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The Azéma-Yor Embedding in Brownian Motion with Drift

Goran Peskir*

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero, let $\mu > 0$ be given and fixed, and let ν be a probability measure on $I\!R$ having a strictly positive density F'. Then there exists a stopping time τ_* of B such that

$$(B_{\tau_*} + \mu \tau_*) \sim \nu$$

if and only if the following condition is satisfied:

$$D_\mu := \int_{I\!R} e^{-2\mu x} \, \nu(dx) \leq 1 \ .$$

Setting in this case $C_\mu = -(2\mu)^{-1} \log(D_\mu)$, the following *explicit* formula is valid:

$$\tau_* = \inf \left\{ \, t > 0 \ \Big| \ (B_t + \mu t) \leq h_\mu\Big(\max_{0 \leq r \leq t} (B_r + \mu r) \Big) \right\}$$

where the map $s \mapsto h_\mu(s)$ for $s > C_\mu$ is expressed through its inverse by

$$h_\mu^{-1}(x) = -\frac{1}{2\mu} \log\left(\frac{1}{1-F(x)} \int_x^\infty e^{-2\mu t} \, dF(t) \right) \quad (x \in I\!R)$$

and we set $h_\mu(s) = -\infty$ for $s \leq C_\mu$. This settles the question raised in [6]. In addition, it is proved that τ_* is pointwise the smallest possible stopping time satisfying $(B_{\tau_*} + \mu \tau_*) \sim \nu$ which generates stochastically the largest possible maximum of the process $(B_t + \mu t)_{t \geq 0}$ up to the time of stopping. This *minimax* property characterizes τ_* uniquely. The result recovers the Azéma-Yor solution of the Skorokhod embedding problem [1] by passing to the limit when $\mu \downarrow 0$. The condition on the existence of a strictly positive density is imposed for simplicity, and more general cases can be treated similarly. The line of arguments used in the proof can be extended to treat the case of more general *nonrecurrent* diffusions.

1. Introduction

This note is motivated by the following question raised in [6]: Given a Brownian motion with drift $X_t = B_t + \mu t$, and a probability measure ν on $I\!R$, find a stopping time τ_* of $(X_t)_{t \geq 0}$ satisfying $X_{\tau_*} \sim \nu$, and determine conditions on ν which make such a construction possible.

It is shown in [6] that under $\mu > 0$, the following condition:

$$\int_{I\!R} e^{-2\mu x} \, \nu(dx) = 1 \tag{1.1}$$

is sufficient to carry out an explicit construction of τ_*. Observe, however, that no $\nu \neq \delta_0$ with support in $I\!R_+$ satisfies (1.1).

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The main purpose of this note is to show that the following condition:

$$\int_{I\!R} e^{-2\mu x}\, \nu(dx) \le 1 \tag{1.2}$$

is necessary and sufficient to carry out an explicit construction of τ_* . In addition, some extremal properties of τ_* are revealed which makes it interesting for applications.

It may be noted that the simple convolution-type argument applied in the proof (following (3.26) below) can also be used to extend the construction given in [6] to the general case of (1.2). This procedure should also lead to the Azéma-Yor stopping time (3.3). The derivation presented below is entirely different and reveals in a clearer manner how (1.2) is actually found.

The problem of embedding the given law ν into a Markov process $X = (X_t)_{t \ge 0}$ has been a subject of many studies. This problem was initiated by Skorokhod [15] when X is Brownian motion, and it is usually referred to as the *Skorokhod embedding problem*. Rost [14] characterizes the existence of an embedding in a general Markov process. Falkner and Fitzsimmons [5] characterize the measures which can arise as stopped distributions of a transient right process for which semipolar sets are polar. These results, however, do not offer an explicit construction of the stopping time. Azéma and Yor [1] give an explicit construction of the stopping time when X is a recurrent diffusion. Bertoin and Le Jan [3] give yet another explicit construction of the stopping time when X is a Hunt process starting at a regular recurrent point. More general embedding problems for martingales are posed and solved by Rogers [13]. In view of these results, the most interesting feature in the problem above is the fact that the Brownian motion with drift *is not recurrent*.

The idea applied in the proof below is simple and well-known: The initial problem is first transformed into an analogous martingale problem by composing X with its scale function (it should be observed that the martingale problem has some new interesting features because the initial diffusion was not recurrent); we then proceed by deriving a differential equation for the optimal stopping boundary (this step is crucial in our approach); solving then the martingale problem, we also solve the initial problem. Note that, in essence, no new argument is used in the proof (if one compares it with [1] for instance) apart from the fact that we insist on deriving a differential equation for the optimal boundary; this line of arguments clarifies the proof and makes it also more tractable for a treatment of other *non-recurrent* diffusions. Actually, the main virtue of this note is to indicate that such extensions can be worked out along the same lines.

