Exotic Option Pricing and Advanced Lévy Models (Eindhoven, 2004), John Wiley, 2005, (217-235)
Research Report No. 436, 2003, *Dept. Theoret. Statist. Aarhus* (19 pp)

On Asian Options of American Type

G. Peskir* *and* N. Uys**

We show that the optimal stopping boundary for the early exercise Asian call option with floating strike can be characterized as the unique solution of a nonlinear integral equation arising from the early exercise premium representation (an explicit formula for the arbitrage-free price in terms of the optimal stopping boundary). The key argument in the proof relies upon a local time-space formula.

1. Introduction

According to financial theory (see e.g. [7] or [18]) the arbitrage-free price of the *early* exercise *Asian call option* with *floating strike* is given as V in (2.1) below where I_τ/τ denotes the *arithmetic* average of the stock price S up to time τ. The problem was first studied in [5] where approximations to the value function V and the optimal boundary b were derived. The main aim of the present paper is to derive exact expressions for V and b.

The optimal stopping problem (2.1) is three-dimensional. When a change-of-measure theorem is applied (as in [16] and [10]) the problem reduces to (2.9) and becomes two-dimensional. The problem (2.9) is more complicated than the well-known problems [12] and [13] since the gain function depends on time in a nonlinear way. From the result of Theorem 3.1 below it follows that the free-boundary problem (2.10)-(2.14) characterizes the value function V and the optimal stopping boundary b in a unique manner. Our main aim, however, is to follow the train of thought initiated by Kolodner [9] where V is initially expressed in terms of b, and b itself is then shown to satisfy a nonlinear integral equation. A particularly simple approach for achieving this goal in the case of the American put option has been suggested in [8], [6], [2] and we will take it up in the present paper. We will moreover see (as in [12] and [13]) that the nonlinear equation derived for b cannot have other solutions. The key argument in the proof relies upon a local time-space formula (see [11]).

---

* Centre for Analytical Finance (funded by the Danish Social Science Research Council).
* Network in Mathematical Physics and Stochastics (funded by the Danish National Research Foundation).
** Programme for Advanced Mathematics of Finance, University of the Witwatersrand, Johannesburg.
Mathematics Subject Classification 2000. Primary 91B28, 45G10. Secondary 60G40, 35R35, 60J60.
Key words and phrases: Asian options of American type, floating strike, early exercise premium, arbitrage-free price, optimal stopping, smooth fit, geometric Brownian motion, free-boundary problem, nonlinear integral equation, forward and backward equations, local time-space calculus, curved boundary.

The latter fact of uniqueness may be seen as the principal result of the paper. The same method of proof can also be used to show the uniqueness of the optimal stopping boundary solving nonlinear integral equations derived in [5] and [19] where this question was not explicitly addressed. These equations arise from the early exercise Asian options (call or put) with floating strike based on *geometric* averaging. The early exercise Asian *put* option with floating strike can be dealt with analogously to the Asian call option treated here. For financial interpretations of the early exercise Asian options and other references on the topic see [5] and [19].

2. Formulation of the problem

The arbitrage-free price of the early exercise Asian call option with floating strike is given by the following expression:

$$V = \sup_{0<\tau\le T} \mathsf{E}\Big(e^{-r\tau}\Big(S_\tau - \frac{1}{\tau}\, I_\tau\Big)^+\Big) \tag{2.1}$$

where τ is a stopping time of the geometric Brownian motion $S=(S_t)_{0\le t\le T}$ solving:

$$dS_t = rS_t\, dt + \sigma S_t\, dB_t \qquad (S_0 = s) \tag{2.2}$$

and $I=(I_t)_{0\le t\le T}$ is the integral process given by:

$$I_t = a + \int_0^t S_s\, ds \tag{2.3}$$

where $s>0$ and $a\ge 0$ are given and fixed. (Throughout $B=(B_t)_{t\ge 0}$ denotes a standard Brownian motion started at zero.) We recall that $T>0$ is the expiration date (maturity), $r>0$ is the interest rate, and $\sigma>0$ is the volatility coefficient.

By the change-of-measure theorem it follows that:

$$V = \sup_{0<\tau\le T} \mathsf{E}\Big(e^{-r\tau} S_\tau \Big(1 - \frac{1}{\tau}\, X_\tau\Big)^+\Big) = s \sup_{0<\tau\le T} \widetilde{\mathsf{E}}\Big(\Big(1 - \frac{1}{\tau}\, X_\tau\Big)^+\Big) \tag{2.4}$$

where following [16] and [10] we set:

$$X_t = \frac{I_t}{S_t} \tag{2.5}$$

and $\widetilde{\mathsf{P}}$ is defined by $d\widetilde{\mathsf{P}} = \exp(\sigma B_T - (\sigma^2/2)T)\, d\mathsf{P}$ so that $\widetilde{B}_t = B_t - \sigma t$ is a standard Brownian motion under $\widetilde{\mathsf{P}}$ for $0\le t\le T$. By Itô's formula one finds that:

$$dX_t = (1 - rX_t)\, dt + \sigma X_t\, d\widehat{B}_t \qquad (X_0 = x) \tag{2.6}$$

under $\widetilde{\mathsf{P}}$ where $\widehat{B} = -\widetilde{B}$ is a standard Brownian motion and $x = a/s$. The infinitesimal generator of X is therefore given by:

$$\mathbb{L}_X = (1 - rx)\, \frac{\partial}{\partial x} + \frac{\sigma^2}{2}\, x^2 \frac{\partial^2}{\partial x^2}\, . \tag{2.7}$$

For further reference recall that the strong solution of (2.2) is given by:

$$S_t = s\exp\Big(\sigma B_t + \Big(r - \frac{\sigma^2}{2}\Big)t\Big) = s\exp\Big(\sigma \widetilde{B}_t + \Big(r + \frac{\sigma^2}{2}\Big)t\Big) \tag{2.8}$$

for $0 \le t \le T$ under P and $\widetilde{\mathsf{P}}$ respectively. When dealing with the process X on its own, however, note that there is no restriction to assume that $s = 1$ and $a = x$ with $x \ge 0$.

Summarizing the preceding facts we see that the early exercise Asian call problem reduces to solving the following optimal stopping problem:

$$V(t,x) = \sup_{0<\tau\le T-t} \widetilde{\mathsf{E}}_{t,x}\Big(\Big(1 - \frac{1}{t+\tau}\, X_{t+\tau}\Big)^+\Big) \tag{2.9}$$

where τ is a stopping time of the diffusion process X solving (2.6) above and $X_t = x$ under $\widetilde{\mathsf{P}}_{t,x}$ with $(t,x) \in [0,T]\times[0,\infty\rangle$ given and fixed.

Standard Markovian arguments indicate that V from (2.9) solves the following free-boundary problem of parabolic type:

$$V_t + \mathbb{L}_X V = 0 \quad \text{in } C \tag{2.10}$$

$$V(t,x) = \Big(1 - \frac{x}{t}\Big)^+ \quad \text{for } x = b(t) \text{ or } t = T \tag{2.11}$$

$$V_x(t,x) = -\frac{1}{t} \quad \text{for } x = b(t) \text{ (smooth fit)} \tag{2.12}$$

$$V(t,x) > \Big(1 - \frac{x}{t}\Big)^+ \quad \text{in } C \tag{2.13}$$

$$V(t,x) = \Big(1 - \frac{x}{t}\Big)^+ \quad \text{in } D \tag{2.14}$$

where the continuation set C and the stopping set $S = \bar{D}$ are defined by:

$$C = \{\, (t,x) \in [0,T\rangle\times[0,\infty\rangle \mid x > b(t) \,\} \tag{2.15}$$

$$D = \{\, (t,x) \in [0,T\rangle\times[0,\infty\rangle \mid x < b(t) \,\} \tag{2.16}$$

and $b : [0,T] \to I\!R$ is the (unknown) optimal stopping boundary, i.e. the stopping time:

$$\tau_b = \inf\{\, 0 \le s \le T - t \mid X_{t+s} \le b(t+s) \,\} \tag{2.17}$$

is optimal in (2.9) (i.e. the supremum is attained at this stopping time). It follows from the result of Theorem 3.1 below that the free-boundary problem (2.10)-(2.14) characterizes the value function V and the optimal stopping boundary b in a unique manner (proving also the existence of the latter).

