Theory Probab. Appl. Vol. 45, No. 1, 2000, (125-136)
Research Report No. 405, 1999, *Dept. Theoret. Statist. Aarhus*

Stopping Brownian Motion without Anticipation as Close as Possible to its Ultimate Maximum

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Let $B = (B_t)_{0 \le t \le 1}$ be standard Brownian motion started at zero, and let $S_t = \max_{0 \le r \le t} B_r$ for $0 \le t \le 1$. Consider the optimal stopping problem

$$V_* = \inf_\tau E(B_\tau - S_1)^2$$

where the infimum is taken over all stopping times of B satisfying $0 \le \tau \le 1$. We show that the infimum is attained at the stopping time

$$\tau_* = \inf \{ 0 \le t \le 1 \mid S_t - B_t \ge z_* \sqrt{1-t} \}$$

where $z_* = 1.12\ldots$ is the unique root of the equation

$$4\Phi(z_*) - 2z_*\varphi(z_*) - 3 = 0$$

with $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^x \varphi(y)\, dy$. The value V_* equals $2\Phi(z_*) - 1$. The method of proof relies upon a stochastic integral representation of S_1, time-change arguments, and the solution of a free-boundary (Stefan) problem.

1. Formulation of the problem

Imagine the real-line movement of a Brownian particle started at 0 during the time interval $[0,1]$. Let S_1 denote the *maximal* positive height that the particle ever reaches during this time interval. As S_1 is a random quantity whose values depend on the entire Brownian path over the time interval, its *ultimate* value is at any given time $t \in [0,1)$ unknown. Following the Brownian particle from the initial time 0 onward, the question arises naturally as to determine a time when the movement should be terminated so that the position of the particle at that time is as 'close' as possible to the *ultimate maximum* S_1. In this paper we present the solution to this problem if 'closeness' is measured by a mean-square distance.

To formulate the problem more precisely, let $B = (B_t)_{0 \le t \le 1}$ be a standard Brownian motion $(B_0 = 0,\, E(B_t) = 0,\, E(B_t^2) = t)$ defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathbb{F}^B = (\mathcal{F}_t^B)_{0 \le t \le 1}$ denote the natural filtration generated by B. Letting $\mathcal{M}$ denote the family of all stopping (Markov) times τ with respect to $\mathbb{F}^B$ satisfying $0 \le \tau \le 1$, the problem is to compute

$$V_* = \inf_{\tau \in \mathcal{M}} E\left(B_\tau - \max_{0 \le t \le 1} B_t\right)^2 \tag{1.1}$$

and to find an optimal stopping time (the one at which the infimum in (1.1) is attained).

*Centre for Mathematical Physics and Stochastics, supported by the Danish National Research Foundation.
MR 1991 Mathematics Subject Classification. Primary 60G40, 60J65, 62L15. Secondary 60J25, 60J60, 34B05.
Key words and phrases: Brownian motion, optimal stopping, anticipation, ultimate maximum, free-boundary (Stefan) problem, the Itô-Clark representation theorem, Markov process, diffusion. © goran@imf.au.dk

The solution of this problem is presented in Theorem 2.1 below. It turns out that the *maximum* process $S = (S_t)_{0 \le t \le 1}$ given by

$$S_t \,=\, \sup_{0 \le s \le t} B_s \tag{1.2}$$

and the *CUSUM-type* reflected process $S - B = (S_t - B_t)_{0 \le t \le 1}$ plays a key role in the solution.

The optimal stopping problem (1.1) is of interest, for example, in financial mathematics and financial engineering where an optimal decision (i.e. optimal stopping time) should be based on a *prediction* of the future behaviour of the observable process (asset price, index, etc.). The argument also carries over to many other applied problems where such predictions play a role.

2. The result and proof

The main result of the paper is contained in the next theorem. Below we let

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\; dy \quad (x \in I\!R) \tag{2.1}$$

denote the density and distribution function of a standard normal variable.

Theorem 2.1

Consider the optimal stopping problem (1.1) where $(B_t)_{0 \le t \le 1}$ is a standard Brownian motion. Then the value V_* is given by the formula

$$V_* \,=\, 2\Phi(z_*) - 1 \,=\, 0.73\ldots \tag{2.2}$$

where $z_* = 1.12\ldots$ is the unique root of the equation

$$4\Phi(z_*) - 2z_*\varphi(z_*) - 3 \,=\, 0 \tag{2.3}$$

and the following stopping time is optimal (see Figures 2-5):

$$\tau_* \,=\, \inf\, \big\{\, 0 \le t \le 1 \;\big|\; S_t - B_t \,\ge\, z_* \sqrt{1-t}\, \big\} \tag{2.4}$$

where S_t is given by (1.2) above.