2. The Azéma-Yor embedding

Since our construction below extends the Azéma-Yor construction [1], we shall recall a few basic facts in this direction for comparison.

1. Let $B = (B_t)_{t \ge 0}$ be standard Brownian motion started at zero, and let ν be a centered probability measure on $I\!R$. Assume for simplicity that ν has a strictly positive density F' , and denote by $S_t = \max_{0 \le r \le t} B_r$ the maximum process associated with B . Then the Azéma-Yor solution [1] of the Skorokhod-embedding problem states that the stopping time

$$\tau_* = \inf \left\{ \, t > 0 \mid B_t \le h_0(S_t) \, \right\} \tag{2.1}$$

satisfies $B_{\tau_*} \sim \nu$, if the map $s \mapsto h_0(s)$ is defined through its inverse by

$$h_0^{-1}(x) = \frac{1}{1-F(x)} \int_x^\infty t \; dF(t) \tag{2.2}$$

for $x \in I\!R$. The map $x \mapsto h_0^{-1}(x)$ is the *barycentre function* of ν .

2. Using that $F(B_{\tau_*}) \sim U(0,1)$, and substituting $F(t) = v$ in (2.2), we see that $S_{\tau_*} = h_0^{-1}(B_{\tau_*})$ is equally distributed as the *Hardy-Littlewood maximal function* of ν :

$$H_0(u) = \frac{1}{1-u} \int_u^1 F^{-1}(v) \; dv \tag{2.3}$$

being defined on the probability space $[0,1]$ with Lebesgue measure (see [7]). From this fact, and the well-known argument of Blackwell and Dubins [4], it follows that

$$P\big(S_\sigma \geq s\big) \leq P\big(S_{\tau_*} \geq s\big) \tag{2.4}$$

for all $s \geq 0$, whenever σ is a stopping time of B satisfying $B_\sigma \sim \nu$ and $E(S_\sigma) < \infty$. These facts were observed in [2], as well as that $E(S_{\tau_*}) < \infty$ if and only if $\int_0^\infty x \log x \; \nu(dx) < \infty$. Assuming then that this holds, it was shown in [11] (see also [16]) that if equality in (2.4) is attained for all $s \geq 0$, then $\tau_* \leq \sigma$ a.s. (Equality actually can be established here if one proceeds further by mimicking the proof of Proposition 3.3 below.) Observe that this result can also be restated as follows: If $\int_0^\infty x \log x \; \nu(dx) < \infty$ and σ is any stopping time of B satisfying $B_\sigma \sim \nu$ and $S_\sigma \sim S_{\tau_*}$, then $\tau_* \leq \sigma$ a.s. Thus, less formally, it says that τ_* is pointwise the smallest possible stopping time satisfying $B_{\tau_*} \sim \nu$ which generates stochastically the largest possible maximum of the process B up to the time of stopping. Another equivalent formulation (closer to terms of Hardy-Littlewood theory [7]) is that τ_* is pointwise the smallest possible stopping time satisfying $B_{\tau_*} \sim \nu$ and $S_{\tau_*} \sim H_0$, whenever $\nu \in L^+ \log L^+$ or $H_0 \in L^1$. The minimax property just described characterizes τ_* uniquely, and makes it also more interesting for applications, e.g. in option pricing theory (see [11]).

3. Yet another extremal property of τ_* follows from the argument of Monroe [9] upon the uniform integrability of $(B_{t \wedge \tau_*})_{t \geq 0}$ established by Azéma and Yor [2]: If $\sigma \leq \tau_*$ is a stopping time of B satisfying $B_\sigma \sim \nu$, then $\sigma = \tau_*$ a.s.

3. The results and proof

The basic result of this note is contained in the following theorem. The explicit formulas (3.3)-(3.4) appearing below should be compared with their relatives (2.1)-(2.2) above. Observe that the latter corresponds to the limiting case $\mu = 0$ in the former. With a view of extending this result to more general non-recurrent diffusions, we would like point out the line of arguments used in the proof which makes the whole construction possible. Below in Proposition 3.2 and Proposition 3.3 we shall delve deeper into the structure of the stopping time (3.3) and reveal some of its extremal properties which make it interesting for applications.