3. The result and proof

In this section we adopt the setting and notation of the early exercise Asian call problem from the previous section. Below we will make use of the following functions:

$$F(t,x) = \widetilde{\mathsf{E}}_{0,x}\Big(\Big(1 - \frac{X_t}{T}\Big)^+\Big) = \int_0^\infty \int_0^\infty \Big(1 - \frac{x+a}{Ts}\Big)^+ f(t,s,a)\, ds\, da \tag{3.1}$$

$$(3.2)\qquad G(t,x,y) = \widetilde{\mathsf{E}}_{0,x}\Big(X_t\, I\big(X_t \le y\big)\Big) = \int_0^\infty\!\int_0^\infty \Big(\frac{x+a}{s}\Big) I\Big(\frac{x+a}{s} \le y\Big) f(t,s,a)\, ds\, da$$

$$(3.3)\qquad H(t,x,y) = \widetilde{\mathsf{P}}_{0,x}\left(X_t \le y\right) = \int_0^\infty\!\int_0^\infty I\Big(\frac{x+a}{s} \le y\Big) f(t,s,a)\, ds\, da$$

for $t>0$ and $x,y \ge 0$, where $(s,a) \mapsto f(t,s,a)$ is the probability density function of (S_t, I_t) under $\widetilde{\mathsf{P}}$ with $S_0 = 1$ and $I_0 = 0$ given by:

$$(3.4)\qquad f(t,s,a) = \frac{2\sqrt{2}}{\pi^{3/2}\sigma^3}\, \frac{s^{r/\sigma^2}}{a^2\sqrt{t}}\, \exp\left(\frac{2\pi^2}{\sigma^2 t} - \frac{(r+\sigma^2/2)^2}{2\sigma^2}\, t - \frac{2}{\sigma^2 a}\,(1+s)\right)$$
$$\times \int_0^\infty \exp\left(-\frac{2z^2}{\sigma^2 t} - \frac{4\sqrt{s}}{\sigma^2 a}\cosh(z)\right) \sinh(z)\, \sin\left(\frac{4\pi z}{\sigma^2 t}\right) dz$$

for $s>0$ and $a>0$. For a derivation of the right-hand side in (3.4) see the Appendix below.

The main result of the paper may be stated as follows.

Theorem 3.1
The optimal stopping boundary in the Asian call problem (2.9) can be characterized as the unique continuous increasing solution $b : [0,T] \rightarrow I\!R$ of the nonlinear integral equation:

$$(3.5)\qquad 1 - \frac{b(t)}{t} = F(T-t, b(t))$$
$$- \int_0^{T-t} \frac{1}{t+u}\bigg(\Big(\frac{1}{t+u} + r\Big) G(u, b(t), b(t{+}u)) - H(u, b(t), b(t{+}u))\bigg)\, du$$

satisfying $0 < b(t) < t/(1{+}rt)$ for all $0<t<T$. [The solution b satisfies $b(0+) = 0$ and $b(T-) = T/(1{+}rT)$, and the stopping time τ_b from (2.17) is optimal in (2.9).]

The arbitrage-free price of the Asian call option (2.9) admits the following 'early exercise premium' representation:

$$(3.6)\qquad V(t,x) = F(T-t, x)$$
$$- \int_0^{T-t} \frac{1}{t+u}\bigg(\Big(\frac{1}{t+u} + r\Big) G\left(u, x, b(t{+}u)\right) - H(u, x, b(t{+}u))\bigg)\, du$$

for all $(t,x) \in [0,T] \times [0,\infty\rangle$. [Further properties of V and b are exhibited in the proof below.]

Proof. The proof will be carried out in several steps. We begin by stating some general remarks which will be freely used below without further mentioning.

1. The reason that we take the supremum in (2.1) and (2.9) over $\tau > 0$ is that the ratio $1/(t{+}\tau)$ is not well defined for $\tau = 0$ when $t = 0$. Note however in (2.1) that $I_\tau/\tau \rightarrow \infty$ as $\tau \downarrow 0$ when $I_0 = a > 0$ and that $I_\tau/\tau \rightarrow s$ as $\tau \downarrow 0$ when $I_0 = a = 0$. Similarly, note in (2.9) that $X_\tau/\tau \rightarrow \infty$ as $\tau \downarrow 0$ when $X_0 = x > 0$ and $X_\tau/\tau \rightarrow 1$ as $\tau \downarrow 0$ when $X_0 = x = 0$. Thus in both cases the gain process (the integrand in (2.1) and (2.9)) tends to 0 as $\tau \downarrow 0$. This shows that in either (2.1) or (2.9) it is never optimal to stop at $t = 0$. To avoid similar (purely technical) complications in the proof to follow we will equivalently

consider $V(t,x)$ only for $t>0$ with the supremum taken over $\tau \geq 0$. The case of $t=0$ will become evident (by continuity) at the end of the proof.

2. Recall that it is no restriction to assume that $s=1$ and $a=x$ so that $X_t=(x+I_t)/S_t$ with $I_0=0$ and $S_0=1$. We will write X_t^x instead of X_t to indicate the dependence on x when needed. It follows that V admits the following representation:

$$V(t,x)=\sup_{0\leq\tau\leq T-t} \widetilde{\mathsf{E}}\bigg(\Big(1-\frac{x+I_\tau}{(t+\tau)\,S_\tau}\Big)^+\bigg) \tag{3.7}$$

for $(t,x)\in\langle 0,T]\times[0,\infty\rangle$. From (3.7) we immediately see that:

$$x\mapsto V(t,x) \text{ is decreasing and convex on } [0,\infty\rangle \tag{3.8}$$

for each $t>0$ fixed.

3. We show that $V:\langle 0,T]\times[0,\infty\rangle\rightarrow I\!\!R$ is continuous. For this, using $\sup(f)-\sup(g)\leq \sup(f-g)$ and $(z-x)^+-(z-y)^+\leq(y-x)^+$ for $x,y,z\in I\!\!R$, we get:

$$\begin{aligned} V(t,x)-V(t,y) &\leq \sup_{0\leq\tau\leq T-t}\bigg(\widetilde{\mathsf{E}}\bigg(\Big(1-\frac{x+I_\tau}{(t+\tau)\,S_\tau}\Big)^+\bigg)-\widetilde{\mathsf{E}}\bigg(\Big(1-\frac{y+I_\tau}{(t+\tau)\,S_\tau}\Big)^+\bigg)\bigg) \\ &\leq (y-x)\sup_{0\leq\tau\leq T-t}\widetilde{\mathsf{E}}\bigg(\frac{1}{(t+\tau)\,S_\tau}\bigg)\leq\frac{1}{t}\,(y-x) \end{aligned} \tag{3.9}$$

for $0\leq x\leq y$ and $t>0$, where in the last inequality we used (2.8) to deduce that $1/S_t=\exp(\sigma\widehat{B}_t-(r+\sigma^2/2)t)\leq\exp(\sigma\widehat{B}_t-(\sigma^2/2)t)$ and the latter is a martingale under $\widetilde{\mathsf{P}}$. From (3.9) with (3.8) we see that $x\mapsto V(t,x)$ is continuous at x_0 uniformly over $t\in[t_0-\delta,t_0+\delta]$ for some $\delta>0$ (small enough) whenever $(t_0,x_0)\in\langle 0,T]\times[0,\infty\rangle$ is given and fixed. Thus to prove that V is continuous on $\langle 0,T]\times[0,\infty\rangle$ it is enough to show that $t\mapsto V(t,x)$ is continuous on $\langle 0,T]$ for each $x\geq 0$ given and fixed. For this, take any $t_1<t_2$ in $\langle 0,T]$ and $\varepsilon>0$, and let τ_1^ε be a stopping time such that $\widetilde{\mathsf{E}}((1-(X^x_{t_1+\tau_1^\varepsilon})/(t_1+\tau_1^\varepsilon))^+)\geq V(t_1,x)-\varepsilon$. Setting $\tau_2^\varepsilon=\tau_1^\varepsilon\wedge(T-t_2)$ we see that $V(t_2,x)\geq\widetilde{\mathsf{E}}((1-(X_{t_2+\tau_2^\varepsilon})/(t_2+\tau_2^\varepsilon))^+)$. Hence we get:

$$\begin{aligned} V(t_1,x)-V(t_2,x) &\leq \widetilde{\mathsf{E}}\bigg(\Big(1-\frac{X^x_{t_1+\tau_1^\varepsilon}}{t_1+\tau_1^\varepsilon}\Big)^+\bigg)-\widetilde{\mathsf{E}}\bigg(\Big(1-\frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2+\tau_2^\varepsilon}\Big)^+\bigg)+\varepsilon \\ &\leq \widetilde{\mathsf{E}}\bigg(\Big(\frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2+\tau_2^\varepsilon}-\frac{X^x_{t_1+\tau_1^\varepsilon}}{t_1+\tau_1^\varepsilon}\Big)^+\bigg)+\varepsilon. \end{aligned} \tag{3.10}$$