Proof. Since $S_1 = \sup_{0 \le s \le 1} B_s$ is a square-integrable functional of the Brownian path on $[0,1]$, by the Itô-Clark representation theorem (see e.g. [2] p.191) there exists a unique $I\!F^B$-adapted process $H = (H_t)_{0 \le t \le 1}$ satisfying $E(\int_0^1 H_t^2\, dt) < \infty$ such that

$$S_1 \,=\, a + \int_0^1 H_t\; dB_t \tag{2.5}$$

where $a = E(S_1)$. Moreover, the following explicit formula is known to be valid:

$$H_t \,=\, 2\left(1 - \Phi\Big(\frac{S_t - B_t}{\sqrt{1-t}}\Big)\right) \tag{2.6}$$

for $0 \le t \le 1$ (see e.g. [3] p.93 and [1] p.365, or Section 3 below for a direct argument).

1. Associate with H the square-integrable martingale $M = (M_t)_{0\le t\le 1}$ given by

$$M_t = \int_0^t H_s \, dB_s \ . \tag{2.7}$$

By the martingale property of M and the optional sampling theorem, we obtain

$$\begin{aligned} E(B_\tau - S_1)^2 &= E|B_\tau|^2 - 2E(B_\tau M_1) + E|S_1|^2 \\ &= E(\tau) - 2E(B_\tau M_\tau) + 1 = E\bigg(\int_0^\tau \big(1-2H_t\big)\, dt \bigg) + 1 \end{aligned} \tag{2.8}$$

for all $\tau \in \mathcal{M}$ (recall that $S_1 \sim |B_1|$). Inserting (2.6) into (2.8) we see that (1.1) can be rewritten as

$$V_* = \inf_{\tau\in\mathcal{M}} E\bigg(\int_0^\tau F\bigg(\frac{S_t - B_t}{\sqrt{1-t}}\bigg)\, dt \bigg) + 1 \tag{2.9}$$

where we denote $F(x) = 4\Phi(x) - 3$.

Since $S - B = (S_t - B_t)_{0\le t\le 1}$ is a Markov process for which the natural filtration $I\!F^{S-B}$ coincides with the natural filtration $I\!F^B$, it follows from general theory of optimal stopping (see [4]) that in (2.9) we need only consider stopping times which are hitting times for $S-B$. Recalling moreover that $S - B \sim |B|$ by Lévy's distributional theorem (see e.g. [2] p.230) and once more appealing to general theory, we see that (2.9) is equivalent to the optimal stopping problem

$$V_* = \inf_{\tau\in\mathcal{M}} E\bigg(\int_0^\tau F\bigg(\frac{|B_t|}{\sqrt{1-t}}\bigg)\, dt \bigg) + 1 \ . \tag{2.10}$$

In our treatment of this problem, we first make use of a deterministic change of time.

2. Motivated by the form of (2.10), consider the process $Z = (Z_t)_{t\ge 0}$ given by

$$Z_t = e^t \, B_{1-e^{-2t}} \ . \tag{2.11}$$

By Itô's formula we find that Z is a (strong) solution of the linear stochastic differential equation

$$dZ_t = Z_t \, dt + \sqrt{2}\, d\beta_t \tag{2.12}$$

where the process $\beta = (\beta_t)_{0\le t\le 1}$ is given by

$$\beta_t = \frac{1}{\sqrt{2}} \int_0^t e^s \, dB_{1-e^{-2s}} = \frac{1}{\sqrt{2}} \int_0^{1-e^{-2t}} \frac{1}{\sqrt{1-s}}\, dB_s \ . \tag{2.13}$$

As β is a continuous Gaussian martingale with mean zero and variance equal to t , it follows by Lévy's characterisation theorem (see e.g. [2] p.142) that β is a standard Brownian motion. We thus may conclude that Z is a diffusion process with the infinitesimal generator given by

$$I\!L_Z = z\,\frac{d}{dz} + \frac{d^2}{dz^2} \ . \tag{2.14}$$

Substituting $t = 1 - e^{-2s}$ in (2.10) and using (2.11), we obtain

$$V_* = 2 \inf_{\tau \in \mathcal{M}} E\left(\int_0^{\sigma_\tau} e^{-2s} F\big(|Z_s|\big)\, ds \right) + 1 \tag{2.15}$$

upon setting $\sigma_\tau = \log(1/\sqrt{1-\tau})$. It is clear from (2.11) that τ is a stopping time with respect to $\mathbb{F}^B$ if and only if σ_τ is a stopping time with respect to $\mathbb{F}^Z$. This shows that our initial problem (1.1) reduces to solve

$$W_* = \inf_{\sigma} E\left(\int_0^{\sigma} e^{-2s} F\big(|Z_s|\big)\, ds \right) \tag{2.16}$$

where the infimum is taken over all $\mathbb{F}^Z$-stopping times σ with values in $[0,\infty]$. This problem belongs to the general theory of optimal stopping for time-homogeneous Markov processes (see [4]).