Theorem 3.1

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero, let $\mu > 0$ be given and fixed, and let ν be a probability measure on $I\!R$ having a strictly positive density F' . Then there exists a stopping time τ_* of B such that

$$(3.1) \qquad (B_{\tau_*}+\mu\tau_*) \sim \nu$$

if and only if the following condition is satisfied:

$$(3.2) \qquad D_\mu := \int_{I\!R} e^{-2\mu x}\,\nu(dx) \le 1 \,.$$

Setting in this case $C_\mu = -(2\mu)^{-1}\log(D_\mu)$ *, the following explicit formula is valid:*

$$(3.3) \qquad \tau_* = \inf\Big\{\, t>0 \;\Big|\; (B_t+\mu t) \le h_\mu\Big(\max_{0\le r\le t}(B_r+\mu r)\Big)\Big\}$$

where the map $s \mapsto h_\mu(s)$ *for* $s > C_\mu$ *is expressed through its inverse by*

$$(3.4) \qquad h_\mu^{-1}(x) = -\frac{1}{2\mu}\log\Big(\frac{1}{1-F(x)}\int_x^\infty e^{-2\mu t}\,dF(t)\Big) \quad (x\in I\!R)$$

and we set $h_\mu(s) = -\infty$ *for* $s \le C_\mu$ *.*

Proof. 1. Set $X_t = B_t + \mu t$. Then $X = (X_t)_{t\ge 0}$ is a diffusion process solving

$$(3.5) \qquad dX_t = \mu\, dt + dB_t$$

with $X_0 = 0$. The scale function of X is given by

$$(3.6) \qquad S(x) = \frac{1}{2\mu}\Big(1-e^{-2\mu x}\Big)$$

for $x \in I\!R$. Thus the process

$$(3.7) \qquad Z_t := S(X_t) = \frac{1}{2\mu}\Big(1-e^{-2\mu X_t}\Big) = \frac{1}{2\mu}\Big(1-\exp\big(-2\mu B_t - 2\mu^2 t\big)\Big)$$

is a continuous local martingale. It is easily verified that $Z = (Z_t)_{t\ge 0}$ solves

$$(3.8) \qquad dZ_t = \big(1-2\mu Z_t\big)\, dB_t$$

with $Z_0 = 0$. Note that $Z_t < (2\mu)^{-1}$ with $Z_t \to (2\mu)^{-1}$ P-a.s. as $t \to \infty$. Observe that the diffusion coefficient in (3.8) takes value 0 at $(2\mu)^{-1}$; however, the solution Z started at zero never reaches $(2\mu)^{-1}$.

2. Let U be a random variable satisfying $U \sim F$, where F is the distribution function of ν . Then $V := S(U) \sim G$ with

$$(3.9) \qquad G(x) = F\Big(S^{-1}(x)\Big)$$

for $x < (2\mu)^{-1}$. Suppose that τ_* is a stopping time of B satisfying $Z_{\tau_*} \sim V$. Then $X_{\tau_*} = S^{-1}(Z_{\tau_*}) \sim S^{-1}(V) = U \sim F$. This shows that the initial (diffusion) problem is reduced to the martingale problem of finding a stopping time τ_* of B satisfying

$$(3.10) \qquad Z_{\tau_*} \sim V \,.$$

Note that $V < (2\mu)^{-1}$ so that $x \mapsto G(x)$ is strictly increasing and continuous on $]-\infty, (2\mu)^{-1}]$ with $G(-\infty) = 0$ and $G((2\mu)^{-1}) = 1$.

3. To solve the problem (3.10) we shall introduce the maximum process:

$$S_t = \max_{0 \le r \le t} Z_r \tag{3.11}$$

and consider the following stopping time:

$$\tau_g = \inf \left\{ t > 0 \mid Z_t \le g(S_t) \right\} \tag{3.12}$$

with a map $s \mapsto g(s)$ being defined on $[0, (2\mu)^{-1}]$. Motivated by the properties of G , it is natural to assume that $s \mapsto g(s)$ is an increasing C^1-function satisfying $g(0) = -\infty$ and $g\big((2\mu)^{-1}\big) = (2\mu)^{-1}$. Our main aim in the sequel is to show how to pick up a map $s \mapsto g_*(s)$ out of all admissible candidates just specified, so that (3.10) holds with $\tau_* = \tau_{g_*}$.