Letting first $t_2-t_1\rightarrow 0$ using $\tau_1^\varepsilon-\tau_2^\varepsilon\rightarrow 0$ and then $\varepsilon\downarrow 0$ we see that $\limsup_{t_2-t_1\rightarrow 0}(V(t_1,x)-V(t_2,x))\leq 0$ by dominated convergence. On the other hand, let τ_2^ε be a stopping time such that $\widetilde{\mathsf{E}}((1-(X^x_{t_2+\tau_2^\varepsilon})/(t_2+\tau_2^\varepsilon))^+)\geq V(t_2,x)-\varepsilon$. Then we have:

$$V(t_1,x)-V(t_2,x)\geq\widetilde{\mathsf{E}}\bigg(\Big(1-\frac{X^x_{t_1+\tau_2^\varepsilon}}{t_1+\tau_2^\varepsilon}\Big)^+\bigg)-\widetilde{\mathsf{E}}\bigg(\Big(1-\frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2+\tau_2^\varepsilon}\Big)^+\bigg)-\varepsilon. \tag{3.11}$$

Letting first $t_2-t_1\rightarrow 0$ and then $\varepsilon\downarrow 0$ we see that $\liminf_{t_2-t_1\rightarrow 0}(V(t_1,x)-V(t_2,x))\geq 0$. Combining the two inequalities we find that $t\mapsto V(t,x)$ is continuous on $\langle 0,T]$. This completes the proof of the initial claim.

4. Denote the gain function by $G(t,x) = (1-x/t)^+$ for $(t,x) \in \langle 0,T] \times [0,\infty\rangle$ and introduce the continuation set $C = \{\, (t,x) \in \langle 0,T\rangle \times [0,\infty\rangle \mid V(t,x) > G(t,x) \,\}$ and the stopping set $S = \{\, (t,x) \in \langle 0,T\rangle \times [0,\infty\rangle \mid V(t,x) = G(t,x) \,\}$. Since V and G are continuous, we see that C is open and S is closed in $\langle 0,T\rangle \times [0,\infty\rangle$. Standard arguments based on the strong Markov property (cf. [17]) show that the first hitting time $\tau_S = \inf\{\, 0 \le s \le T-t \mid (t+s, X_{t+s}) \in S \,\}$ is optimal in (2.9) as well as that V is $C^{1,2}$ on C and satisfies (2.10). In order to determine the structure of the optimal stopping time τ_S (i.e. the shape of the sets C and S) we will first examine basic properties of the diffusion process X solving (2.6) under $\widetilde{\mathsf{P}}$.

5. The state space of X equals $[0,\infty\rangle$ and it is clear from the representation (2.5) with (2.8) that 0 is an entrance boundary point. The drift of X is given by $\mu(x) = 1 - rx$ and the diffusion coefficient of X is given by $\sigma(x) = \sigma x$ for $x \ge 0$. Hence we see that $\mu(x)$ is greater/less than 0 if and only if x is less/greater than $1/r$. This shows that there is a permanent push (drift) of X towards the constant level $1/r$ (when X is above $1/r$ the push of X is downwards and when X is below $1/r$ the push of X is upwards). The scale function of X is given by $s(x) = \int_1^x y^{2r/\sigma^2} e^{2/\sigma^2 y}\, dy$ for $x > 0$, and the speed measure of X is given by $m(dx) = (2/\sigma^2)\, x^{-2(1+r/\sigma^2)}\, e^{-2/\sigma^2 x}\, dx$ on the Borel σ-algebra of $\langle 0,\infty\rangle$. Since $s(0) = -\infty$ and $s(\infty) = +\infty$ we see that X is recurrent. Moreover, since $\int_0^\infty m(dx) = (2/\sigma^2)^{-2r/\sigma^2}\, \Gamma(1+2r/\sigma^2)$ is finite we find that X has an invariant probability density function given by:

$$f(x) = \frac{(2/\sigma^2)^{1+2r/\sigma^2}}{\Gamma(1+2r/\sigma^2)}\, \frac{1}{x^{2(1+r/\sigma^2)}}\, e^{-2/\sigma^2 x} \tag{3.12}$$

for $x > 0$. In particular, it follows that $X_t/t \to 0$ $\widetilde{\mathsf{P}}$-a.s. as $t \to \infty$. This fact has an important consequence for the optimal stopping problem (2.9): If the horizon T is infinite, then it is never optimal to stop. Indeed, in this case letting $\tau \equiv t$ and passing to the limit for $t \to \infty$ we see that $V \equiv 1$ on $\langle 0,\infty\rangle \times [0,\infty\rangle$. This shows that the infinite horizon formulation of the problem (2.9) provides no useful information to the finite horizon formulation (such as in [12] and [13] for example). To examine the latter beyond the trivial fact that all points (t,x) with $x \ge t$ belong to C (which is easily seen by considering the hitting times $\tau_\varepsilon = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \le (t+s) - \varepsilon \,\}$ and noting that $\widetilde{\mathsf{P}}_{t,x}(0 < \tau_\varepsilon < T-t) > 0$ if $x \ge t$ with $0 < t < T$) we will examine the gain process in the problem (2.9) using stochastic calculus as follows.

6. Setting $\alpha(t) = t$ for $0 \le t \le T$ to denote the diagonal in the state space and applying the local time-space formula (cf. [11]) under $\widetilde{\mathsf{P}}_{t,x}$ when $(t,x) \in \langle 0,T\rangle \times [0,\infty\rangle$ is given and fixed, we get:

$$\begin{aligned} G(t+s, X_{t+s}) &= G(t,x) + \int_0^s G_t(t+u, X_{t+u})\, du \\ &\quad + \int_0^s G_x(t+u, X_{t+u})\, dX_{t+u} + \frac{1}{2}\int_0^s G_{xx}(t+u, X_{t+u})\, d\langle X, X\rangle_{t+u} \\ &\quad + \frac{1}{2}\int_0^s \Big(G_x(t+u, \alpha(t+u)+) - G_x(t+u, \alpha(t+u)-) \Big)\, d\ell^{\alpha}_{t+u}(X) \\ &= G(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < \alpha(t+u)\big)\, du \end{aligned} \tag{3.13}$$

$$- \sigma \int_0^s \frac{X_{t+u}}{t+u}\, I\big(X_{t+u} < \alpha(t{+}u)\big)\, d\widehat{B}_u + \frac{1}{2} \int_0^s \frac{d\ell^\alpha_{t+u}(X)}{t+u}$$

where $\ell^\alpha_{t+u}(X)$ is the local time of X on the curve α given by:

$$\ell^\alpha_{t+u}(X) = \widetilde{\mathsf{P}}-\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(\alpha(t{+}v){-}\varepsilon < X_{t+v} < \alpha(t{+}v){+}\varepsilon\big)\, d\langle X, X\rangle_{t+v} \tag{3.14}$$
$$= \widetilde{\mathsf{P}}-\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(\alpha(t{+}v){-}\varepsilon < X_{t+v} < \alpha(t{+}v){+}\varepsilon\big)\, \frac{\sigma^2}{2} X^2_{t+v}\, dv$$

and $d\ell^\alpha_{t+u}(X)$ refers to the integration with respect to the continuous increasing function $u \mapsto \ell^\alpha_{t+u}(X)$. From (3.13) we respectively read:

$$G(t+s, X_{t+s}) = G(t,x) + A_s + M_s + L_s \tag{3.15}$$

where A and L are processes of bounded variation (L is increasing) and M is a continuous (local) martingale. We note moreover that $s \mapsto L_s$ is strictly increasing only when $X_s = \alpha(s)$ for $0 \le s \le T-t$ i.e. when X visits α. On the other hand, when X is below α then the integrand $a(t+u, X_{t+u})$ of A_s may be either positive or negative. To determine both regions exactly we need to examine the sign of the expression $a(t,x) = x/t^2 - (1-rx)/t$. It follows that $a(t,x)$ is larger/less than 0 if and only if x is larger/less than $\gamma(t)$ where $\gamma(t) = t/(1{+}rt)$ for $0 \le t \le T$. By considering the exit times from small balls in $\langle 0,T \rangle {\times} [0,\infty\rangle$ with centre at (t,x) and making use of (3.13) with the optional sampling theorem (to get rid of the martingale part), upon observing that $\gamma(t) < \alpha(t)$ for all $0 < t \le T$ so that the local time part is zero, we see that all points (t,x) lying above the curve γ (i.e. $x > \gamma(t)$ for $0 < t < T$) belong to the continuation set C. Exactly the same arguments (based on the fact that the favourable regions above γ and on α are far away from X) show that for each $x < \gamma(T) = T/(1+rT)$ given and fixed, all points (t,x) belong to the stopping set S when t is close to T. Moreover, recalling (3.8) and the fact that $V(t,x) \ge G(t,x)$ for all $x \ge 0$ with $t \in \langle 0,T \rangle$ fixed, we see that for each $t \in \langle 0,T \rangle$ there is a point $b(t) \in [0, \gamma(t)]$ such that $V(t,x) > G(t,x)$ for $x > b(t)$ and $V(t,x) = G(t,x)$ for $x \in [0, b(t)]$. Combining it with the previous conclusion on S we find that $b(T-) = \gamma(T) = T/(1+rT)$. (Yet another argument for this identity will be given below. Note that this identity is different from the identity $b(T-) = T$ used in [5, page 1126].) This establishes the existence of the non-trivial (non-zero) optimal stopping boundary b on a left-neighborhood of T. We will now show that b extends (continuously and decreasingly) from the initial neighborhood of T backward in time as long as it visits 0 at some time $t_0 \in [0,T\rangle$, and later in the second part of the proof below we will deduce that this t_0 is equal to 0. The key argument in the proof is provided by the following inequality. Notice that this inequality is not obvious a priori (unlike in [12] and [13]) since $t \mapsto G(t,x)$ is increasing and the supremum in (2.9) is taken over a smaller class of stopping times $\tau \in [0, T{-}t]$ when t is larger.

7. We show that the inequality is satisfied:

$$V_t(t,x) \le G_t(t,x) \tag{3.16}$$

for all $(t,x) \in C$. (It may be noted from (2.10) that $V_t = -(1-rx)V_x - (\sigma^2/2)x^2 V_{xx} \le (1-rx)/t$ since $V_x \ge -1/t$ and $V_{xx} \ge 0$ by (3.8), so that $V_t \le G_t$ holds above γ because

$(1-rx)/t \le x/t^2$ if and only if $x \ge t/(1+rt)$. Hence the main issue is to show that (3.16) holds below γ and above b . Any analytic proof of this fact seems difficult and we resort to probabilistic arguments.)

To prove (3.16) fix $0<t<t+h<T$ and $x \ge 0$ so that $x \le \gamma(t)$. Let $\tau = \tau_S(t+h,x)$ be the optimal stopping time for $V(t+h,x)$. Since $\tau \in [0,T-t-h] \subseteq [0,T-t]$ we see that $V(t,x) \ge \widetilde{\mathsf{E}}_{t,x}((1-X_{t+\tau}/(t+\tau))^+)$ so that using the inequality stated prior to (3.9) above (and the convenient refinement by an indicator function), we get:

$$
\begin{aligned}
(3.17)\qquad & V(t+h,x)-V(t,x)-\Big(G(t+h,x)-G(t,x)\Big)\\
&\le \widetilde{\mathsf{E}}\bigg(\Big(1-\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\Big)^+\bigg)-\widetilde{\mathsf{E}}\bigg(\Big(1-\frac{x+I_\tau}{(t+\tau)\,S_\tau}\Big)^+\bigg)-\Big(\frac{x}{t}-\frac{x}{t+h}\Big)\\
&\le \widetilde{\mathsf{E}}\bigg(\Big(\frac{x+I_\tau}{(t+\tau)\,S_\tau}-\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\Big)\,I\Big(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\le 1\Big)\bigg)-\frac{xh}{t\,(t+h)}\\
&= \widetilde{\mathsf{E}}\bigg(\frac{x+I_\tau}{S_\tau}\Big(\frac{1}{t+\tau}-\frac{1}{t+h+\tau}\Big)\,I\Big(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\le 1\Big)\bigg)-\frac{xh}{t\,(t+h)}\\
&= \widetilde{\mathsf{E}}\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\,\frac{h}{t+\tau}\,I\Big(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\le 1\Big)\bigg)-\frac{xh}{t\,(t+h)}\\
&\le \frac{h}{t}\,\widetilde{\mathsf{E}}\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\,I\Big(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\le 1\Big)\bigg)-\frac{xh}{t\,(t+h)}\le 0
\end{aligned}
$$

where the final inequality follows from the fact that with $Z := (x+I_\tau)/((t+h+\tau)S_\tau)$ we have $V(t+h,x) = \widetilde{\mathsf{E}}((1-Z)^+) = \widetilde{\mathsf{E}}((1-Z)\,I(Z\le 1)) = \widetilde{\mathsf{P}}(Z\le 1) - \widetilde{\mathsf{E}}(Z\,I(Z\le 1)) \ge G(t+h,x) = 1-x/(t+h)$ so that $\widetilde{\mathsf{E}}(Z\,I(Z\le 1)) \le \widetilde{\mathsf{P}}(Z\le 1)-1+x/(t+h) \le x/(t+h)$ as claimed. Dividing the initial expression in (3.17) by h and letting $h \downarrow 0$ we obtain (3.16) for all $(t,x) \in C$ such that $x \le \gamma(t)$. Since $V_t \le G_t$ above γ (as stated following (3.16) above) this completes the proof of (3.16).

8. We show that $t \mapsto b(t)$ is increasing on $\langle 0,T\rangle$. This is an immediate consequence of (3.17). Indeed, if (t_1,x) belongs to C and t_0 from $\langle 0,T\rangle$ satisfies $t_0 < t_1$, then by (3.17) we have that $V(t_0,x)-G(t_0,x) \ge V(t_1,x)-G(t_1,x) > 0$ so that (t_0,x) must belong to C . It follows that b cannot be strictly decreasing thus proving the claim.

9. We show that the smooth-fit condition (2.12) holds, i.e. that $x \mapsto V(t,x)$ is C^1 at $b(t)$. For this, fix a point $(t,x) \in \langle 0,T\rangle \times \langle 0,\infty\rangle$ lying at the boundary so that $x = b(t)$. Then $x \le \gamma(t) < \alpha(t)$ and for all $\varepsilon > 0$ such that $x+\varepsilon < \alpha(t)$ we have:

$$
(3.18)\qquad \frac{V(t,x+\varepsilon)-V(t,x)}{\varepsilon} \ge \frac{G(t,x+\varepsilon)-G(t,x)}{\varepsilon} = -\frac{1}{t}.
$$

Letting $\varepsilon \downarrow 0$ and using that the limit on the left-hand side exists (since $x \mapsto V(t,x)$ is convex), we get the inequality:

$$
(3.19)\qquad \frac{\partial^+ V}{\partial x}(t,x) \ge \frac{\partial G}{\partial x}(t,x) = -\frac{1}{t}.
$$

To prove the converse inequality, fix $\varepsilon > 0$ such that $x+\varepsilon < \alpha(t)$, and consider the stopping times $\tau_\varepsilon = \tau_S(t, x+\varepsilon)$ being optimal for $V(t, x+\varepsilon)$. Then we have:

$$\tag{3.20} \begin{aligned} \frac{V(t,x+\varepsilon)-V(t,x)}{\varepsilon} &\le \frac{1}{\varepsilon}\Big(\widetilde{\mathsf{E}}\Big(\Big(1-\frac{x+\varepsilon+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\Big)^+ - \Big(1-\frac{x+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\Big)^+\Big)\Big) \\ &\le \frac{1}{\varepsilon}\,\widetilde{\mathsf{E}}\Big(\frac{x+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}} - \frac{x+\varepsilon+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\Big) = -\,\widetilde{\mathsf{E}}\Big(\frac{1}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\Big). \end{aligned}$$

Since each point x in $\langle 0,\infty\rangle$ is regular for X , and the boundary b is increasing, it follows that $\tau_\varepsilon \downarrow 0$ $\widetilde{\mathsf{P}}-a.s.$ as $\varepsilon \downarrow 0$. Letting $\varepsilon \downarrow 0$ in (3.20) we get:

$$\tag{3.21} \frac{\partial^+ V}{\partial x}(t,x) \le -\frac{1}{t}$$

by dominated convergence. It follows from (3.19) and (3.21) that $(\partial^+ V/\partial x)(t,x) = -1/t$ implying the claim.

10. We show that b is continuous. Note that the same proof also shows that $b(T-) = T/(1+rT)$ as already established above by a different method.