3. To calculate (2.16) define

$$W_*(z) = \inf_{\sigma} E_z\left(\int_0^{\sigma} e^{-2s} F\big(|Z_s|\big)\, ds \right) \tag{2.17}$$

for $z \in \mathbb{R}$, where $Z_0 = z$ under P_z , and the infimum is taken as above. General theory combined with basic properties of the map $z \mapsto F(|z|)$ prompts that the stopping time

$$\sigma_* = \inf\{\, t > 0 \,:\, |Z_t| \geq z_* \,\} \tag{2.18}$$

should be optimal in (2.17), where $z_* > 0$ is a constant to be found.

To determine z_* and compute the value function $z \mapsto W_*(z)$ in (2.17), it is a matter of routine to formulate the following *free-boundary* (*Stefan*) problem:

$$\big(\mathbb{L}_Z - 2\big) W(z) = -F(|z|) \quad \text{for} \quad z \in (-z_*, z_*) \tag{2.19}$$

$$W(\pm z_*) = 0 \qquad \text{(instantaneous stopping)} \tag{2.20}$$

$$W'(\pm z_*) = 0 \qquad \text{(smooth fit)} \tag{2.21}$$

where $\mathbb{L}_Z$ is given by (2.14) above. We shall extend the solution of (2.19)-(2.21) by setting its value equal to 0 for $z \notin (-z_*, z_*)$, and thus the map so obtained will be C^2 everywhere on $\mathbb{R}$ but at $-z_*$ and z_* where it is C^1 .

Inserting $\mathbb{L}_Z$ from (2.14) into (2.19) leads to the following equation:

$$W''(z) + zW'(z) - 2W(z) = -F(|z|) \tag{2.22}$$

for $z \in (-z_*, z_*)$. The form of the equation (2.12) and the value (2.16) indicates that $z \mapsto W_*(z)$ should be even; thus we shall additionally impose

$$W'(0) = 0 \tag{2.23}$$

and consider (2.22) only for $z \in [0, z_*)$.

The general solution of the equation (2.22) for $z \geq 0$ is given by

$$W(z) = C_1(1+z^2) + C_2\big(z\varphi(z) + (1+z^2)\Phi(z)\big) + 2\Phi(z) - 3/2 \ . \tag{2.24}$$

The three conditions $W(z_*) = W'(z_*) = W'(0) = 0$ determine constants C_1 , C_2 and z_* uniquely; it is easily verified that $C_1 = \Phi(z_*)$, $C_2 = -1$, and z_* is the unique root of the equation (2.3). Inserting this back into (2.22), we obtain the following candidate for the value (2.17):

$$W(z) = \Phi(z_*)(1+z^2) - z\varphi(z) + (1-z^2)\Phi(z) - 3/2 \tag{2.25}$$

when $z \in [0, z_*]$, upon extending it to an even function on $\mathbb{R}$ as indicated above (see *Figure 1*).

To verify that this solution $z \mapsto W(z)$ coincides with the value function (2.17), and that σ_* from (2.18) is an optimal stopping time, we shall note that $z \mapsto W(z)$ is C^2 everywhere but at $\pm z_*$ where it is C^1 . Thus by the Itô-Tanaka formula we find:

$$e^{-2t}W(Z_t) = W(Z_0) + \int_0^t e^{-2s}\Big(\mathbb{L}_Z W(Z_s) - 2\,W(Z_s)\Big)\,ds + \sqrt{2}\int_0^t e^{-2s}W'(Z_s)\,d\beta_s\ . \tag{2.26}$$

Hence by (2.22) and the fact that $\mathbb{L}_Z W(z) - 2\,W(z) = 0 > -F(|z|)$ for $z \notin [-z_*, z_*]$, upon extending W'' to $\pm z_*$ as we please and using that the Lebesgue measure of those $t > 0$ for which $Z_t = \pm z_*$ is zero, we get:

$$e^{-2t}W(Z_t) \geq W(Z_0) - \int_0^t e^{-2s}F(|Z_s|)\,ds + M_t \tag{2.27}$$

where $M = (M_t)_{t\geq 0}$ is a continuous local martingale given by $M_t = \sqrt{2}\int_0^t e^{-2s}W'(Z_s)\,d\beta_s$. Using further that $W(z) \leq 0$ for all z , a simple application of the optional sampling theorem in the stopped version of (2.27) under P_z shows that $W_*(z) \geq W(z)$ for all z . To prove equality one may note that the passage from (2.26) to (2.27) also yields:

$$0 = W(Z_0) - \int_0^{\sigma_*} e^{-2s}F(|Z_s|)\,ds + M_{\sigma_*} \tag{2.28}$$

upon using (2.19) and (2.20). Since clearly $E_z(\sigma_*) < \infty$ and thus $E_z(\sqrt{\sigma_*}) < \infty$ as well, and $z \mapsto W'(z)$ is bounded on $[-z_*, z_*]$, we can again apply the optional sampling theorem and conclude that $E_z(M_{\sigma_*}) = 0$. Taking the expectation under P_z on both sides in (2.28) enables one therefore to conclude $W_*(z) = W(z)$ for all z , and the proof of the claim is complete.