4. We show that $Z_{\tau_g} \sim V$ if and only if the inverse $x \mapsto g^{-1}(x)$ of the map $s \mapsto g(s)$ solves the following differential equation:

$$\big(g^{-1}\big)'(x) - \frac{G'(x)}{1 - G(x)}\, g^{-1}(x) = -\frac{G'(x)}{1 - G(x)}\, x \tag{3.13}$$

for $x < (2\mu)^{-1}$. To prove this claim, we shall first verify that (see [8] and [1])

$$F_g(s) := P\{S_{\tau_g} \le s\} = 1 - \exp\left(-\int_0^s \frac{dt}{t - g(t)} \right) \tag{3.14}$$

for all $0 \le s < (2\mu)^{-1}$. It is important to realize that this fact generally holds only for $s < (2\mu)^{-1}$, and depending on the magnitude of the left-hand derivative of $s \mapsto g(s)$ at $(2\mu)^{-1}$, the distribution function of S_{τ_g} may have a jump at $(2\mu)^{-1}$. Observe, however, that

$$\Big\{ S_{\tau_g} = (2\mu)^{-1} \Big\} = \Big\{ Z_{\tau_g} = (2\mu)^{-1} \Big\} = \big\{ \tau_g = \infty \big\} \, . \tag{3.15}$$

Hence the fact that $x \mapsto G(x)$ is continuous at $(2\mu)^{-1}$ will have for a consequence in our derivation below that the set in (3.14) is of P-measure zero. This is a quick way of establishing that $\tau_g < \infty$ P-a.s. upon a proper choice of the boundary $s \mapsto g(s)$ for $s < (2\mu)^{-1}$.

5. To derive (3.14), we may use the fact that

$$H(S_t) - (S_t - Z_t)\, H'(S_t) \tag{3.16}$$

is a continuous local martingale whenever $H \in C^2$, which is easily verified by Itô formula. We shall apply this fact with $H_\varepsilon(s) = \int_0^s h_\varepsilon(r)\, dr$ where $h : [0, (2\mu)^{-1}[\rightarrow I\!R_+$ is any bounded C^1-function which is zero on $[0, \varepsilon]$. Then the process (3.16) is uniformly bounded and therefore it is a uniformly integrable martingale. Thus by the optional sampling theorem we get

$$E\Big(H_\varepsilon(S_{\tau_g}) \Big) = E\Big((S_{\tau_g} - Z_{\tau_g})\, H_\varepsilon'(S_{\tau_g}) \Big) \, . \tag{3.17}$$

Integrating by parts, the left-hand side in (3.17) becomes

(3.18) $$E\Big(H_\varepsilon(S_{\tau_g})\Big) = \int_0^{1/2\mu} H_\varepsilon(s)\ dF_g(s) = H_\varepsilon(s)\,F_g(s)\,\Big|_0^{1/2\mu} - \int_0^{1/2\mu} F_g(s-)\ dH_\varepsilon(s)$$
$$= H_\varepsilon(1/2\mu) - \int_0^{1/2\mu} F_g(s-)\ dH_\varepsilon(s) = \int_0^{1/2\mu}\Big(1-F_g(s-)\Big)\,h_\varepsilon(s)\ ds\ .$$

On the other hand, the right-hand side in (3.17) is equal to

(3.19) $$E\Big((S_{\tau_g}-Z_{\tau_g})\,H'_\varepsilon(S_{\tau_g})\Big) = E\Big(\big(S_{\tau_g}-g(S_{\tau_g})\big)\,H'_\varepsilon(S_{\tau_g})\Big)$$
$$= \int_0^{1/2\mu}\big(s-g(s)\big)\,h_\varepsilon(s)\ dF_g(s)\ .$$

Letting $\varepsilon \downarrow 0$ in (3.18) and (3.19), we find that

(3.20) $$\int_0^{1/2\mu}\Big(1-F_g(s-)\Big)\,h(s)\ ds = \int_0^{1/2\mu}\big(s-g(s)\big)\,h(s)\ dF_g(s)$$

for all bounded C^1-functions $h : [0,(2\mu)^{-1}[\ \to I\!R_+$. This shows that

(3.21) $$\frac{dF_g}{ds}(s) = \frac{1-F_g(s)}{s-g(s)}$$

for $0 < s < (2\mu)^{-1}$. The equation (3.21) is easily solved, and since clearly $F_g(0) = 0$, this leads to (3.14) above.

6. Suppose now that $Z_{\tau_g} \sim V$ for some g . Then by (3.15) we find that $P(S_{\tau_g} = (2\mu)^{-1}) = 0$, and therefore (3.14) holds for $s = (2\mu)^{-1}$ as well. This implies that

(3.22) $$1 - G(x) = P\big\{Z_{\tau_g} > x\big\} = P\big\{g(S_{\tau_g}) > x\big\} = P\big\{S_{\tau_g} > g^{-1}(x)\big\}$$
$$= \exp\left(-\int_0^{g^{-1}(x)} \frac{dt}{t-g(t)}\right) = \exp\left(-\int_{-\infty}^{x} \frac{dr}{g'\big(g^{-1}(r)\big)\big(g^{-1}(r)-r\big)}\right)$$

for all $x < (2\mu)^{-1}$ upon substituting $g(t) = r$. Differentiating over x in (3.22), we see that

(3.23) $$\frac{G'(x)}{1-G(x)} = \frac{1}{g'\big(g^{-1}(x)\big)\big(g^{-1}(x)-x\big)}$$

for all $x < (2\mu)^{-1}$, and this equation is equivalent to (3.13). On the other hand, if g solves (3.13), or equivalently (3.23), then the final equality in (3.22) follows upon integrating in (3.23). This proves the claim above.