Let us first show that b is right-continuous. For this, fix $t \in \langle 0,T\rangle$ and consider a sequence $t_n \downarrow t$ as $n\to\infty$. Since b is increasing, the right-hand limit $b(t+)$ exists. Because $(t_n, b(t_n)) \in S$ for all $n\ge 1$, and S is closed, it follows that $(t, b(t+)) \in S$. Hence by (2.16) we see $b(t+) \le b(t)$. Since the reverse inequality follows obviously from the fact that b is increasing, this completes the proof of the first claim.

Let us next show that b is left-continuous. Suppose that there exists $t\in\langle 0,T\rangle$ such that $b(t-) < b(t)$. Fix a point x in $\langle b(t-), b(t)]$ and note by (2.12) that for $s<t$ we have:

$$\tag{3.22} V(s,x) - G(s,x) = \int_{b(s)}^{x}\int_{b(s)}^{y} \big(V_{xx}(s,z) - G_{xx}(s,z)\big)\,dz\,dy$$

upon recalling that V is $C^{1,2}$ on C . Note that $G_{xx}=0$ below α so that if $V_{xx}\ge c$ on $R = \{\,(u,y)\in C \mid s\le u<t \text{ and } b(u)<y\le x\,\}$ for some $c>0$ (for all $s<t$ close enough to t and some $x>b(t-)$ close enough to $b(t-)$) then by letting $s\uparrow t$ in (3.22) we get:

$$\tag{3.23} V(t,x) - G(t,x) \ge c\,\frac{\big(x-b(t)\big)^2}{2} > 0$$

contradicting the fact that (t,x) belongs to $\bar{D}$ and thus is an optimal stopping point. Hence the proof reduces to showing that $V_{xx}\ge c$ on small enough R for some $c>0$.

To derive the latter fact we may first note from (2.10) upon using (3.16) that $V_{xx} = (2/(\sigma^2x^2))(-V_t-(1-rx)V_x) \ge (2/(\sigma^2x^2))(-x/t^2-(1-rx)V_x)$. Suppose now that for each $\delta>0$ there is $s<t$ close enough to t and there is $x>b(t-)$ close enough to $b(t-)$ such that $V_x(u,y)\le -1/u+\delta$ for all $(u,y)\in R$ (where we recall that $-1/u = G_x(u,y)$ for all $(u,y)\in R$). Then from the previous inequality we find that $V_{xx}(u,y) \ge (2/(\sigma^2y^2))(-y/u^2 + (1-ry)(1/u-\delta)) = (2/(\sigma^2y^2))((u-y(1+ru))/u^2 - \delta(1-ru)) \ge c > 0$ for $\delta>0$ small enough since $y<u/(1+ru)=\gamma(u)$ and $y<1/r$ for all $(u,y)\in R$. Hence the proof reduces to showing that $V_x(u,y)\le -1/u+\delta$ for all $(u,y)\in R$ with R small enough when $\delta>0$ is given and fixed.

To derive the latter inequality we can make use of the estimate (3.20) to conclude that

$$\frac{V(u,y+\varepsilon)-V(u,y)}{\varepsilon} \le -\,\widetilde{\mathsf{E}}\Big(\frac{1}{(u+\sigma_\varepsilon)\,M_{\sigma_\varepsilon}}\Big) \tag{3.24}$$

where $\sigma_\varepsilon = \inf\{\, 0 \le v \le T-u \mid X_{u+v}^{y+\varepsilon} = b(u)\,\}$ and $M_t = \sup_{0\le s\le t} S_s$. A simple comparison argument (based on the fact that b is increasing) shows that the supremum over all $(u,y)\in R$ on the right-hand side of (3.24) is attained at $(s, x+\varepsilon)$. Letting $\varepsilon \downarrow 0$ in (3.24) we thus get:

$$V_x(u,y) \le -\,\widetilde{\mathsf{E}}\Big(\frac{1}{(u+\sigma)\,M_\sigma}\Big) \tag{3.25}$$

for all $(u,y)\in R$ where $\sigma = \inf\{\, 0\le v\le T-s \mid X_{s+v}^x = b(s)\,\}$. Since by regularity of X we find that $\sigma\downarrow 0$ $\widetilde{\mathsf{P}}$-a.s. as $s\uparrow t$ and $x\downarrow b(t-)$, it follows from (3.25) that:

$$V_x(u,y) \le -\frac{1}{u} + \widetilde{\mathsf{E}}\Big(\frac{(u+\sigma)\,M_\sigma - u}{u\,(u+\sigma)\,M_\sigma}\Big) \le -\frac{1}{u} + \delta \tag{3.26}$$

for all $s<t$ close enough to t and some $x>b(t-)$ close enough to $b(t-)$. This completes the proof of the second claim, and thus the initial claim is proved as well.

11. We show that V is given by the formula (3.6) and that b solves equation (3.5). For this, note that V satisfies the following conditions:

$$V \text{ is } C^{1,2} \text{ on } C\cup D \tag{3.27}$$
$$V_t + \mathbb{L}_X V \text{ is locally bounded} \tag{3.28}$$
$$x\mapsto V(t,x) \text{ is convex} \tag{3.29}$$
$$t\mapsto V_x(t,b(t)\pm) \text{ is continuous.} \tag{3.30}$$

Indeed, the conditions (3.27) and (3.28) follow from the facts that V is $C^{1,2}$ on C and $V=G$ on D upon recalling that D lies below γ so that $G(t,x)=1-x/t$ for all $(t,x)\in D$ and thus G is $C^{1,2}$ on D . [When we say in (3.28) that $V_t+\mathbb{L}_X V$ is locally bounded, we mean that $V_t+\mathbb{L}_X V$ is bounded on $K\cap(C\cup D)$ for each compact set K in $[0,T]\times I\!R_+$.] The condition (3.29) was established in (3.8) above. The condition (3.30) follows from (2.12) since according to the latter we have $V_x(t,b(t)\pm) = -1/t$ for $t>0$.

Since (3.27)-(3.30) are satisfied we know that the local time-space formula (cf. Theorem 3.1 in [11]) can be applied. This gives:

$$\begin{aligned} V(t+s,X_{t+s}) &= V(t,x) + \int_0^s \big(V_t+\mathbb{L}_X V\big)(t+u,X_{t+u})\, I\big(X_{t+u}\neq b(t+u)\big)\,du \\ &\quad + \int_0^s \sigma\, X_{t+u}\, V_x(t+u,X_{t+u})\, I\big(X_{t+u}\neq b(t+u)\big)\,dB_u \\ &\quad + \frac{1}{2}\int_0^s \big(V_x(t+u,X_{t+u}+) - V_x(t+u,X_{t+u}-)\big)\, I\big(X_{t+u}=b(t+u)\big)\,d\ell^b_{t+u}(X) \\ &= \int_0^s \big(G_t+\mathbb{L}_X G\big)(t+u,X_{t+u})\, I\big(X_{t+u}<b(t+u)\big)\,du + M_s \end{aligned} \tag{3.31}$$

where the final equality follows by the smooth-fit condition (2.12) and $M_s = \int_0^s \sigma X_{t+u} V_x(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, dB_u$ is a continuous martingale for $0 \le s \le T-t$ with $t > 0$. Noting that $(G_t + \mathbb{L}_X G)(t,x) = x/t^2 - (1-rx)/t$ for $x < t$ we see that (3.31) yields:

$$V(t+s, X_{t+s}) = V(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < b(t+u)\big)\, du + M_s. \tag{3.32}$$

Setting $s = T-t$, using that $V(T,x) = G(T,x)$ for all $x \ge 0$, and taking the $\widetilde{\mathsf{P}}_{t,x}$-expectation in (3.32), we find by the optional sampling theorem that:

$$\begin{aligned} &\widetilde{\mathsf{E}}_{t,x}\left(\left(1 - \frac{X_T}{T}\right)^+ \right) \\ &= V(t,x) + \int_0^{T-t} \widetilde{\mathsf{E}}_{t,x}\left(\left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < b(t+u)\big) \right) du. \end{aligned} \tag{3.33}$$

Making use of (3.1)-(3.3) we see that (3.33) is the formula (3.6). Moreover, inserting $x = b(t)$ in (3.33) and using that $V(t, b(t)) = G(t, b(t)) = 1 - b(t)/t$, we see that b satisfies the equation (3.5) as claimed.

