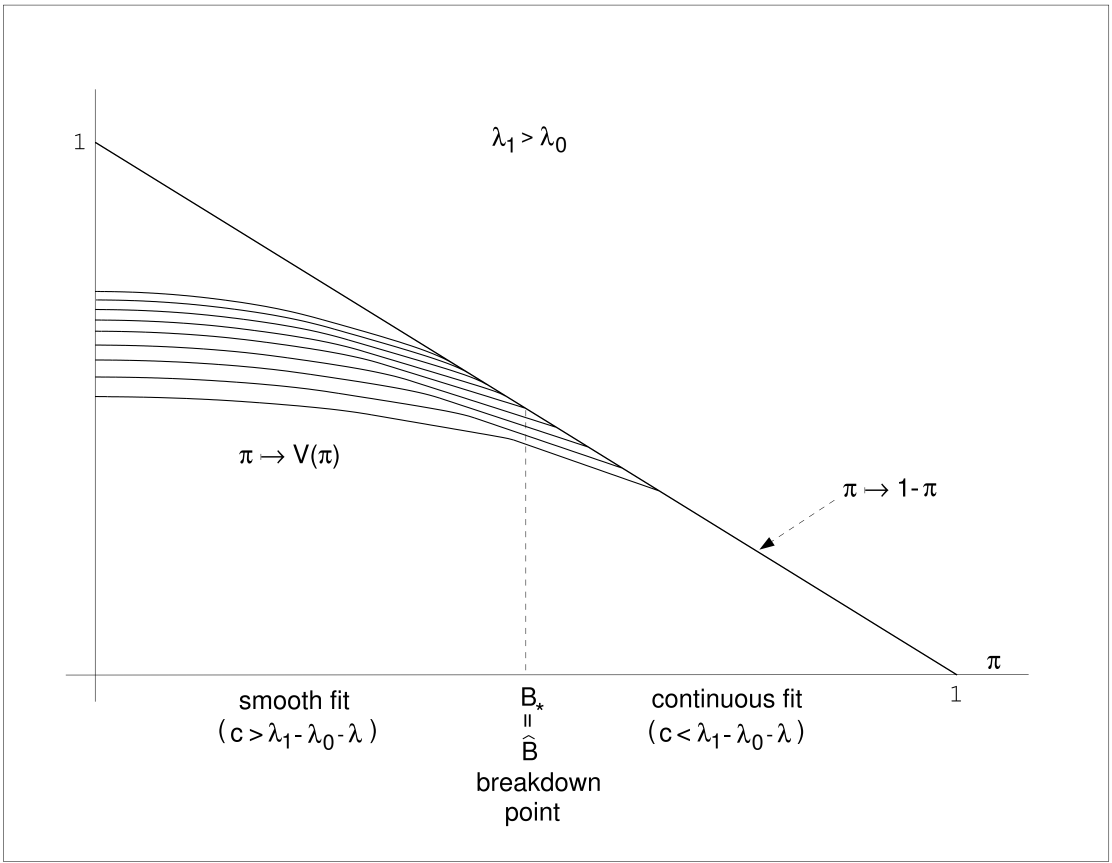

Figure 3. A computer drawing of the optimal cost functions $\pi \mapsto V(\pi)$ from (2.5+2.7) in the case $\lambda_1 = 4$, $\lambda_0 = 2$, $\lambda = 1$ and $c = 1.4,\ 1.3,\ 1.2,\ 1.1,\ 1,\ 0.9,\ 0.8,\ 0.7,\ 0.6$. The given $V(\pi)$ equals $V(\pi; B_*)$ from (3.14) for all $0 < \pi \le B_*$ where B_* as a function of c is given by (4.2) and (4.5). The smooth-fit condition (3.4) holds in the cases $c = 1.4,\ 1.3,\ 1.2,\ 1.1$. The point $c = 1$ is a breakdown point when the optimal threshold B_* equals the singularity point $\widehat{B}$ from (3.16), and the smooth-fit condition gets replaced by the condition (3.18) with $\bar{B} = B_* = \widehat{B} = 0.5$ in this case. For $c = 0.9,\ 0.8,\ 0.7,\ 0.6$ the smooth-fit condition (3.4) does not hold. In these cases the continuous-fit condition (3.3) is satisfied. Moreover, numerical computations suggest that the mapping $B_* \mapsto V'(B_*-; B_*)$ which equals -1 for $0 < B_* < \widehat{B}$ and jumps to $-\lambda_0/\lambda_1 = -0.5$ for $B_* = \widehat{B}$ is decreasing on $[\widehat{B}, 1\rangle$ and tends to a value slightly larger than -0.6 when $B_* \uparrow 1$ that is $c \downarrow 0$. The general case $\lambda_1 > \lambda_0$ looks very much the same.