From (2.15)-(2.17) and (2.25) we find that $V_* = 2W_*(0) + 1 = 2(\Phi(z_*) - 1) + 1 = 2\Phi(z_*) - 1$. This establishes (2.2). Transforming σ_* from (2.18) back to the initial problem via the equivalence of (2.9), (2.10) and (2.15), we see that τ_* from (2.4) is optimal. The proof is complete. □

Remarks:

1. Recalling that $S - B \sim |B|$ we see that τ_* is identically distributed as the stopping time $\widetilde{\tau} = \inf\{\, t > 0 : |B_t| = z_*\sqrt{1-t}\,\}$. This implies $E(\tau_*) = E(\widetilde{\tau}) = E|B_{\widetilde{\tau}}|^2 = (z_*)^2 E(1-\widetilde{\tau}) = (z_*)^2(1 - E(\tau_*))$, and hence we obtain

$$E(\tau_*) = \frac{(z_*)^2}{1 + (z_*)^2} = 0.55\ldots \tag{2.29}$$

Moreover, using that $(B_t^4 - 6tB_t^2 + 3t^2)_{t\geq 0}$ is a martingale, similar arguments show that

$$E(\tau_*)^2 = \frac{(z_*)^6+5(z_*)^4}{(1+(z_*)^2)(3+6(z_*)^2+(z_*)^4)} = 0.36\ldots \tag{2.30}$$

From (2.29) and (2.30) we find

$$Var(\tau_*) = \frac{2(z_*)^4}{(1+(z_*)^2)^2(3+6(z_*)^2+(z_*)^4)} = 0.05\ldots \tag{2.31}$$

2. For the sake of comparison with (2.2) and (2.29) it is interesting to note that

$$V_0 = \inf_{0\le t\le 1} E\left(\left(B_t - \max_{0\le s\le 1} B_s\right)^2\right) = \frac{1}{\pi}+\frac{1}{2} = 0.81\ldots \tag{2.32}$$

with the infimum being attained at $t=1/2$. For this, recall from (2.8) and (2.6) that

$$E(B_t-S_1)^2 = E\left(\int_0^t F\left(\frac{S_s-B_s}{\sqrt{1-s}}\right)ds\right)+1 \tag{2.33}$$

where $F(x) = 4\Phi(x)-3$. Using further that $S-B\sim|B|$, elementary calculations show

$$\begin{aligned} E(B_t-S_1)^2 &= 4\left(\int_0^t E\left(\Phi\left(\frac{|B_s|}{\sqrt{1-s}}\right)\right)ds\right)-3t+1 \\ &= 4\int_0^t\left(1-\frac{1}{\pi}\arctan\sqrt{\frac{1-s}{s}}\right)ds-3t+1 \\ &= -\frac{4}{\pi}\left(t\arctan\sqrt{\frac{1-t}{t}}+\frac{1}{2}\arctan\sqrt{\frac{t}{1-t}}-\frac{1}{2}\sqrt{t\,(1-t)}\right)+t+1\ . \end{aligned} \tag{2.34}$$

Hence (2.32) is easily verified by standard means.

3. In view of the fact that σ_* from (2.18) with $z_* = 1.12\ldots$ from (2.3) is optimal in the problem (2.17), it is interesting to observe that the unique solution of the equation $F(\hat{z}) = 0$ is given by $\hat{z} = 0.67\ldots$ Noting moreover that the map $z\mapsto F(z)$ is increasing on $[0,\infty)$ and satisfies $F(0) = -1$, we see that $F(z)<0$ for all $z\in[0,\hat{z})$ and $F(z)>0$ for all $z>\hat{z}$. The size of the gap between $\hat{z}$ and z_* quantifies the tendency of the process $|Z|$ to return back to the 'favourable' region $[0,\hat{z})$ where clearly it is never optimal to stop.