12. We show that $b(t) > 0$ for all $0 < t \le T$ and that $b(0+) = 0$. For this, suppose that $b(t_0) = 0$ for some $t_0 \in \langle 0, T\rangle$ and fix $t \in \langle 0, t_0\rangle$. Then $(t,x) \in C$ for all $x > 0$ as small as desired. Taking any such $(t,x) \in C$ and denoting by $\tau_S = \tau_S(t,x)$ the first hitting time to S under $\widetilde{\mathsf{P}}_{t,x}$, we find by (3.32) that:

$$\begin{aligned} V(t+\tau_S, X_{t+\tau_S}) &= G(t+\tau_S, X_{t+\tau_S}) = \left(1 - \frac{X_{t+\tau_S}}{t+\tau_S}\right)^+ = V(t,x) + M_{t+\tau_S} \\ &= 1 - \frac{x}{t} + M_{t+\tau_S}. \end{aligned} \tag{3.34}$$

Taking the $\widetilde{\mathsf{P}}_{t,x}$-expectation and letting $x \downarrow 0$ we get:

$$\widetilde{\mathsf{E}}_{t,0}\left(1 - \frac{X_{t+\tau_S}}{t+\tau_S}\right)^+ = 1 \tag{3.35}$$

where $\tau_S = \tau_S(t,0)$. As clearly $\widetilde{\mathsf{P}}_{t,0}(X_{t+\tau_S} \ge T) > 0$ we see that the left-hand side of (3.35) is strictly smaller than 1 thus contradicting the identity. This shows that $b(t)$ must be strictly positive for all $0 < t \le T$. Combining this conclusion with the known inequality $b(t) \le \gamma(t)$ which is valid for all $0 < t \le T$ we see that $b(0+) = 0$ as claimed.

13. We show that b is the unique solution of the nonlinear integral equation (3.5) in the class of continuous functions $c : \langle 0, T\rangle \to I\!R$ satisfying $0 < c(t) < t/(1+rt)$ for all $0 < t < T$. (Note that this class is larger than the class of functions having the established properties of b which is moreover known to be increasing.) The proof of the uniqueness will be presented in the final three steps of the main proof as follows.

14. Let $c : \langle 0, T] \to I\!R$ be a continuous solution of the equation (3.5) satisfying $0 < c(t) < t$ for all $0 < t < T$. We want to show that this c must then be equal to the optimal stopping boundary b.

Motivated by the derivation (3.31)-(3.33) which leads to the formula (3.6), let us consider the function $U^c : \langle 0,T] \times [0,\infty\rangle \rightarrow I\!R$ defined as follows:

$$U^c(t,x) = \widetilde{\mathsf{E}}_{t,x}\bigg(\Big(1-\frac{X_T}{T}\Big)^+\bigg) - \int_0^{T-t} \widetilde{\mathsf{E}}_{t,x}\bigg(\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\bigg)\, du \tag{3.36}$$

for $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. In terms of (3.1)-(3.3) note that U^c is explicitly given by:

$$U^c(t,x) = F(T-t,x) - \int_0^{T-t} \frac{1}{t+u}\bigg(\Big(\frac{1}{t+u}+r\Big) G\big(u,x,c(t+u)\big) - H\big(u,x,c(t+u)\big)\bigg)\, du \tag{3.37}$$

for $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. Observe that the fact that c solves (3.5) on $\langle 0,T\rangle$ means exactly that $U^c(t,c(t)) = G(t,c(t))$ for all $0<t<T$. We will now moreover show that $U^c(t,x) = G(t,x)$ for all $x \in [0,c(t)]$ with $t \in \langle 0,T\rangle$. This is the key point in the proof (cf. [12] and [13]) that can be derived using a martingale argument as follows.

If $X = (X_t)_{t \ge 0}$ is a Markov process (with values in a general state space) and we set $F(t,x) = \mathsf{E}_x(G(X_{T-t}))$ for a (bounded) measurable function G with $\mathsf{P}_x(X_0 = x) = 1$, then the Markov property of X implies that $F(t,X_t)$ is a martingale under P_x for $0 \le t \le T$. Similarly, if we set $F(t,x) = \mathsf{E}_x(\int_0^{T-t} H(X_u)\, du)$ for a (bounded) measurable function H with $\mathsf{P}_x(X_0 = x) = 1$, then the Markov property of X implies that $F(t,X_t) + \int_0^t H(X_u)\, du$ is a martingale under P_x for $0 \le t \le T$. Combining these two martingale facts applied to the time-space Markov process $(t+s, X_{t+s})$ instead of X_s , we find that:

$$U^c(t+s, X_{t+s}) - \int_0^s \Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\, du \tag{3.38}$$

is a martingale under $\widetilde{\mathsf{P}}_{t,x}$ for $0 \le s \le T-t$. We may thus write:

$$U^c(t+s, X_{t+s}) - \int_0^s \Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\, du = U^c(t,x) + N_s \tag{3.39}$$

where $(N_s)_{0 \le s \le T-t}$ is a martingale with $N_0 = 0$ under $\widetilde{\mathsf{P}}_{t,x}$.

On the other hand, we know from (3.13) that:

$$G(t+s, X_{t+s}) = G(t,x) + \int_0^s \Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < \alpha(t+u)\big)\, du + M_s + L_s \tag{3.40}$$

where $M_s = -\sigma \int_0^s (X_{t+u}/(t+u))\, I(X_{t+u} < \alpha(t+u))\, d\widehat{B}_u$ is a continuous martingale under $\widetilde{\mathsf{P}}_{t,x}$ and $L_s = (1/2) \int_0^s d\ell^\alpha_{t+u}(X)/(t+u)$ is an increasing process for $0 \le s \le T-t$.

For $0 \le x \le c(t)$ with $t \in \langle 0,T\rangle$ given and fixed consider the stopping time:

$$\sigma_c = \inf\, \{\, 0 \le s \le T-t \mid X_{t+s} \ge c(t+s)\, \}. \tag{3.41}$$

Using that $U^c(t,c(t)) = G(t,c(t))$ for all $0 < t < T$ (since c solves (3.5) as pointed out above) and that $U^c(T,x) = G(T,x)$ for all $x \ge 0$, we see that $U^c(t+\sigma_c, X_{t+\sigma_c}) = G(t+\sigma_c, X_{t+\sigma_c})$. Hence from (3.39) and (3.40) using the optional sampling theorem we find:

$$
\begin{aligned}
(3.42)\qquad U^c(t,x) &= \widetilde{\mathsf{E}}_{t,x}\Big(U^c(t+\sigma_c, X_{t+\sigma_c})\Big) \\
&\qquad - \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_c}\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\, du\bigg) \\
&= \widetilde{\mathsf{E}}_{t,x}\Big(G(t+\sigma_c, X_{t+\sigma_c})\Big) \\
&\quad - \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_c}\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\, du\bigg) \\
&= G(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_c}\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < \alpha(t+u)\big)\, du\bigg) \\
&\quad - \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_c}\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\, du\bigg) \\
&= G(t,x)
\end{aligned}
$$

since $X_{t+u} < \alpha(t+u)$ and $X_{t+u} < c(t+u)$ for all $0 \le u < \sigma_c$. This proves that $U^c(t,x) = G(t,x)$ for all $x \in [0,c(t)]$ with $t \in \langle 0,T\rangle$ as claimed.

15. We show that $U^c(t,x) \le V(t,x)$ for all $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. For this, consider the stopping time:

$$
(3.43)\qquad \tau_c = \inf\,\{\, 0 \le s \le T-t \mid X_{t+s} \le c(t+s) \,\}
$$

under $\widetilde{\mathsf{P}}_{t,x}$ with $(t,x) \in \langle 0,T] \times [0,\infty\rangle$ given and fixed. The same arguments as those given following (3.41) above show that $U^c(t+\tau_c, X_{t+\tau_c}) = G(t+\tau_c, X_{t+\tau_c})$. Inserting τ_c instead of s in (3.39) and using the optional sampling theorem we get:

$$
(3.44)\qquad U^c(t,x) = \widetilde{\mathsf{E}}_{t,x}\Big(U^c(t+\tau_c, X_{t+\tau_c})\Big) = \widetilde{\mathsf{E}}_{t,x}\Big(G(t+\tau_c, X_{t+\tau_c})\Big) \le V(t,x)
$$

where the final inequality follows from the definition of V proving the claim.