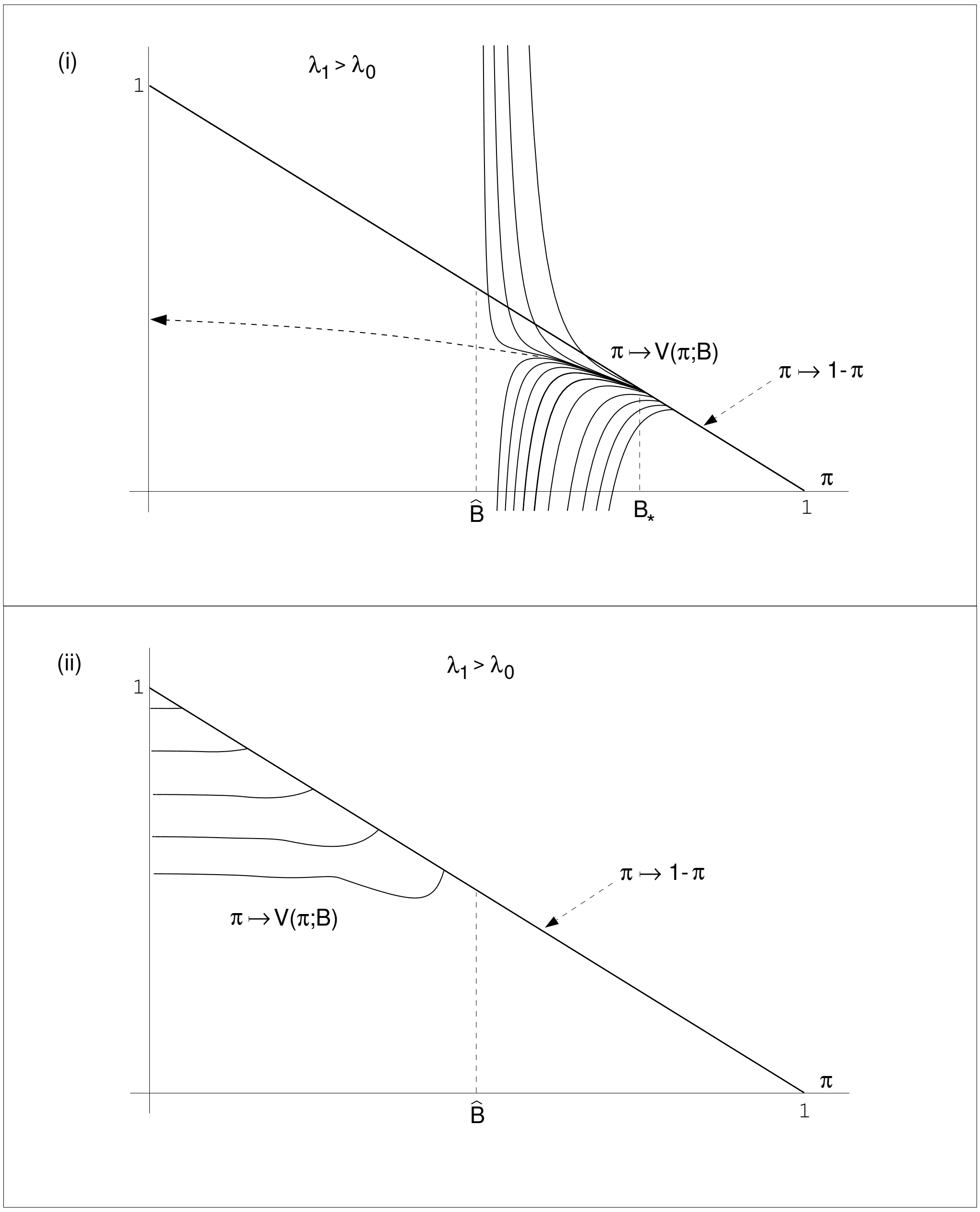

Figure 4. A computer drawing of the maps $\pi \mapsto V(\pi; B)$ from (3.14) for different B from $\langle 0,1 \rangle$ in the case $\lambda_1 = 4$, $\lambda_0 = 2$, $\lambda = 1$, $c = 2/5$. The singularity point $\widehat{B}$ from (3.16) equals $1/2$. The optimal threshold B_* can be determined from the fact that all solutions $\pi \mapsto V(\pi; B)$ for $B > B_*$ hit zero for some $\pi > \widehat{B}$, and all solutions $\pi \mapsto V(\pi; B)$ for $B < B_*$ hit $1-\pi$ for some $\pi > \widehat{B}$. (This is shown in part (i) above.) A simple numerical method based on the preceding fact suggests the following estimates $0.750 < B_* < 0.752$. The optimal cost function $\pi \mapsto V(\pi)$ from (2.5+2.7) equals $\pi \mapsto V(\pi; B_*)$ for $0 \le \pi \le B_*$ and $1-\pi$ for $B_* \le \pi \le 1$. The solutions $\pi \mapsto V(\pi; B)$ for $B \le \widehat{B}$ are ruled out since they fail