4. The case of a general time interval $[0,T]$ easily reduces to the case of a unit time interval treated above by using the scaling property of Brownian motion implying

$$\inf_{0\le\tau\le T} E\left(\left(B_\tau - \max_{0\le t\le T} B_t\right)^2\right) = T\inf_{0\le\tau\le 1} E\left(\left(B_\tau - \max_{0\le t\le 1} B_t\right)^2\right) \tag{2.35}$$

which further equals to $T(2\Phi(z_*)-1)$ by (2.2). Moreover, the same argument shows that the optimal stopping time in (2.35) is given by

$$\tau_* = \inf\left\{\, 0\le t\le T \mid S_t-B_t \ge z_*\sqrt{T-t}\,\right\} \tag{2.36}$$

where z_* is the same as in Theorem 2.1.

5. The maximum functional in the argument above can be replaced by other functionals. The

integral functional is an example which turns out to have a trivial solution.

Setting $I_1 = \int_0^1 B_t\,dt$ we find by Itô's formula that the following analogue of (2.5) is valid:

$$I_1 \,=\, \int_0^1 (1-t)\,dB_t\ . \tag{2.37}$$

Denoting $M_t \,=\, \int_0^t (1-s)\,dB_s$ it follows as in (2.8) that

$$E(B_\tau - I_1)^2 = E|B_\tau|^2 - 2E(B_\tau M_1) + E|I_1|^2 = E(\tau^2-\tau) + 1/3 \tag{2.38}$$

for all $\tau \in \mathcal{M}$. Hence we see that (cf. (2.40) below):

$$\inf_{\tau\in\mathcal{M}} E(B_\tau - I_1)^2 \,=\, 1/12 \,=\, 0.08\ldots \tag{2.39}$$

and that the infimum is attained at $\tau_* \equiv 1/2$.

6. From the point of view of mathematical statistics, the "estimator" B_τ of S_1 is *biased*, since $E(B_\tau) = 0$ for all $0 \le \tau \le 1$ but $E(S_1) \neq 0$. It is thus desirable to consider the values

$$\widetilde{V}_* = \inf_{a\in I\!R\,,\,\tau\in\mathcal{M}} E\Big(a + B_\tau - S_1\Big)^2 \quad\text{and}\quad \widetilde{V}_0 = \inf_{a\in I\!R\,,\,0\le t\le 1} E\Big(a + B_t - S_1\Big)^2 \tag{2.40}$$

and compare them with the values from (1.1) and (2.32). However, by using that $E(B_\tau) = 0$ we also find at once that $a_* = E(S_1)$ is optimal in (2.40) with $\widetilde{V}_* = V_* - 2/\pi = 0.09\ldots$ and $\widetilde{V}_0 = V_0 - 2/\pi = 0.18\ldots$

3. Stochastic integral representation of the maximum process

In this section we present a *direct* derivation of the stochastic integral representation (2.5) and (2.6) (cf. [3] p.89-93 and [1] p.363-369). For the sake of comparison we shall deal with a standard Brownian motion with drift given by

$$B_t^\mu \,=\, B_t + \mu t \tag{3.1}$$

where μ is a real number. The maximum process S^μ associated with B^μ is given by

$$S_t^\mu \,=\, \sup_{0\le s\le t} B_s^\mu\ . \tag{3.2}$$

1. To derive the analogue of (2.5) and (2.6) in this case, we shall first note that *stationary independent increments* of B^μ imply

$$\begin{aligned} E\big(S_1^\mu \,|\, \mathcal{F}_t^B\big) &= S_t^\mu + E\bigg(\Big(\sup_{t\le s\le 1} B_s^\mu - S_t^\mu\Big)^+ \,\Big|\, \mathcal{F}_t^B\bigg) \\ &= S_t^\mu + E\bigg(\Big(\sup_{t\le s\le 1} (B_s^\mu - B_t^\mu) - (S_t^\mu - B_t^\mu)\Big)^+ \,\Big|\, \mathcal{F}_t^B\bigg) \\ &= S_t^\mu + E\Big(S_{1-t}^\mu - (z-x)\Big)^+\Big|_{z=S_t^\mu\,,\,x=B_t^\mu}\ . \end{aligned} \tag{3.3}$$

Using further the formula $E(X-c)^+ = \int_c^\infty P\{X>z\}\,dz$, we see that (3.3) reads as

$$E\big(S_1^\mu \,|\, \mathcal{F}_t^B\big) = S_t^\mu + \int_{S_t^\mu - B_t^\mu}^\infty \big(1 - F_{1-t}^\mu(z)\big)\,dz := f(t, B_t^\mu, S_t^\mu) \tag{3.4}$$

where we use the following notation:

$$F_{1-t}^\mu(z) = P\big\{S_{1-t}^\mu \le z\big\} \tag{3.5}$$

and the map $f = f(t,x,s)$ is defined accordingly.