16. We show that $c \ge b$ on $[0,T]$. For this, consider the stopping time:

$$
(3.45)\qquad \sigma_b = \inf\,\{\, 0 \le s \le T-t \mid X_{t+s} \ge b(t+s) \,\}
$$

under $\widetilde{\mathsf{P}}_{t,x}$ where $(t,x) \in \langle 0,T\rangle \times [0,\infty\rangle$ such that $x < b(t) \wedge c(t)$. Inserting σ_b in place of s in (3.32) and (3.39) and using the optional sampling theorem we get:

$$
(3.46)\qquad \widetilde{\mathsf{E}}_{t,x}\Big(V(t+\sigma_b, X_{t+\sigma_b})\Big) = G(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_b}\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big)\, du\bigg)
$$

$$
\begin{aligned}
(3.47)\qquad &\widetilde{\mathsf{E}}_{t,x}\Big(U^c(t+\sigma_b, X_{t+\sigma_b})\Big) \\
&\quad = G(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_b}\Big(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\Big) I\big(X_{t+u} < c(t+u)\big)\, du\bigg)
\end{aligned}
$$

where we also use that $V(t,x) = U^c(t,x) = G(t,x)$ for $x < b(t) \wedge c(t)$. Since $U^c \le V$ it follows from (3.46) and (3.47) that:

$$\widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\sigma_b} \bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \bigg) I\big(X_{t+u} \ge c(t{+}u)\big)\, du \bigg) \ge 0. \tag{3.48}$$

Due to the fact that $b(t) < t/(1{+}rt)$ for all $0 < t < T$, we see that $X_{t+u}/(t+u)^2 - (1 - rX_{t+u})/(t+u) < 0$ in (3.48) so that by the continuity of b and c it follows that $c \ge b$ on $[0,T]$ as claimed.

17. We show that c must be equal to b . For this, let us assume that there is $t \in \langle 0,T \rangle$ such that $c(t) > b(t)$. Pick $x \in \langle b(t), c(t) \rangle$ and consider the stopping time τ_b from (2.17). Inserting τ_b instead of s in (3.32) and (3.39) and using the optional sampling theorem we get:

$$\widetilde{\mathsf{E}}_{t,x}\Big(G(t+\tau_b, X_{t+\tau_b}) \Big) = V(t,x) \tag{3.49}$$

$$\begin{aligned} &\widetilde{\mathsf{E}}_{t,x}\Big(G(t+\tau_b, X_{t+\tau_b}) \Big) \\ &\quad = U^c(t,x) + \widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\tau_b} \bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \bigg) I\big(X_{t+u} < c(t{+}u)\big)\, du \bigg) \end{aligned} \tag{3.50}$$

where we also use that $V(t+\tau_b, X_{t+\tau_b}) = U^c(t+\tau_b, X_{t+\tau_b}) = G(t+\tau_b, X_{t+\tau_b})$ upon recalling that $c \ge b$ and $U^c = G$ either below c or at T . Since $U^c \le V$ we see from (3.49) and (3.50) that:

$$\widetilde{\mathsf{E}}_{t,x}\bigg(\int_0^{\tau_b} \bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \bigg) I\big(X_{t+u} < c(t{+}u)\big)\, du \bigg) \ge 0. \tag{3.51}$$

Due to the fact that $c(t) < t/(1{+}rt)$ for all $0 < t < T$ by assumption, we see that $X_{t+u}/(t+u)^2 - (1 - rX_{t+u})/(t+u) < 0$ in (3.51) so that by the continuity of b and c it follows that such a point (t,x) cannot exist. Thus c must be equal to b , and the proof is complete. $\square$

4. Remarks on numerics

1. The following method can be used to calculate the optimal stopping boundary b numerically by means of the integral equation (3.5). Note that the formula (3.6) can be used to calculate the arbitrage-free price V when b is known.

Set $t_i = ih$ for $i = 0, 1, \ldots, n$ where $h = T/n$ and denote:

$$J(t, b(t)) = 1 - \frac{b(t)}{t} - F(T{-}t, b(t)) \tag{4.1}$$

$$K(t, b(t); t{+}u, b(t{+}u)) = \frac{1}{t{+}u}\bigg(\Big(\frac{1}{t{+}u} + r \Big) G(u, b(t), b(t{+}u)) - H(u, b(t), b(t{+}u)) \bigg). \tag{4.2}$$

Then the following discrete approximation of the integral equation (3.5) is valid:

$$J(t_i, b(t_i)) = \sum_{j=i+1}^{n} K(t_i, b(t_i); t_j, b(t_j))\, h \tag{4.3}$$

for $i = 0, 1, \ldots, n-1$. Letting $i = n-1$ and $b(t_n) = T/(1+rT)$ we can solve equation (4.3) numerically and get a number $b(t_{n-1})$. Letting $i = n-2$ and using the values of $b(t_{n-1})$ and $b(t_n)$ we can solve equation (4.3) numerically and get a number $b(t_{n-2})$. Continuing the recursion we obtain $b(t_n), b(t_{n-1}), \ldots, b(t_1), b(t_0)$ as an approximation of the optimal stopping boundary b at points $0, h, \ldots, T-h, T$.

It is an interesting numerical problem to show that the approximation converges to the true function b on $[0,T]$ as $h \downarrow 0$. Another interesting problem is to derive the rate of convergence.

2. To perform the previous recursion we need to compute the functions F , G , H from (3.1)-(3.3) as efficiently as possible. Simply by observing the expressions (3.1)-(3.4) it is apparent that finding these functions numerically is not trivial. Moreover, the nature of the probability density function f in (3.4) presents a further numerical challenge. Part of this probability density function is the Hartman-Watson density discussed in [1]. As t tends to zero, the numerical estimate of the Hartman-Watson density oscillates, with the oscillations increasing rapidly in both amplitude and frequency as t gets closer to zero. The authors of [1] mention that this may be a consequence of the fact that $t \mapsto \exp(2\pi^2/\sigma^2 t)$ rapidly increases to infinity while $z \mapsto \sin(4\pi z/\sigma^2 t)$ oscillates more and more frequently. This rapid oscillation makes accurate estimation of $f(t,s,a)$ with t close to zero very difficult.

The problems when dealing with t close to zero are relevant to pricing the early exercise Asian call option. To find the optimal stopping boundary b as the solution to the implicit equation (4.3) it is necessary to work backward from T to 0 . Thus to get an accurate estimate for b when $b(T)$ is given, the next estimate of $b(u)$ must be found for some value of u close to T so that $t = T-u$ will be close to zero.

Even if we get an accurate estimate for f , to solve (3.1)-(3.3) we need to evaluate two nested integrals. This is slow computationally. A crude attempt has been made at storing values for f and using these to estimate F , G , H in (3.1)-(3.3) but this method has not produced reliable results.

3. Another approach to finding the functions F , G , H from (3.1)-(3.3) can be based on numerical solutions of partial differential equations. Two distinct methods are available.

Consider the transition probability density of the process X given by:

$$p(s,x;t,y) = \frac{d}{dy} \widetilde{\mathsf{P}}(X_t \le y \mid X_s = x) \tag{4.4}$$

where $0 \le s < t$ and $x, y \ge 0$. Since $p(s,x;t,y) = p(0,x;t-s,y)$ we see that there is no restriction to assume that $s = 0$ in the sequel.

4. The *forward equation* approach leads to the initial-value problem:

$$p_t = -((1-ry)\,p)_y + (D\,y\,p)_{yy} \quad (\,t > 0\,,\ y > 0\,) \tag{4.5}$$

$$p(0,x;0+,y) = \delta(y-x) \quad (\,y \ge 0\,) \tag{4.6}$$

where $D = \sigma^2/2$ and $x \ge 0$ is given and fixed (recall that δ denotes the Dirac delta function). Standard results (cf. [4]) imply that there is a unique non-negative solution $(t,y) \mapsto p(0,x;t,y)$ of (4.5)-(4.6). The solution p satisfies the following boundary conditions:

$$p(0,x;t,0+) = 0 \quad (\,0 \text{ is entrance}\,) \tag{4.7}$$

$$(4.8) \qquad p(0,x;t,\infty-) = 0 \quad (\,\infty \text{ is normal}\,).$$

The solution p satisfies the following integrability condition:

$$(4.9) \qquad \int_0^\infty p(0,x;t,y)\,dy = 1$$

for all $x \geq 0$ and all $t \geq 0$. Once the solution $(t,y) \mapsto p(0,x;t,y)$ of (4.5)-(4.6) has been found, the functions F, G, H from (3.1)-(3.3) can be computed using the general formula:

$$(4.10) \qquad \widetilde{\mathsf{E}}_{0,x}(g(X_t)) = \int_0^\infty g(y)\,p(0,x;t,y)\,dy$$

upon choosing the appropriate function $g : I\!\!R_+ \to I\!\!R_+$.