to be concave. (This is shown in part (ii) above.) The general case $\lambda_1 > \lambda_0$ with $c < \lambda_1 - \lambda_0 - \lambda$ looks very much the same.

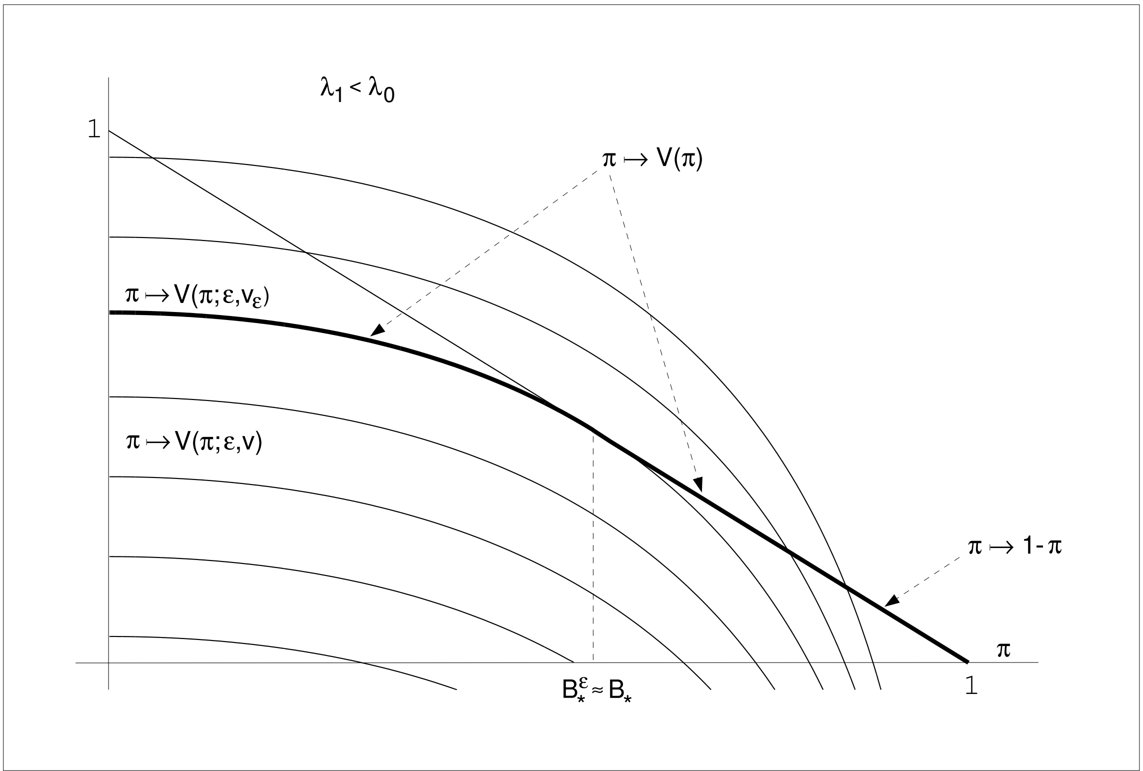

Figure 5. A computer drawing of the maps $\pi \mapsto V(\pi;\varepsilon,v)$ from (3.15) for different v from $\langle 0,1\rangle$ with $\varepsilon = 0.001$ in the case $\lambda_1 = 2$, $\lambda_0 = 4$, $\lambda = 1$, $c = 1$. For each $\varepsilon > 0$ there is a unique number $v_\varepsilon \in \langle 0,1\rangle$ such that the map $\pi \mapsto V(\pi;\varepsilon,v_\varepsilon)$ hits the map $\pi \mapsto 1-\pi$ smoothly at some $B_*^\varepsilon \in \langle 0,1\rangle$. Letting $\varepsilon \downarrow 0$ we obtain $B_*^\varepsilon \to B_*$ and $V(\pi;\varepsilon,v_\varepsilon) \to V(\pi)$ for all $\pi \in [0,1]$ where B_* is the optimal threshold from (2.16) and $\pi \mapsto V(\pi)$ is the optimal cost function from (2.5+2.7).