2. Applying Itô's formula to the right-hand side of (3.4), and using that the left-hand side defines a continuous martingale, we find upon setting $a_\mu = E(S_1^\mu)$ that

$$\begin{aligned} E\big(S_1^\mu \,|\, \mathcal{F}_t^B\big) &= a_\mu + \int_0^t \frac{\partial f}{\partial x}(s, B_s^\mu, S_s^\mu)\,dB_s \\ &= a_\mu + \int_0^t \big(1 - F_{1-s}^\mu(S_s^\mu - B_s^\mu)\big)\,dB_s \end{aligned} \tag{3.6}$$

as a non-trivial continuous martingale cannot have paths of bounded variation. This reduces the initial problem to the problem of calculating (3.5).

3. The following explicit formula is well-known (see e.g. [1] p.368 or [5] p.759-760):

$$F_{1-t}^\mu(z) = \Phi\left(\frac{z-\mu(1-t)}{\sqrt{1-t}}\right) - e^{2\mu z}\Phi\left(\frac{-z-\mu(1-t)}{\sqrt{1-t}}\right). \tag{3.7}$$

Inserting this into (3.6) we obtain the representation

$$S_1^\mu = a_\mu + \int_0^1 H_t^\mu\,dB_t \tag{3.8}$$

where the process H^μ is explicitly given by

$$H_t^\mu = 1 - \Phi\left(\frac{(S_t^\mu - B_t^\mu) - \mu(1-t)}{\sqrt{1-t}}\right) + e^{2\mu(S_t^\mu - B_t^\mu)}\,\Phi\left(\frac{-(S_t^\mu - B_t^\mu) - \mu(1-t)}{\sqrt{1-t}}\right). \tag{3.9}$$

Setting $\mu = 0$ in this expression, we recover (2.5) and (2.6).

4. Note that the argument above extends to a large class of processes with stationary independent increments (including Lévy processes) by reducing the initial problem to calculating the analogue of (3.5). In particular, the following "prediction" result deserves a special note. It is derived in exactly the same way as (3.4) above.

Let $X = (X_t)_{0\le t\le T}$ be a process with stationary independent increments started at zero, and let us denote $S_t = \max_{0\le s\le t} X_s$ for $0 \le t \le T$. If $E(S_T) < \infty$ then the predictor $E(S_T \,|\, \mathcal{F}_t^X)$ of S_T based on the observations $\{\, X_s \,|\, 0 \le s \le t \,\}$ is given by the following formula:

$$E\big(S_T \,|\, \mathcal{F}_t^X\big) = S_t + \int_{S_t - X_t}^\infty \big(1 - F_{T-t}(z)\big)\,dz \tag{3.10}$$

where $F_{T-t}(z) = P\{S_{T-t} \le z\}$.

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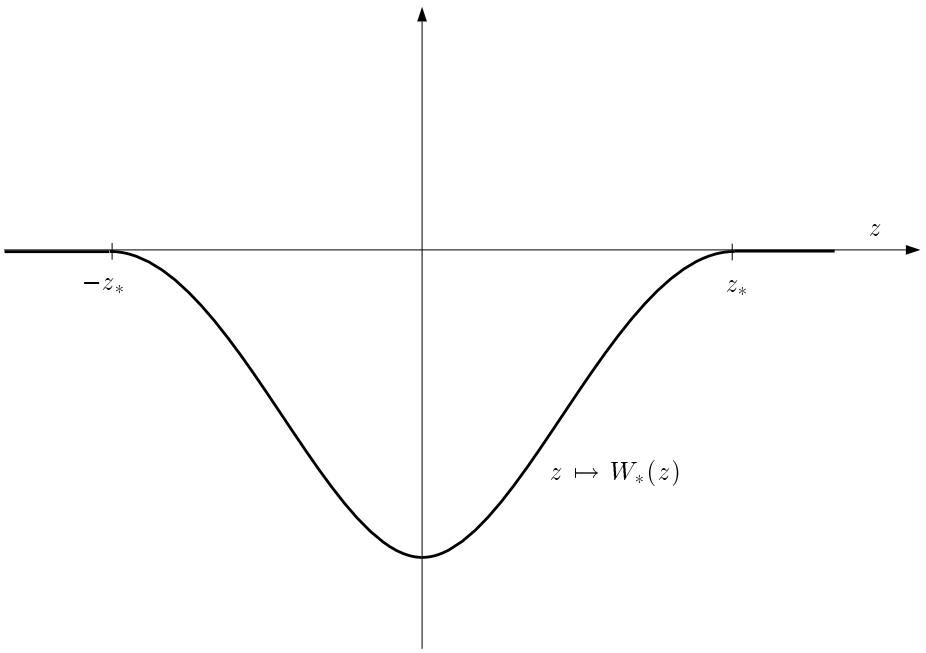

Figure 1. A computer drawing of the map (2.17). The smooth fit (2.21) holds at $-z_*$ and z_* .

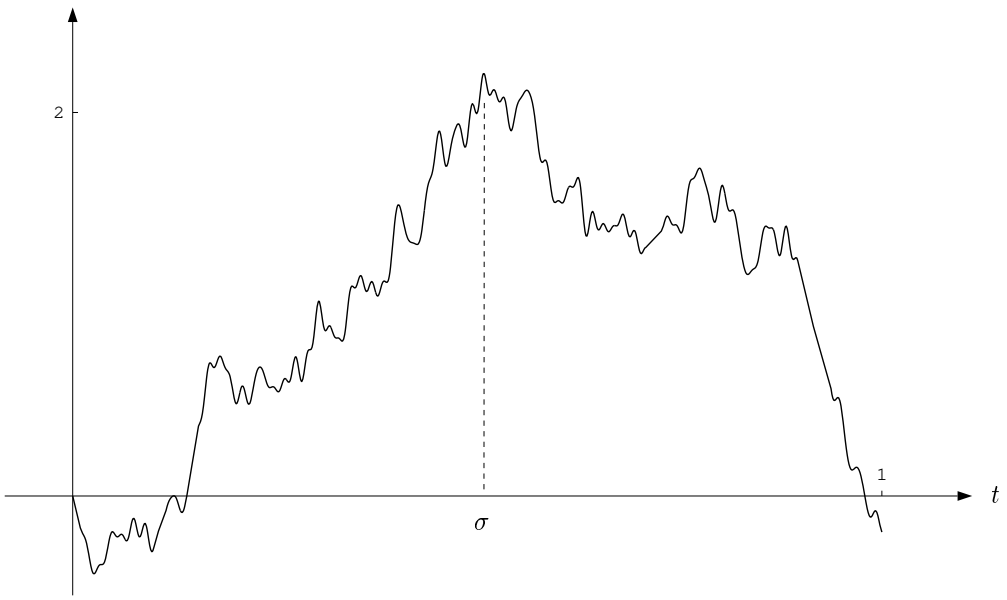

Figure 2. A computer simulation of a Brownian path $(B_t(\omega))_{0\le t\le 1}$ with the maximum being attained at $\sigma = 0.51$.

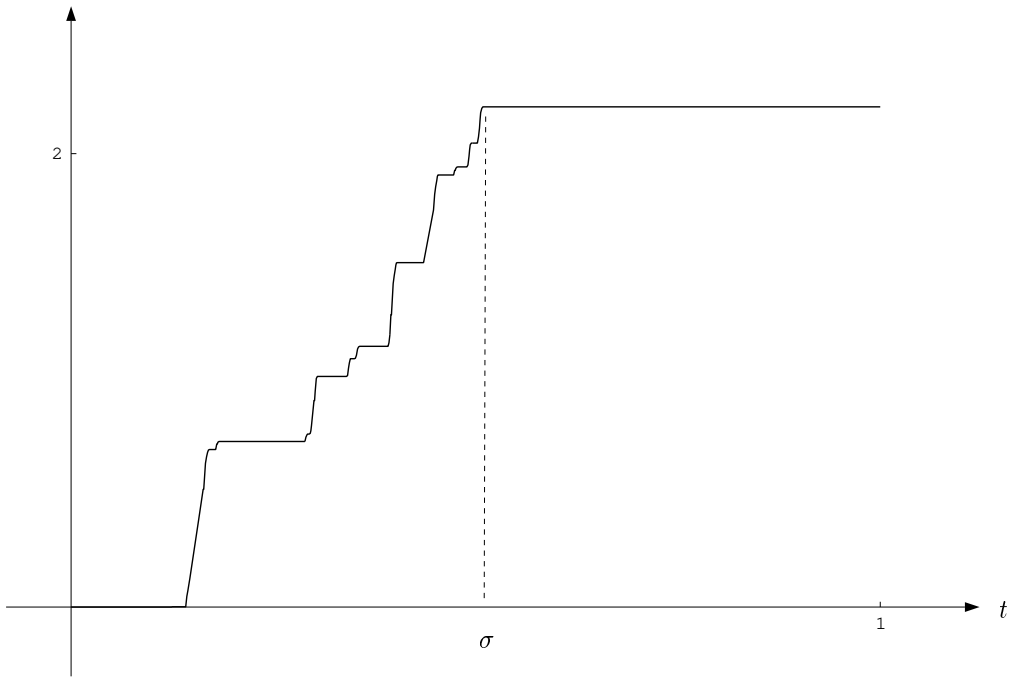

Figure 3. A computer drawing of the maximum process $(S_t(\omega))_{0\leq t\leq 1}$ associated with the Brownian path from *Figure 2*.