7. The general solution of (3.13) is given by

(3.24) $$g^{-1}(x) = \frac{1}{1-G(x)}\left(C - \int_{-\infty}^{x} t\ dG(t)\right)$$

where $C \in I\!R$ is a constant. Since $G((2\mu)^{-1}) = 1$, and we want $g^{-1}\big((2\mu)^{-1}\big) = (2\mu)^{-1}$, we see that the following identity must hold:

(3.25) $$C = E(V) = \int_{-\infty}^{1/2\mu} t \; dG(t) \; .$$

This gives the following explicit expression:

(3.26) $$g_*^{-1}(x) = \frac{1}{1-G(x)} \int_x^{1/2\mu} t \; dG(t)$$

for $x < (2\mu)^{-1}$. Hence we see that $g_*^{-1}(-\infty) = E(V)$, and thus $g_*\big(E(V)\big) = -\infty$, so everything agrees fine if $E(V) = 0$. Observe, however, that the whole preceding construction can be carried out in exactly the same way also for those functions g_* for which $0 \le E(V) < (2\mu)^{-1}$. The identity (3.14) holds then for all $s \in [E(V), (2\mu)^{-1}[$ with $F_g(E(V)) = 0$. The given τ_{g_*} is then described as follows: Let Z first hit $E(V)$, and then starting afresh, the time τ_{g_*} is obtained by adding on that time also the time needed for $Z_t \ge g_*(S_t)$ to happen. More formally, this can be described by $\tau_* = \tau_{s_*} + \tau_{g_*} \circ \theta_{\tau_{s_*}}$, where we set $s_* = E(V)$ and $\tau_{s_*} = \inf\{\, t > 0 \mid Z_t = s_* \,\}$. Observe that the more complicated form of this stopping time when $E(V) > 0$ has been formalised in terms of the initial process X by assigning the value $-\infty$ to h_μ in (3.3) above.

8. These considerations showed that the following condition:

(3.27) $$0 \le E(V) < (2\mu)^{-1}$$

is sufficient to solve the martingale problem (3.10). Note that the second inequality is trivial, and the first one is equivalently written as

(3.28) $$E\Big(e^{-2\mu U}\Big) \le 1$$

which is exactly the condition (3.2). However, this condition is necessary as well, as is easily seen from the fact that the process

(3.29) $$\exp\Big(-2\mu B_t - 2\mu^2 t\Big)$$

is a continuous local martingale (by the optional sampling theorem and Fatou's lemma). Thus the condition (3.2) is necessary and sufficient, and the martingale problem (3.10) is solved.

9. It remains to transfer this solution back to the initial problem. For this, note that from the definition of the stopping time τ_* and the process Z , we have

(3.30) $$\begin{aligned} \tau_* &= \inf\,\big\{\, t > 0 \mid Z_t \le g_*(S_t) \,\big\} \\ &= \inf\,\big\{\, t > 0 \mid X_t \le \big(S^{-1} \circ g_* \circ S\big)(X_t^*) \,\big\} \end{aligned}$$

where we denote $X_t^* = \sup_{0 \le r \le t} X_r$ and define $g_*(s) = -\infty$ for $s \le E(V)$ if $E(V) > 0$. Setting $h_\mu = S^{-1} \circ g_* \circ S$, we see that

(3.31) $$h_\mu^{-1}(x) = S^{-1}\Big(g_*^{-1}\big(S(x)\big)\Big) = S^{-1}\bigg(\frac{1}{1-F(x)} \int_x^{\infty} S(t) \; dF(t)\bigg)$$

by means of (3.9). Now using that $S^{-1}(z) = -(2\mu)^{-1} \log(1-2\mu z)$, we end up with the formula

(3.4). This can be then formally expressed in terms of the map $s \mapsto h_\mu(s)$ by assigning the value $-\infty$ as described above in the lines following (3.26). The proof is complete. $\square$

Remarks: 1. It is easily verified by Jensen's inequality that under condition (3.2) any random variable $U \not\equiv 0$ satisfying $U \sim \nu$ must also satisfy $E(U) > 0$. This expectation may be $+\infty$. Moreover, it easily follows by the Burkholder-Davis-Gundy inequality and the optional sampling theorem (see e.g. [12]) that $E(U) < \infty$ if and only if $E(\tau_*) < \infty$; then $E(\tau_*) = \mu^{-1} E(U)$. In this case $E(\max_{0 \le t \le \tau_*} |X_t|) < \infty$ and the process $(X_{t \wedge \tau_*})_{t \ge 0}$ is uniformly integrable.