5. The *backward equation* approach leads to the terminal-value problem:

$$(4.11) \qquad q_t = (1-rx)\,q_x + D\,x^2\,q_{xx} \quad (\,t>0\,,\ x>0\,)$$

$$(4.12) \qquad q(T,x) = h(x) \quad (\,x \geq 0\,)$$

where $h : I\!\!R_+ \to I\!\!R_+$ is a given function. Standard results (cf. [4]) imply that there is a unique non-negative solution $(t,x) \mapsto q(t,x)$ of (4.11)-(4.12). Taking $x \mapsto h(x)$ to be $x \mapsto (1-x/T)^+$ (with T fixed), $x \mapsto x\,I(x \leq y)$ (with y fixed), $x \mapsto I(x \leq y)$ (with y fixed) it follows that the unique non-negative solution q of (4.11)-(4.12) coincides with F, G, H from (3.1)-(3.3) respectively. (For numerical results of a similar approach see [14].)

6. It is an interesting numerical problem to carry out either of the two methods described above and produce approximations to the optimal stopping boundary b using (4.3). Another interesting problem is to derive the rate of convergence.

5. Appendix

In this section we derive the explicit expression for the probability density function f of (S_t, I_t) under $\widetilde{\mathsf{P}}$ with $S_0 = 1$ and $I_0 = 0$ given in (3.4) above.

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. With $t > 0$ and $\nu \in I\!\!R$ given and fixed recall from [20, page 527] that the random variable $A_t^{(\nu)} = \int_0^t e^{2(B_s+\nu s)}ds$ has the conditional distribution:

$$(5.1) \qquad \mathsf{P}\Big(A_t^{(\nu)} \in dy \,\Big|\, B_t + \nu t = x\Big) = a(t,x,y)\,dy$$

where the density function a for $y > 0$ is given by:

$$(5.2) \qquad a(t,x,y) = \frac{1}{\pi y^2}\exp\left(\frac{x^2+\pi^2}{2t} + x - \frac{1}{2y}\Big(1+e^{2x}\Big)\right)$$
$$\times \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{e^x}{y}\cosh(z)\right)\sinh(z)\,\sin\left(\frac{\pi z}{t}\right)dz.$$

This implies that the random vector $\big(2(B_t+\nu t), A_t^{(\nu)}\big)$ has the distribution:

$$(5.3) \qquad \mathsf{P}\Big(2(B_t+\nu t) \in dx, A_t^{(\nu)} \in dy\Big) = b(t,x,y)\,dx\,dy$$

where the density function b for $y > 0$ is given by:

$$
\begin{aligned}
(5.4) \qquad b(t,x,y) &= a\left(t, \frac{x}{2}, y\right) \frac{1}{2\sqrt{t}}\, \varphi\left(\frac{x-2\nu t}{2\sqrt{t}}\right) \\
&= \frac{1}{(2\pi)^{3/2} y^2 \sqrt{t}} \exp\left(\frac{\pi^2}{2t} + \Big(\frac{\nu+1}{2}\Big)x - \frac{\nu^2}{2}\, t - \frac{1}{2y}\Big(1+e^x\Big)\right) \\
&\quad \times \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{e^{x/2}}{y}\cosh(z)\right) \sinh(z)\, \sin\left(\frac{\pi z}{t}\right) dz
\end{aligned}
$$

and we set $\varphi(z) = (1/\sqrt{2\pi}) e^{-z^2/2}$ for $z \in I\!R$ (for related expressions in terms of Hermite functions see [3] and [15]).

Denoting $K_t = \alpha B_t + \beta t$ and $L_t = \int_0^t e^{\alpha B_s + \beta s} ds$ with $\alpha \neq 0$ and $\beta \in I\!R$ given and fixed, and using that the scaling property of B implies:

$$
(5.5) \qquad \mathsf{P}\left(\alpha B_t + \beta t \le x, \int_0^t e^{\alpha B_s + \beta s}\, ds \le y\right) = \mathsf{P}\left(2(B_{t'} + \nu t') \le x, \int_0^{t'} e^{2(B_s + \nu s)}\, ds \le \frac{\alpha^2}{4}\, y\right)
$$

with $t' = \alpha^2 t/4$ and $\nu = 2\beta/\alpha^2$, it follows by applying (5.3) and (5.4) that the random vector (K_t, L_t) has the distribution:

$$
(5.6) \qquad \mathsf{P}\Big(K_t \in dx, L_t \in dy\Big) = c(t,x,y)\, dx\, dy
$$

where the density function c for $y > 0$ is given by:

$$
\begin{aligned}
(5.7) \qquad c(t,x,y) &= \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, x,\, \frac{\alpha^2}{4}\, y\right) \\
&= \frac{2\sqrt{2}}{\pi^{3/2}\alpha^3} \frac{1}{y^2\sqrt{t}} \exp\left(\frac{2\pi^2}{\alpha^2 t} + \Big(\frac{\beta}{\alpha^2} + \frac{1}{2}\Big)x - \frac{\beta^2}{2\alpha^2}\, t - \frac{2}{\alpha^2 y}\Big(1+e^x\Big)\right) \\
&\quad \times \int_0^\infty \exp\left(-\frac{2z^2}{\alpha^2 t} - \frac{4e^{x/2}}{\alpha^2 y}\cosh(z)\right) \sinh(z)\, \sin\left(\frac{4\pi z}{\alpha^2 t}\right) dz.
\end{aligned}
$$

From (2.8) and (2.3) we see that f satisfies:

$$
(5.8) \qquad f(t,s,a) = \frac{1}{s}\, c(t, \log(s), a) = \frac{1}{s}\, \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, \log(s),\, \frac{\alpha^2}{4}\, a\right)
$$

with $\alpha = \sigma$ and $\beta = r + \sigma^2/2$. Hence (3.4) follows by the final expression in (5.4).

References

[1] Barrieu, P. Rouault, A. *and* Yor, M. (2003). A study of the Hartman-Watson distribution motivated by numerical problems related to Asian options pricing. *Prépublication PMA* 813, *Université Pierre et Marie Curie, Paris.*

[2] Carr, P. Jarrow, R. *and* Myneni, R. (1992). Alternative characterizations of American put options. *Math. Finance* 2 (78–106).

[3] Dufresne, D. (2001). The integral of geometric Brownian motion. *Adv. Appl. Probab.* 33 (223–241).

[4] Feller, W. (1952). The parabolic differential equations and the associated semi-groups of transformations. *Ann. of Math.* 55 (468-519).

[5] Hansen, A. T. *and* Jørgensen, P. L. (2000). Analytical valuation of American-style Asian options. *Management Sci.* 46 (1116–1136).

[6] Jacka, S. D. (1991). Optimal stopping and the American put. *Math. Finance* 1 (1–14).

[7] Karatzas, I. *and* Shreve, S. E. (1998). *Methods of Mathematical Finance.* Springer-Verlag.

[8] Kim, I. J. (1990). The analytic valuation of American options. *Rev. Financial Stud.* 3 (547–572).

[9] Kolodner, I. I. (1956). Free boundary problem for the heat equation with applications to problems of change of phase I. General method of solution. *Comm. Pure Appl. Math.* 9 (1–31).

[10] Kramkov, D. O. *and* Mordecky, E. (1994). Integral option. *Theory Probab. Appl.* 39 (162–173).

[11] Peskir, G. (2005). A change-of-variable formula with local time on curves. *J. Theoret. Probab.* 18 (499–535).

[12] Peskir, G. (2005). On the American option problem. *Math. Finance.* 15 (169–181).

[13] Peskir, G. (2005). The Russian option: Finite horizon. *Finance Stoch.* 9 (251–267).

[14] Rogers, L. C. G. *and* Shi, Z. (1995). The value of an Asian option. *J. Appl. Probab.* 32 (1077-1088).

[15] Schröder, M. (2003). On the integral of geometric Brownian motion. *Adv. Appl. Probab.* 35 (159–183).

[16] Shepp, L. A. *and* Shiryaev, A. N. (1994). A new look at the Russian option. *Theory Probab. Appl.* 39 (103–119).

[17] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer-Verlag.

[18] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance (Facts, Models, Theory).* World Scientific.

[19] Wu, L. Kwok, Y. K. *and* Yu, H. (1999). Asian options with the American early exercise feature. *Int. J. Theor. Appl. Finance* 1 (101–111).

[20] Yor, M. (1992). On some exponential functionals of Brownian motion. *Adv. Appl. Probab.* 24 (509–531).

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk

Nadia Uys
Programme for Advanced Mathematics of Finance
School of Computational and Applied Mathematics
University of the Witwatersrand, South Africa
Private Bag 3, Wits, 2050
nuys@cam.wits.ac.za