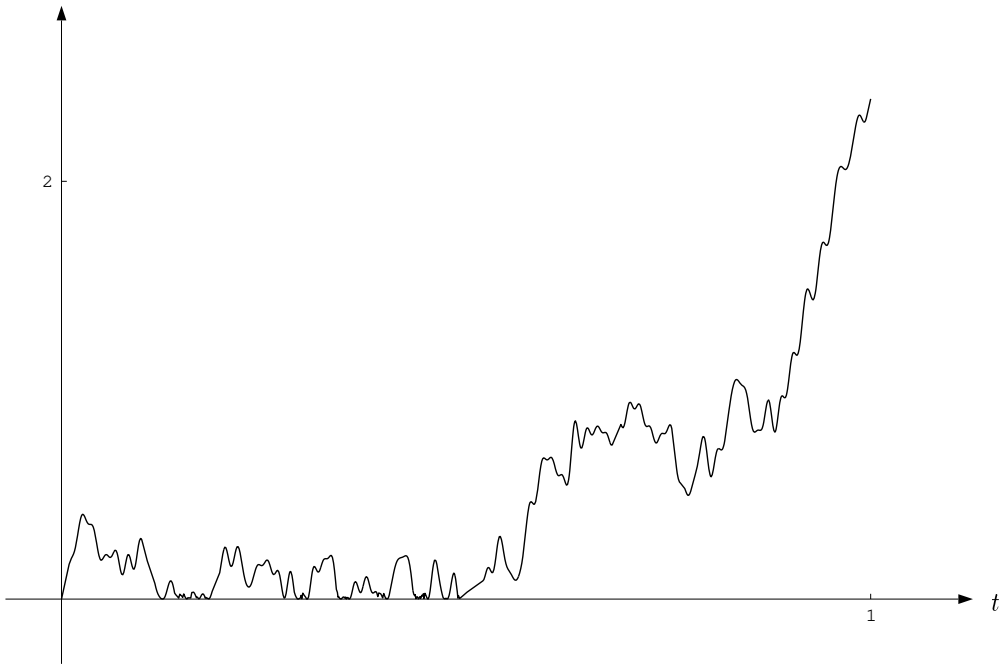

Figure 4. A computer drawing of the difference process $(S_t(\omega) - B_t(\omega))_{0\leq t\leq 1}$ from *Figures 2-3*.

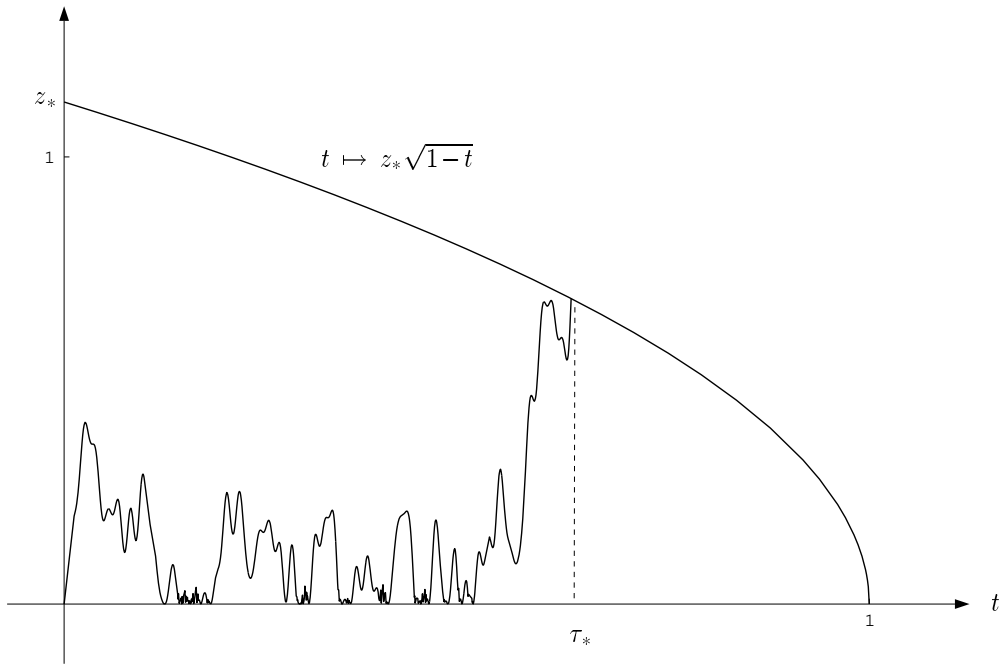

Figure 5. A computer drawing of of the optimal stopping strategy (2.4) for the Brownian path from *Figures 2-4*. It turns out that $\tau_* = 0.62$ in this case (cf. *Figure 2*).