2. Observe that the function $x \mapsto h_\mu^{-1}(x)$ appearing in (3.4) may be viewed as an extension of the Hardy-Littlewood function (2.2) from the case of standard Brownian motion (when $\mu = 0$) to the case of Brownian motion with drift $\mu > 0$. It may be interesting to examine a role of this extension in the context of Hardy-Littlewood theory [7].

Our next aim is to explore the minimax property stated in Section 2 above in the case of Brownian motion with drift.

Proposition 3.2 (The minimax property)

Under the assumptions of Theorem 3.1, let σ *be any stopping time of* B *satisfying*

$$(B_\sigma + \mu\sigma) \sim \nu \ . \tag{3.32}$$

Then the following inequality holds:

$$P\Big\{ \max_{0 \le t \le \sigma} (B_t + \mu t) \ge s \Big\} \le P\Big\{ \max_{0 \le t \le \tau_*} (B_t + \mu t) \ge s \Big\} \tag{3.33}$$

for all $s \ge 0$. *Suppose, moreover, that* ν *satisfies*

$$\int_0^\infty x \ \nu(dx) < \infty \ . \tag{3.34}$$

If σ *is any stopping time of* B *satisfying (3.32) and*

$$E\Big(\max_{0 \le t \le \sigma} (B_t + \mu t) \Big) = E\Big(\max_{0 \le t \le \tau_*} (B_t + \mu t) \Big) \tag{3.35}$$

then the following inequality holds:

$$\tau_* \le \sigma \quad P\text{-a.s.} \tag{3.36}$$

Remarks: 1. Observe from (3.33) and integration by parts that the left-hand expectation in (3.35) is always smaller than the right-hand one. Clearly, under (3.34) the condition (3.35) is equivalent to the fact that equality in (3.33) is attained for all $s \ge 0$, or in other words, that

$$\max_{0 \le t \le \sigma} (B_t + \mu t) \sim \max_{0 \le t \le \tau_*} (B_t + \mu t) \ . \tag{3.37}$$

Thus, the result above shows that τ_* is *pointwise the smallest possible stopping time which generates stochastically the largest possible maximum* of the Brownian motion with drift up to the time of stopping. Observe that this minimax property characterizes τ_* uniquely, and that equality in (3.36) actually holds (see the proof of Proposition 3.3 below). Moreover, a slight modification of the argument below (if dealing with $(Z_t)_{t\geq 0}$ instead of $(X_t)_{t\geq 0}$ in part (II) of the proof) shows that (3.36) (with equality) is still valid under (3.37) even if (3.34) fails. The minimax property is of interest for applications whenever one wishes to stop the process at a prescribed law as soon as possible after the highest point feasible was reached (for more details see e.g. [11]).

2. Note that (3.33) holds without additional assumptions on the size of $\max_{0\leq t\leq\sigma}(B_t+\mu t)$ (compare it with Proposition 2.2 in [11]). This fact is intuitively clear as there is no recurrence.

Proof. (I): Set $X_t = B_t + \mu t$ and let $Z_t = S(X_t)$ where $x \mapsto S(x)$ is the scale function of X given by (3.6). Clearly, for (3.33) it is enough to prove that

$$P\Big\{ \max_{0\leq t\leq\sigma} Z_t \geq s \Big\} \leq P\Big\{ \max_{0\leq t\leq\tau_*} Z_t \geq s \Big\} \tag{3.38}$$

for all $0 < s < (2\mu)^{-1}$. Recall also from the proof above that $Z_\sigma \sim G$, and that

$$\tau_* = \inf\, \big\{\, t > 0 \mid Z_t \leq g_*(S_t)\, \big\} \tag{3.39}$$

where $s \mapsto g_*(s)$ for $s > E(V)$ is given through (3.26), and we set $g_*(s) = -\infty$ for $s \leq E(V)$.

To prove (3.38) we will modify the well-known argument of Blackwell and Dubins [4]. Observe that our proof below in essence is the same as the proof of Proposition 2.2 in [11], and we include it here merely for completeness. In the sequel we use the notation (3.11).

By Doob's maximal inequality (see e.g. [12]) we have

$$sP\{S_{\sigma\wedge t} \geq s\} \leq \int_{\{S_{\sigma\wedge t}\geq s\}} Z_{\sigma\wedge t}\; dP \tag{3.40}$$

for all $t > 0$ with $s > 0$ given and fixed. Since $Z_t < (2\mu)^{-1}$ for all t , we may use Fatou's lemma, and by letting $t \to \infty$ in (3.40), we get

$$sP\{S_\sigma \geq s\} \leq \int_{\{S_\sigma\geq s\}} Z_\sigma\; dP\ . \tag{3.41}$$

Setting $A = \{S_\sigma \geq s\}$, we have

$$\begin{aligned} \int_A Z_\sigma\; dP &\leq \int_A \big(Z_\sigma \vee x\big)\; dP = \int_A \big(x + (Z_\sigma - x)^+\big)\; dP \\ &\leq x\,P(A) + \int (Z_\sigma - x)^+\, dP = x\,P(A) + \int_{\{Z_\sigma > x\}} Z_\sigma\; dP - x\,P\{Z_\sigma > x\} \\ &= x\Big(P(A) - P\{Z_\sigma > x\}\Big) + \int_{\{Z_\sigma > x\}} Z_\sigma\; dP \end{aligned} \tag{3.42}$$

for all $x \in I\!R$. Thus, if we choose x such that $P(A) - P\{Z_\sigma > x\} = 0$, or in other words, $G(x) = 1 - P(A)$, then by (3.41) and (3.42) we get

$$sP\{S_\sigma \geq s\} \leq \int_{\{Z_\sigma > x\}} Z_\sigma\; dP\ . \tag{3.43}$$

Since $G(Z_\sigma) \sim U(0,1)$ by our assumption (3.32), we have

$$\int_{\{Z_\sigma > x\}} Z_\sigma \; dP = \int_{\{G(Z_\sigma) > G(x)\}} G^{-1}\big(G(Z_\sigma)\big) \; dP = \int_{G(x)}^1 G^{-1}(v) \; dv = \int_{1-P(A)}^1 G^{-1}(v) \; dv \; . \tag{3.44}$$

On the other hand, since $G(Z_{\tau_*}) \sim U(0,1)$, we have

$$P\{S_{\tau_*} \geq s\} = P\{g_*^{-1}(Z_{\tau_*}) \geq s\} = P\bigg(\frac{1}{1-G(Z_{\tau_*})} \int_{G(Z_{\tau_*})}^1 G^{-1}(v) \; dv \geq s \bigg) = \lambda\bigg(u \in [0,1] \; \Big| \; \frac{1}{1-u} \int_u^1 G^{-1}(v) \; dv \geq s \bigg) = 1 - u_* \tag{3.45}$$

where u_* is the *smallest* u in $[0,1]$ for which

$$\frac{1}{1-u} \int_u^1 G^{-1}(v) \; dv \; \geq \; s \tag{3.46}$$

since the function on the left-hand side in (3.46) is increasing in u .

From (3.43) and (3.44) we now see that

$$s P(A) \leq \int_{1-P(A)}^1 G^{-1}(v) \; dv \tag{3.47}$$

and therefore $u_* \leq 1-P(A)$. This by (3.45) shows that

$$P\{S_\sigma \geq s\} = P(A) \leq 1-u_* = P\{S_{\tau_*} \geq s\} \tag{3.48}$$

and the proof of (3.38) is complete. Thus (3.33) is established.

(II): Due to its complexity and length, we shall only outline a proof of (3.36) which follows the line of arguments from [11]. Once the idea is properly understood, we believe that the text indicated below should be completed with no difficulties.

From Remark 1 following the proof of Theorem 3.1 above we know that under the condition (3.34) the right-hand side expectation in (3.35) is finite. Motivated by facts from [11], consider the following optimal stopping problem:

$$\sup_\tau E\bigg(\max_{0 \leq t \leq \tau} X_t - \int_0^\tau c(X_t) \, dt \bigg) \tag{3.49}$$

where the map $x \mapsto c(x)$ is defined by

$$c(x) = \frac{F'(x) \, S'(x)}{2\,\big(1-F(x)\big)\, S'\bigg(S^{-1}\Big(\frac{1}{1-F(x)} \int_x^\infty S(t) \; dF(t) \Big) \bigg)} \; . \tag{3.50}$$

Then it is possible to verify that the map $s \mapsto h_*(s)$ given through its inverse by

(3.51) $$h_*^{-1}(x) = S^{-1}\left(\frac{1}{1-F(x)}\int_x^\infty S(t)\ dF(t)\right) \quad (x \in I\!R)$$

is the maximal solution of the following first-order nonlinear differential equation:

(3.52) $$g'(s) = \frac{S'\big(g(s)\big)}{2\,c\big(g(s)\big)\Big(S(s)-S\big(g(s)\big)\Big)}$$

which stays strictly below the diagonal in $I\!R^2$, and thus by the maximality principle (see [10]) the stopping time τ_* is optimal for the problem (3.49). (Observe that the maps in (3.51) and (3.4) coincide.) Then under the assumption (3.32) above, together with (3.35) being finite, one can extend the idea and argument used in Corollary 2.3 in [11] and prove first that

(3.53) $$E\left(\int_0^\sigma c(X_t)\ dt\right) = E\left(\int_0^{\tau_*} c(X_t)\ dt\right)$$

as well that (3.36) then in essence follows from the optimality of τ_* within the problem (3.49). The key equation in this process is

(3.54) $$\frac{1}{2}H''(x) + \mu H'(x) = c(x)$$

which is treated easily in this context. We omit all remaining details for simplicity. □

We shall conclude our considerations by examining yet another extremal property of τ_* which extends the argument of Monroe [9] to the Brownian motion with drift.

Proposition 3.3
Under the assumptions of Theorem 3.1, let σ be any stopping time of B satisfying (3.32). If $\sigma \le \tau_*$, then $\sigma = \tau_*$ P-a.s.

Proof. Set $X_t = B_t + \mu t$ and let $Z_t = S(X_t)$ where $x \mapsto S(x)$ is the scale function of X given by (3.6). Recall that $Z = (Z_t)_{t\ge0}$ is a continuous local martingale satisfying $Z_t < (2\mu)^{-1}$ for all t . Choose a localization sequence of bounded stopping times $(\gamma_n)_{n\ge1}$ for Z . Then $(Z_{t\wedge\gamma_n})_{t\ge0}$ is a uniformly integrable martingale for all $n \ge 1$. Thus by the fact that $Z_{\tau_*} \sim Z_\sigma$ and the optional sampling theorem, we get

(3.55) $$\int_{\{Z_{\tau_*}\ge z\}} Z_{\tau_*}\, dP = \int_{\{Z_\sigma\ge z\}} Z_\sigma\, dP = \lim_{n\to\infty}\int_{\{Z_{\sigma\wedge\gamma_n}\ge z\}} Z_{\sigma\wedge\gamma_n}\, dP$$
$$= \lim_{n\to\infty}\int_{\{Z_{\sigma\wedge\gamma_n}\ge z\}} Z_{\tau_*\wedge\gamma_n}\, dP = \int_{\{Z_\sigma\ge z\}} Z_{\tau_*}\, dP$$

for all $z \in I\!R$. We now claim that this implies that $\{Z_{\tau_*}\ge z\} = \{Z_\sigma \ge z\}$ P-a.s. for all $z \in I\!R$, which in turn easily shows that $Z_{\tau_*} = Z_\sigma$ P-a.s.

For this, note by (3.32) and (3.55) that

(3.56) $$\int_{\{Z_{\tau_*}\ge z\}} (Z_{\tau_*}-z)\, dP = \int_{\{Z_\sigma\ge z\}} (Z_{\tau_*}-z)\, dP = \int_{\{Z_\sigma\ge z\,,\, Z_{\tau_*}\ge z\}} (Z_{\tau_*}-z)\, dP$$

$$+\int_{\{Z_\sigma \ge z\,,\, Z_{\tau_*}<z\}} (Z_{\tau_*}-z)\, dP \le \int_{\{Z_{\tau_*}\ge z\}} (Z_{\tau_*}-z)\, dP$$

with a strict inequality if $P\{Z_\sigma \ge z,\, Z_{\tau_*}<z\}>0$. Thus $\{Z_\sigma \ge z\} \subseteq \{Z_{\tau_*} \ge z\}$ P-a.s., and since these two sets have equal P-probability, this proves the claim above.

Let now ρ be any stopping time of B satisfying $\sigma \le \rho \le \tau_*$. Then by the optional sampling theorem, and the fact that $Z_{\tau_*} = Z_\sigma$ P-a.s., we get

$$(Z_\rho - n)^+ \le E\Big((Z_{\tau_*}-n)^+ \,\Big|\, \mathcal{F}_\rho\Big) = E\Big((Z_\sigma - n)^+ \,\Big|\, \mathcal{F}_\rho\Big) = (Z_\sigma - n)^+ \tag{3.57}$$

for all $n \ge 1$. Letting $n \to \infty$ we see that $Z_\rho \le Z_\sigma$ P-a.s. Clearly, this is only possible if $\sigma = \tau_*$ P-a.s. The proof is complete. □

Remark: Observe that no uniform integrability condition is needed for this result (recall Remark 1 following Theorem 3.1). This is in contrast with the case when the drift is zero (recall Subsection 3 of Section 2) and can be intuitively explained by absence of recurrence (recall also Remark 2 stated after Proposition 3.2).

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Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk