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The Existence of Measurable Approximating Maximums

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Certain statistical models are described by a parametrized family of sequences of random functions adapted to the fixed sequence of σ-algebras. The main problem is to estimate maximum points of the associated information function which is the limit of the sequence of random functions. For this various sequences of maximum functions are used. The main theorem of this paper establishes the existence of measurable approximating maximums associated to such a family which is indexed by the second countable Hausdorff space satisfying the Blackwell property. These spaces might be characterized as the second countable analytic ones. It is moreover shown that for separable families of sequences of adapted random functions the same theorem remains valid without Blackwell property as well.

1. Introduction

Some statistical models are described by a family $\mathcal{H} = (\{ h_n(\omega,\theta) , \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0)$ of reversed submartingales defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the analytic metric space Θ_0 (see [2], [3], [4], [5]). From the general theory of reversed submartingales we know that $h_n(\theta)$ converges P-almost surely to a random variable $h_\infty(\theta)$ as $n \to \infty$, for all $\theta \in \Theta_0$. If the tail σ-algebra $\mathcal{S}_\infty = \bigcap_{n=1}^\infty \mathcal{S}_n$ is degenerated, that is $P(A) \in \{0,1\}$ for all $A \in \mathcal{S}_\infty$, then $h_\infty(\theta)$ is also degenerated, that is P-almost surely equal to some constant which depends on θ for $\theta \in \Theta_0$. In this case the *information function* associated to $\mathcal{H}$:

$$I(\theta) = \lim_{n\to\infty} h_n(\theta) \ P\text{-a.s.} = \lim_{n\to\infty} Eh_n(\theta)$$

may be well-defined for all $\theta \in \Theta_0$. The main problem in the context of statistical models just mentioned is to estimate the maximum points of $I(\theta)$ on Θ_0 using only information on the sequence $\{ h_n \mid n \geq 1 \}$. For this, two concepts of maximum functions might be introduced as follows. Let $\{ \hat{\theta}_n \mid n \geq 1 \}$ be a sequence of functions from Ω into Θ , where (Θ, d) is a compact metric space containing Θ_0 . Then $\{ \hat{\theta}_n \mid n \geq 1 \}$ is called a *sequence of empirical maximums* associated to $\mathcal{H}$, if there exist a function $q : \Omega \to \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying the following two conditions:

(1.1) $$\hat{\theta}_n(\omega) \in \Theta_0 \ , \quad \forall \omega \in \Omega \setminus N \ , \ \forall n \geq q(\omega)$$

(1.2) $$h_n(\omega, \hat{\theta}_n(\omega)) = h_n^*(\omega, \Theta_0) \ , \quad \forall \omega \in \Omega \setminus N \ , \ \forall n \geq q(\omega)$$

where $h_n^*(\omega, B) = \sup_{\theta \in B} h_n(\omega,\theta)$ for $n \geq 1$, $\omega \in \Omega$ and $B \subset \Theta_0$. The sequence

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$\{\,\hat{\theta}_n \mid n \geq 1\,\}$ is called a *sequence of approximating maximums* associated to $\mathcal{H}$, if there exist a function $q : \Omega \rightarrow \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying the following two conditions:

$$\hat{\theta}_n(\omega) \in \Theta_0 \;,\quad \forall \omega \in \Omega \setminus N \;,\; \forall n \geq q(\omega) \tag{1.3}$$

$$\liminf_{n\rightarrow\infty}\; h_n(\omega, \hat{\theta}_n(\omega)) \;\geq\; \sup_{\theta\in\Theta_0} I(\theta) \;,\quad \forall \omega \in \Omega \setminus N \;. \tag{1.4}$$

Note that even though $h_n(\omega, \cdot)$ need not attain its maximal value on Θ_0 , and therefore (1.2) fails in this case, we can always find a sequence $\{\,\hat{\theta}_n \mid n \geq 1\,\}$ satisfying (1.4). However, the statistical nature lying behind requires $\hat{\theta}_n$ to be measurable relative to $\mathcal{S}_n$ for all $n \geq 1$. This requirement makes the establishment of the existence of approximating maximums much harder, and calls for assumptions on Θ_0 in order to make the existence of suitable measurable selections available. Furthermore, one of the main interests in the context of statistical models just mentioned is to characterize the sets of all possible accumulation and limit points of all possible sequences of approximating maximums associated to $\mathcal{H}$. In this direction a certain convergence uniformization is important to be established, as shown in the proof of the main Theorem 4.1 below.

Both of the problems just mentioned, namely the existence and the uniformization, are solved in a fundamental theorem due to J. Hoffmann-Jørgensen (see [3] p.42) in the case where the parameter set Θ_0 is an analytic metric space. In this paper we show that a little stronger version of the same theorem remains valid, if the parameter set Θ_0 is only assumed to be a second countable Hausdorff space satisfying the Blackwell property. These spaces might be characterized as the second countable analytic ones. Actually, the proof carries out without submartingale property as well, and the only assumption which is essentially used on $\mathcal{H}$ is the $\mathcal{S}_n \times \mathcal{B}(\Theta_0)$-measurability of h_n being valid for all $n \geq 1$. Thus the results below are formulated and stated for general families of sequences of adapted random functions. (We think, however, that the statistical background just explained is important, and the reader should be aware of it.) Finally, it is shown that for separable families the Blackwell property is not needed.

2. Preliminary facts

A measurable space $(X, \mathcal{A})$ is called *Blackwell*, if $f(X)$ is an analytic subset of the real line, whenever f is an $\mathcal{A}$-measurable real valued function on X . A Hausdorff space X is called *Blackwell*, if X together with its Borel σ-algebra $\mathcal{B}(X)$ forms a Blackwell space. In this case we shall often say that X satisfies the *Blackwell property*. It is well-known that every analytic space is a separable Hausdorff space satisfying the Blackwell property, and in particular every analytic metric space is a second countable Hausdorff space satisfying the Blackwell property. Moreover, one can verify that every second countable Blackwell space is analytic.

Blackwell spaces posses nice stability properties and we shall refer the reader to [1] for an extensive treatment of this subject. In this paper we shall need the statement of the following well-known theorem (see [1]):

(2.1) (*The projection theorem*)
Let $(X, \mathcal{A})$ be a Blackwell space, and let $(Y, \mathcal{B})$ be a measurable space. If A is an $\mathcal{A} \times \mathcal{B}$-Souslin set in $X \times Y$, then the projection $\pi_Y(A)$ of A onto Y is a $\mathcal{B}$-Souslin set.

Also, we shall need the following version of the so-called measurable selection theorem, which follows easily by Theorem 3.4 with (3.7.10) in [1]:

(2.2) (*The measurable selection theorem*)
Let $(\Omega, \mathcal{F}, P)$ *be a probability space, and let* $(X, \mathcal{A})$ *be a countable generated Blackwell space. Then for every* $\mathcal{F} \times \mathcal{A}$*-Souslin set* C *in* $\Omega \times X$ *, there exists a* P*-measurable map* $\xi : \Omega \to X$ *such that:*

$$\big(\omega, \xi(\omega)\big) \in C \ , \quad \forall \omega \in \pi_\Omega(C)$$

where $\pi_\Omega(C)$ *is the projection of* C *onto* Ω *.*

Let us say that the usefulness of these two theorems follows mainly from the well-known fact that for some σ-algebra $\mathcal{A}$, every $\mathcal{A}$-Souslin set is universally $\mathcal{A}$-measurable. Note that every second countable Blackwell space is a countably generated Blackwell space. In other words, every second countable analytic space (for example, every analytic metric space) is a countable generated Blackwell space. Also note that every countable set together with any σ-algebra of its subsets forms a countably generated Blackwell space. Moreover, it is well-known that every $\mathcal{A}$-Souslin subset of a Blackwell space $(X, \mathcal{A})$ together with the trace σ-algebra forms also a Blackwell space. All these facts will be used more or less explicitly in the rest of the paper.

We shall now introduce the main object under our consideration in this paper. For this, let $(\Omega, \mathcal{F}, P)$ be a probability space, let Θ_0 be a topological space, let $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \ \ldots$ be a *decreasing* sequence of σ-algebras on Ω all being contained in $\mathcal{F}$, and let $h_n(\cdot, \theta) : \Omega \to \mathbf{R}$ be a random variable which is mesurable relative to $\mathcal{S}_n$ for all $n \geq 1$ and all $\theta \in \Theta_0$. Under these conditions we will shortly say that $\mathcal{H} = (\{\, h_n(\omega, \theta) \, , \, \mathcal{S}_n \mid n \geq 1 \,\} \mid \theta \in \Theta_0)$ is an *adapted family* (of random functions and σ-algebras) defined on $(\Omega, \mathcal{F}, P)$ and indexed by Θ_0 . Let $\mathcal{B}(\Theta_0)$ denote the Borel σ-algebra on Θ_0 . According to [4], we will say that $\mathcal{H}$ is:

(2.3) *measurable*, if h_n is $\mathcal{S}_n \times \mathcal{B}(\Theta_0)$-measurable for all $n \geq 1$

(2.4) *degenerated*, if the tail σ-algebra $\mathcal{S}_\infty = \bigcap_{n=1}^\infty \mathcal{S}_n$ is degenerated, that is $P(A) \in \{0, 1\}$ for all $A \in \mathcal{S}_\infty$

(2.5) *separable relative to* $\mathcal{S} \subset 2^{\Theta_0}$ and $\mathcal{C} \subset 2^{\mathbf{R}}$, if $\forall B \in \mathcal{S}$ there exists a sequence $\{\, \theta_i \mid i \geq 1 \,\}$ in Θ_0 such that $\forall C \in \mathcal{C}$ we have:

$$P^*\bigg(\bigcup_{n=1}^{\infty} \{\, h_n(\theta_i) \in C \, , \, \forall \theta_i \in B \,\} \triangle \{\, h_n(\theta) \in C \, , \, \forall \theta \in B \,\} \bigg) = 0$$

(2.6) *separable*, if it is separable relative to the family $\mathcal{G}(\Theta_0)$ of all open sets in Θ_0 and the family $\mathcal{C}(\mathbf{R})$ of all closed sets in $\mathbf{R}$

(2.7) *P-a.s. upper (lower) semicontinuous*, if there exists a P-null set $N \in \mathcal{F}$ such that the function $\theta \mapsto h_n(\omega, \theta)$ is upper (lower) semicontinuous on Θ_0 for all $\omega \in \Omega \setminus N$ and all $n \geq 1$.

By Proposition 3.1 in [4] and slight modifications of Proposition 3.3 and Corollary 3.6 in [4] one can easily verify that the following statements are satisfied:

(2.8) Let $\mathcal{H} = (\{\, h_n(\omega, \theta) \, , \, \mathcal{S}_n \mid n \geq 1 \,\} \mid \theta \in \Theta_0)$ be an adapted family defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the second countable Hausdorff space Θ_0 .

If $\mathcal{H}$ is separable, then there exists a dense sequence $\{ \theta_i \mid i \geq 1 \}$ in Θ_0 and a P-null set $N \in \mathcal{F}$ such that $\forall G \in \mathcal{G}(\Theta_0)$, $\forall \omega \in \Omega \setminus N$ and $\forall n \geq 1$ we have:

(2.8.1) $$\sup_{\theta \in G} h_n(\omega, \theta) = \sup_{\theta_i \in G} h_n(\omega, \theta_i)$$

(2.8.2) $$\inf_{\theta \in G} h_n(\omega, \theta) = \inf_{\theta_i \in G} h_n(\omega, \theta_i) \ .$$

Moreover, (2.8.1) remains true if $\mathcal{H}$ is separable relative to $\mathcal{G}(\Theta_0)$ and $\mathcal{C}_{-\infty}(\mathbf{R}) = \{ (-\infty, p] \mid p \in \mathbf{Q} \}$, and (2.8.2) remains true if $\mathcal{H}$ is separable relative to $\mathcal{G}(\Theta_0)$ and $\mathcal{C}_{+\infty}(\mathbf{R}) = \{ [q, +\infty) \mid q \in \mathbf{Q} \}$. A slight modification of (4.18) in Proposition 4.3 in [4] yields:

(2.9) If an adapted family $\mathcal{H}$ is P-a.s. lower semicontinuous, then it is separable relative to $\mathcal{G}(\Theta_0)$ and $\mathcal{C}_{-\infty}(\mathbf{R})$.

For more information and details in this direction we shall refer the reader to [4].

3. Uniformization

In order to formulate a general version of the existence theorem in the next section (Theorem 4.1), we need a definition of the uniform convergence of a given sequence of random functions with values in a topological space to a given subset of it. One of its particular forms, which will be suitable for our purposes, is presented in Lemma 3.1 below. In Lemma 3.2 we state the essential point of the convergence uniformization in the existence theorem.

Let X be a topological space, and let A be a subset of X . Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{ \hat{\theta}_n \mid n \geq 1 \}$ be a sequence of functions from Ω into X . We shall say that the sequence $\{ \hat{\theta}_n \mid n \geq 1 \}$ *converges uniformly* to A , if for every open set G in X containing A , there exists $n_0 \geq 1$ such that $\forall n \geq n_0$ we have:

(3.1) $$\hat{\theta}_n(\omega) \in G \ , \quad \forall \omega \in \Omega \ .$$

In this case we shall write $\hat{\theta}_n \rightrightarrows A$ on Ω . If (3.1) is satisfied for all $\omega \in \Omega \setminus N$ where N is a P-null set in $\mathcal{F}$, then we shall say that the sequence $\{ \hat{\theta}_n \mid n \geq 1 \}$ *converges uniformly P-almost surely* to A and in this case we shall write $\hat{\theta}_n \rightrightarrows A$ P-a.s.

Lemma 3.1

Let X be a second countable topological space, and let K be a compact subset of X . Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{ \hat{\theta}_n \mid n \geq 1 \}$ be a sequence of functions from Ω into X . Then there exists a decreasing sequence $\mathcal{G}(K) = \{ G_j \mid j \geq 1 \}$ of open sets in X containing K and satisfying $\hat{\theta}_n \rightrightarrows K$ on Ω , if and only if $\forall G \in \mathcal{G}(K)$, $\exists n_0 \geq 1$ such that $\forall n \geq n_0$ we have $\hat{\theta}_n(\omega) \in G$ for all $\omega \in \Omega$. The analogous equivalence relation holds for the P-a.s. uniform convergence as well.

Proof. Let $\mathcal{E} = \{ E_i \mid i \geq 1 \}$ be a countable base for the topology on X , and let $\mathcal{E}^*$ be the smallest family of subsets of X which contains $\mathcal{E}$ and is closed under the formation of finite unions of its elements, that is $E^* \in \mathcal{E}^*$ if and only if $E^* = \bigcup_{j=1}^{n} E_{i_j}$ for some $E_{i_1}, \ldots, E_{i_n} \in \mathcal{E}$ and some $n \geq 1$. Put $\mathcal{G}^*(K) = \{ E^* \in \mathcal{E}^* \mid K \subset E^* \} = \{ E_i^* \mid i \geq 1 \}$, and define $\mathcal{G}(K) = \{ G_j \mid j \geq 1 \}$ with $G_j = \bigcap_{i=1}^{j} E_i^*$ for $j \geq 1$. Using the compactness of K and the definition of $\mathcal{E}$ it is easily verified that for every open set G in X containing K there exists $E_k^* \in \mathcal{E}^*$ with $k \geq 1$ such that $K \subset E_k^* \subset G$. The proof hence follows straightforward. $\square$

If X is a Hausdorff space and a sequence $\{ \theta_n \mid n \geq 1 \}$ in X converges uniformly to some compact subset K of X , then the set of all accumulation points $\mathcal{C}\{\theta_n\}$ of the sequence $\{ \theta_n \mid n \geq 1 \}$ is evidently contained in K . Hence we may easily conclude:

(3.2) Let $\{ \hat{\theta}_n \mid n \geq 1 \}$ be a sequence of functions from a probability space $(\Omega, \mathcal{F}, P)$ into a Hausdorff space X , and let K be a compact subset of X . If $\hat{\theta}_n \rightrightarrows K$ on Ω , then $\mathcal{C}\{\hat{\theta}_n\} \subset K$. Similarly, if $\hat{\theta}_n \rightrightarrows K$ P-a.s., then $\mathcal{C}\{\hat{\theta}_n\} \subset K$ P-a.s.

Let us also note if (X, d) is a metric space, then for any two subsets A and B of X the following three statements are equivalent:

$$\hat{\theta}_n \rightrightarrows A \cup B \tag{3.3}$$

$$d(\hat{\theta}_n(\omega), A \cup B) \rightrightarrows 0 \ \text{ uniformly for } \ \omega \in \Omega \tag{3.4}$$

$$d(\hat{\theta}_n(\omega), \bar{A} \cup \bar{B}) \rightrightarrows 0 \ \text{ uniformly for } \ \omega \in \Omega \tag{3.5}$$

where $d(\theta, C)$ is the distance between a point $\theta \in X$ and a set $C \in 2^X$. Of course, the analogous equivalence relation holds for the P-a.s. uniform convergence as well.

Lemma 3.2

Let $\{ \alpha_n^i \mid n \geq 1 \}$ *and* $\{ \beta_n^j \mid n \geq 1 \}$ *be two sequences of random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *satisfying* $\lim_{n\to\infty} \alpha_n^i = \limsup_{n\to\infty} \beta_n^j = +\infty$ *P-a.s. for every* $i = 1, 2, \ldots, N$ *and every* $j = 1, 2, \ldots, M$ *. Then there exists an increasing surjection* $\sigma : \mathbf{N} \to \mathbf{N}$ *satisfying:*

$$P\Big(\liminf_{n\to\infty} \{ \alpha_n^i \geq \sigma(n) \}\Big) = P\Big(\limsup_{n\to\infty} \{ \beta_n^j \geq \sigma(n) \}\Big) = 1$$

for all $i = 1, 2, \ldots, N$ *and all* $j = 1, 2, \ldots, M$.

Proof. Since $\inf_{j \geq n} \alpha_j^1 \to +\infty$ P-a.s. as $n \to \infty$ and $\{ \inf_{j\geq n} \alpha_j^1 \geq c \} \subset \{ \alpha_n^1 \geq c \}$ for all $c \in \mathbf{R}$, we shall without loss of generality suppose that $\alpha_n^1 \uparrow +\infty$ P-a.s. as $n \to \infty$. Hence for every fixed $k \geq 1$ we can find $n_k \geq 1$ such that:

$$P\{ \alpha_{n_k}^1 < k \} < 2^{-k} \ .$$

Thus by the first Borel-Cantelli lemma we may conclude:

$$P\Big(\liminf_{k\to\infty} \{ \alpha_{n_k}^1 \geq k \}\Big) = 1 - P\Big(\limsup_{k\to\infty} \{ \alpha_{n_k}^1 < k \}\Big) = 1 \ . \tag{3.6}$$

Let us now define a surjection $\sigma_1 : \mathbf{N} \to \mathbf{N}$ by putting:

$$\sigma_1(j) = k \ , \ \forall n_k \leq j < n_{k+1}$$

for all $k \geq 1$, and put $\sigma_1(j) = 1$ for $1 \leq j < n_1$, if $n_1 > 1$. Since $\{ \alpha_k^1 \mid k \geq 1 \}$ is increasing, then we have:

$$\{ \; \alpha^1_{n_k} \geq \sigma_1(n_k) \; \} \subset \{ \; \alpha^1_{n_k+1} \geq \sigma_1(n_k) \; \} \subset \ldots \subset \{ \; \alpha^1_{n_{k+1}-1} \geq \sigma_1(n_k) \; \}$$

for all $k \geq 1$. Using these relations and (3.6) we easily find:

$$P\left(\liminf_{n\to\infty} \{ \, \alpha^1_n \geq \sigma_1(n) \, \} \right) = 1 \; . \tag{3.7}$$

By our hypotheses we have $\sup_{k\leq j\leq n} \beta^1_j \uparrow +\infty$ P-a.s. as $n \to \infty$ for all $k \geq 1$. Therefore for every fixed $k \geq 1$ we can find $n_k \geq 1$ such that:

$$P\Big\{ \sup_{k\leq j\leq n_k} \beta^1_j \, < \, k \Big\} \, < \, 2^{-k} \; .$$

Thus by the first Borel-Cantelli lemma we may conclude:

$$P\bigg(\liminf_{k\to\infty} \Big\{ \sup_{k\leq j\leq n_k} \beta^1_j \, \geq \, k \Big\} \bigg) \, = 1 \; . \tag{3.8}$$

Let us define a surjection $\tau_1 \, : \mathbf{N} \to \mathbf{N}$ by putting:

$$\tau_1(j) \, = \, k \;\; , \, \forall n_{k-1} \, < \, j \, \leq \, n_k$$

for all $k \geq 1$ with $n_0 := 0$. Then by the increase of τ_1 from (3.8) we get:

$$1 = P\bigg(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \Big\{ \sup_{k\leq j\leq n_k} \beta^1_j \geq k \Big\} \bigg) = \lim_{n\to\infty} P\bigg(\bigcap_{k=n}^{\infty} \bigcup_{j=k}^{n_k} \Big\{ \, \beta^1_j \, \geq \, \tau_1(n_k) \, \Big\} \bigg) \leq$$

$$\leq \lim_{n\to\infty} P\bigg(\bigcap_{k=n}^{\infty} \bigcup_{j=k}^{n_k} \Big\{ \, \beta^1_j \, \geq \, \tau_1(j) \, \Big\} \bigg) \; .$$

Hence we easily conclude:

$$P\left(\limsup_{n\to\infty} \{ \; \beta^1_n \geq \tau_1(n) \; \} \right) = 1 \; . \tag{3.9}$$

In exactly the same way we can find increasing surjections σ_i and τ_j associated to the sequences $\{ \; \alpha^i_n \mid n \geq 1 \; \}$ and $\{ \; \beta^j_n \mid n \geq 1 \; \}$ for $i = 2, \ldots, N$ and $j = 2, \ldots, M$. Then the proof follows straightforward by putting $\sigma = \min\{ \; \sigma_1, \sigma_2, \ldots, \sigma_N, \tau_1, \tau_2, \ldots, \tau_M \; \}$ and using the statements that correspond to (3.7) and (3.9) in these cases. □

4. The existence of measurable approximating maximums theorem

Let $\mathcal{H} = (\{ \; h_n(\omega,\theta) \; , \; \mathcal{S}_n \; \mid \; n \; \geq \; 1 \; \} \; \mid \; \theta \; \in \; \Theta_0)$ be an adapted family of random functions and σ-algebras defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the topological space Θ_0 . Let $\mathcal{G}(\Theta_0)$ denote the family of all open sets in Θ_0 , and let $\mathcal{K}(\Theta_0)$ denote the family of all compact sets in Θ_0 . Let us define:

$$h^*_n(\omega, G) \; = \; \sup_{\theta\in G} h_n(\omega,\theta)$$

$$H^*(\omega, G) = \limsup_{n\to\infty} \; h^*_n(\omega, G)$$

$$\bar{H}(\omega, K) = \inf_{G\in\mathcal{G}(\Theta_0), K\subset G} H^*(\omega, G)$$

$$H_0^*(\omega, G) = \liminf_{n\to\infty} \; h_n^*(\omega, G)$$

$$\bar{H}_0(\omega, K) = \inf_{G\in\mathcal{G}(\Theta_0), K\subset G} H_0^*(\omega, G)$$

for all $\omega \in \Omega$, all $G \in \mathcal{G}(\Theta_0)$, and all $K \in \mathcal{K}(\Theta_0)$. Let $\mathcal{S}_\infty = \cap_{n=1}^\infty \mathcal{S}_n$ and let $\mathcal{S}_n^P$ denote the completion of the σ-algebra $\mathcal{S}_n$ relative to the restriction of P to $\mathcal{S}_n$ for all $1 \leq n \leq \infty$. Then we have $\mathcal{S}_\infty^P = \cap_{n=1}^\infty \mathcal{S}_n^P$. If $\mathcal{H}$ is measurable and Θ_0 is second countable satisfying the Blackwell property, then by the projection theorem (2.1) and the proof of Lemma 3.1 we see that $H^*(G)$, $\bar{H}(K)$, $H_0^*(G)$ and $\bar{H}_0(K)$ are $\mathcal{S}_\infty^P$-measurable functions from Ω into Θ_0 , for all $G \in \mathcal{G}(\Theta_0)$ and all $K \in \mathcal{K}(\Theta_0)$. The main theorem of the paper may now be stated as follows.

Theorem 4.1

Let $\mathcal{H} = (\{\, h_n(\omega,\theta) \, , \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \Theta_0)$ be a measurable adapted family defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the topological space Θ_0 , let K and L be compact subsets of Θ_0 , and let us suppose that one of the following four conditions is satisfied:

(4.1) *Θ_0 is a second countable Hausdorff space satisfying the Blackwell property, or in other words Θ_0 is a second countable analytic space*

(4.2) *Θ_0 is a second countable Hausdorff space, and $\mathcal{H}$ is separable relative to $\mathcal{G}(\Theta_0)$ and $\mathcal{C}_{-\infty}(\mathbf{R})$*

(4.3) *Θ_0 is a second countable Hausdorff space, and $\mathcal{H}$ is separable*

(4.4) *Θ_0 is a second countable Hausdorff space, and $\mathcal{H}$ is P-a.s. lower semicontinuous.*

Then there exists a sequence of random functions $\{\, \hat{\theta}_n \mid n \geq 1 \,\}$ from Ω into Θ_0 and a P-null set $N \in \mathcal{F}$ satisfying the following properties:

(4.5) $\hat{\theta}_n \rightrightarrows K \cup L$ on Ω

(4.6) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \subset K \cup L$, for all $\omega \in \Omega$

(4.7) $\mathcal{C}\{\hat{\theta}_n(\omega)\} \cap K \neq \emptyset$ and $\mathcal{C}\{\hat{\theta}_n(\omega)\} \cap L \neq \emptyset$, for all $\omega \notin N$

(4.8) $\bar{H}_0(\omega, K) \wedge \bar{H}(\omega, L) \leq \liminf_{n\to\infty} \; h_n(\omega, \hat{\theta}_n(\omega)) \leq \bar{H}(\omega, K) \vee \bar{H}(\omega, L)$, for all $\omega \notin N$

(4.9) $\bar{H}_0(\omega, K) \vee \bar{H}(\omega, L) \leq \limsup_{n\to\infty} \; h_n(\omega, \hat{\theta}_n(\omega)) \leq \bar{H}(\omega, K) \vee \bar{H}(\omega, L)$, for all $\omega \notin N$

(4.10) $\hat{\theta}_n$ is $\mathcal{S}_n$-measurable for all $n \geq 1$.

Proof. Without loss of generality we shall suppose that all functions involved are with values in $\mathbf{R}$. Let $\mathcal{G}(K) = \{\, G_j \mid j \geq 1 \,\}$ and $\mathcal{G}(L) = \{\, H_j \mid j \geq 1 \,\}$ be decreasing sequences of open sets in Θ_0 associated to K and L in the sense of Lemma 3.1. Choose $\mathcal{S}_\infty^P$-measurable functions $\varepsilon_j : \Omega \to]0, \infty[$ for $j \geq 1$ to satisfy $\varepsilon_j(\omega) \to 0$ as $j \to \infty$ and $H_0^*(\omega, G_j) - \varepsilon_j(\omega) \leq H_0^*(\omega, G_{j+1}) - \varepsilon_{j+1}(\omega)$ for all $j \geq 1$ and all $\omega \in \Omega$. (For instance, we could obtain ε_j's from the equation $H_0^*(\omega, G_j) - \varepsilon_j(\omega) = H_0^*(\omega, K) - 1/j$ for $j \geq 1$ and $\omega \in \Omega$. Note that $H_0^*(\omega, G_j) \downarrow H_0^*(\omega, K)$ as $j \to \infty$ for $\omega \in \Omega$, and thus ε_j just defined is strictly positive and $\mathcal{S}_\infty^P$-measurable for all $j \geq 1$.) Similarly, choose $\mathcal{S}_\infty^P$-

measurable functions $\delta_j : \Omega \to]0,\infty[$ for $j \geq 1$ to satisfy $\delta_j(\omega) \to 0$ as $j \to \infty$ and $H^*(\omega,H_j) - \delta_j(\omega) \leq H^*(\omega,H_{j+1}) - \delta_{j+1}(\omega)$ for all $j \geq 1$ and all $\omega \in \Omega$. (For instance, we could obtain δ_j's from the equation $H^*(\omega,H_j) - \delta_j(\omega) = H^*(\omega,L) - 1/j$ for $j \geq 1$ and $\omega \in \Omega$. Note that $H^*(\omega,H_j) \downarrow H^*(\omega,L)$ as $j \to \infty$ for $\omega \in \Omega$, and thus δ_j just defined is strictly positive and $\mathcal{S}_\infty^P$-measurable for all $j \geq 1$.) Let us now define:

$$G_{nj} = \{\ (\omega,\theta) \in \Omega \times G_j \ \mid\ h_n(\omega,\theta) > H_0^*(\omega,G_j) - \varepsilon_j(\omega)\ \}$$

$$H_{nj} = \{\ (\omega,\theta) \in \Omega \times H_j \ \mid\ h_n(\omega,\theta) > H^*(\omega,H_j) - \delta_j(\omega)\ \}$$

$$G_{nj}^\omega = \{\ \theta \in G_j \ \mid\ (\omega,\theta) \in G_{nj}\ \}$$

$$H_{nj}^\omega = \{\ \theta \in H_j \ \mid\ (\omega,\theta) \in H_{nj}\ \}$$

for $n\, , j \geq 1$ and $\omega \in \Omega$. Since $\mathcal{H}$ is measurable, then we have $G_{nj} \in \mathcal{S}_n^P \times \mathcal{B}(G_j)$ and $H_{nj} \in \mathcal{S}_n^P \times \mathcal{B}(H_j)$, and hence $G_{nj}^\omega \in \mathcal{B}(G_j)$ and $H_{nj}^\omega \in \mathcal{B}(H_j)$ for all $n, j \geq 1$ and all $\omega \in \Omega$

Let us in addition define:

$$\alpha_n(\omega) = \sup\{\ j \geq 1 \ \mid\ G_{nj}^\omega \neq \emptyset\ \}$$

$$\beta_n(\omega) = \sup\{\ j \geq 1 \ \mid\ H_{nj}^\omega \neq \emptyset\ \}$$

for $n \geq 1$ and $\omega \in \Omega$. Since $\{\, \alpha_n \geq j\,\} = \bigcup_{k=j}^\infty \pi_\Omega(G_{nk})$ and $\{\, \beta_n \geq j\,\} = \bigcup_{k=j}^\infty \pi_\Omega(H_{nk})$ for all $n\, , j \geq 1$, then by the projection theorem (2.1) we have:

(4.11) $$\alpha_n\ ,\ \beta_n : \Omega \to \bar{\mathbf{N}}_0 \ \text{ are } \ \mathcal{S}_n^P\text{-measurable for all } \ n \geq 1\ .$$

By definition of the functions $H_0^*(G_j)$ and $H^*(H_j)$ we see that for any given $j \geq 1$ and $\omega \in \Omega$, there exist $n_{j,\omega} \geq 1$ and infinite $C_{j,\omega} \subset \mathbf{N}$ satisfying:

(4.12) $$h_n(\omega,\theta_{1,j,\omega,n}) > H_0^*(\omega,G_j) - \varepsilon_j(\omega)\quad ,\ \ \forall n \geq n_{j,\omega}$$

(4.13) $$h_n(\omega,\theta_{2,j,\omega,n}) > H^*(\omega,H_j) - \delta_j(\omega)\quad ,\ \ \forall n \in C_{j,\omega}$$

for some $\theta_{1,j,\omega,n} \in G_j$ and $\theta_{2,j,\omega,n} \in H_j$ with $n \geq 1$. From (4.12) and (4.13) we easily find:

(4.14) $$\lim_{n\to\infty} \alpha_n(\omega) = \limsup_{n\to\infty} \beta_n(\omega) = +\infty\ ,\quad \forall \omega \in \Omega\ .$$

Hence by Lemma 3.2 there exists an increasing surjection $\sigma : \mathbf{N} \to \mathbf{N}$ satisfying:

(4.15) $$P\Big(\liminf_{n\to\infty}\{\, \alpha_n \geq \sigma(n)\,\}\Big) = P\Big(\limsup_{n\to\infty}\{\, \beta_n \geq \sigma(n)\,\}\Big) = 1\ .$$

Now we apply the measurable selection theorem (2.2). For this, if (4.1) holds, then the preceding setting of the proof can stay unchanged. Moreover, if any of conditions (4.2), (4.3) or (4.4) holds, then according to (2.8) and (2.9) we can find a dense sequence $\{\, \theta_i \mid i \geq 1\}$ in Θ_0 and a P-null set $N_1 \in \mathcal{F}$ such that $\forall G \in \mathcal{G}(\Theta_0)$, $\forall \omega \in \Omega \setminus N_1$ and $\forall n \geq 1$ we have:

$$\sup_{\theta \in G} h_n(\omega,\theta) = \sup_{\theta_i \in G} h_n(\omega,\theta_i)\ .$$

Hence we can easily verify that the preceding setting and all of the conclusions remain valid if we replace the starting space Θ_0 by $\tilde{\Theta}_0 = \{ \theta_i \mid i \geq 1 \}$ equipped with the relative topology which is inherited from Θ_0 . Since $\tilde{\Theta}_0$ is countable, then it is a Blackwell space . This observation shows that under hypothesis (4.2), (4.3) or (4.4) it is no restriction to assume that Θ_0 satisfies (4.1). Therefore we can apply measurable selection theorem (2.2) which implies the existence of $\mathcal{S}_n^P$-measurable maps $\xi_{nj} : \Omega \to G_j$ and $\psi_{nj} : \Omega \to H_j$ for $n, j \geq 1$ satisfying:

$$h_n(\omega, \xi_{nj}(\omega)) > H_0^*(\omega, G_j) - \varepsilon_j(\omega) \quad , \quad \forall \omega \in \pi_\Omega(G_{nj})$$
$$h_n(\omega, \psi_{nj}(\omega)) > H^*(\omega, H_j) - \delta_j(\omega) \quad , \quad \forall \omega \in \pi_\Omega(H_{nj}) \; .$$

In order to establish (4.7) we shall define:

$$\tau_k(\omega) = \inf \{ \; n \geq k \; \mid \; \sigma(n) \leq \beta_n(\omega) \; \}$$

for all $k \geq 1$. Then by (4.11) we see that τ_k is $\mathcal{S}_k^P$-measurable for all $k \geq 1$. Denote $A_k = \{ \tau_k = k \}$ for $k \geq 1$, and put $B_n = \bigcap_{k=n}^\infty A_k$ for $n \geq 1$. Let us now define:

$$\begin{aligned} \check{\theta}_n(\omega) &= \xi_{n\sigma(n)}(\omega) \;\; \text{if} \;\; n \;\; \text{is odd, and} \;\; \omega \in B_n \; ; \\ &= \psi_{n\sigma(n)}(\omega) \;\; \text{if} \;\; n \;\; \text{is even, and} \;\; \omega \in B_n \; ; \\ &= \xi_{n\sigma(n)}(\omega) \;\; \text{if} \;\; \omega \in A_n^c \cap B_n^c \\ &= \psi_{n\sigma(n)}(\omega) \;\; \text{if} \;\; \omega \in A_n \cap B_n^c \end{aligned}$$

for all $n \geq 1$ and all $\omega \in \Omega$. Then by construction and the fact that $P(\limsup_{n\to\infty} A_n) = 1$, it is easily verified that each of the terms $\xi_{n\sigma(n)}(\omega)$ and $\psi_{n\sigma(n)}(\omega)$ on the right-hand side occurs infinitely often, for all $\omega \in N_2$, where $N_2 \in \mathcal{S}_\infty$ is a P-null set containing the exceptional sets from (4.15). Hence by the first part of (4.15) we can easily verify that (4.7), (4.8) and (4.9) hold for the sequence $\{ \check{\theta}_n \mid n \geq 1 \}$ with N_2 instead of N . Moreover, by the definition of the sequences $\mathcal{G}(K) = \{ G_j \mid j \geq 1 \}$ and $\mathcal{G}(L) = \{ H_j \mid j \geq 1 \}$ it is easily verified that (4.5) holds, and thus (4.6) follows as well. It is evident that $\check{\theta}_n$ is $\mathcal{S}_n^P$-measurable for all $n \geq 1$, thus in order to obtain (4.10), let us single out points $\theta_j \in G_j \cup H_j$ for all $j \geq 1$. Since the Borel σ-algebra $\mathcal{B}(\Theta_0)$ is countably generated, then there exists a $\mathcal{S}_n$-measurable function $\tilde{\theta}_n$ from Ω into Θ_0 satisfying $\tilde{\theta}_n(\omega) = \check{\theta}_n(\omega)$ for all $\omega \notin M_n$, where $M_n \in \mathcal{S}_n$ is a P-null set. Let us define:

$$\begin{aligned} \hat{\theta}_n(\omega) &= \tilde{\theta}_0(\omega) \;\; \text{if} \;\; \omega \in M_n^c \; ; \\ &= \theta_n \;\; \text{if} \;\; \omega \in M_n \end{aligned}$$

for all $\omega \in \Omega$ and all $n \geq 1$. Then evidently the sequence $\{ \hat{\theta}_n \mid n \geq 1 \}$ satisfies (4.5)-(4.10) with $N = N_1 \cup N_2 \cup \left(\bigcup_{n=1}^\infty M_n \right)$. This completes the proof. $\square$

Remark 4.2 Suppose that the family $\mathcal{H}$ in Theorem 4.1 is degenerated. Then the functions $\bar{H}_0(K)$, $\bar{H}(K)$, $\bar{H}_0(L)$ and $\bar{H}(L)$ are degenerated, that is P-almost surely equal to constants which depend on K and L respectively, as well as the inferior and the superior limit appearing in (4.8) and (4.9). Also, putting $K = L$ in (4.8) and (4.9) we find:

$$\bar{H}_0(\omega, K) \leq \liminf_{n\to\infty} h_n(\omega, \hat{\theta}_n(\omega)) \leq \limsup_{n\to\infty} h_n(\omega, \hat{\theta}_n(\omega)) = \bar{H}(\omega, K)$$

for all $\omega \notin N$. Moreover, if we make a small change in the preceding proof by putting:

$$G_{nj}^{+} = \{\ (\omega,\theta) \in \Omega \times G_j \mid H_0^*(\omega,G_j) + j^{-1} > h_n(\omega,\theta) > H_0^*(\omega,G_j) - \varepsilon_j(\omega)\ \}$$

and taking a $\mathcal{S}_n^P$-measurable map $\xi_{nj} : \Omega \to G_j$ satisfying:

$$h_n(\omega,\xi_{nj}(\omega)) > H_0^*(\omega,G_j) - \varepsilon_j(\omega) \quad , \quad \forall \omega \in \pi_\Omega(G_{nj})$$
$$H_0^*(\omega,G_j) + j^{-1} > h_n(\omega,\xi_{nj}(\omega)) > H_0^*(\omega,G_j) - \varepsilon_j(\omega) \quad , \quad \forall \omega \in \pi_\Omega(G_{nj}^{+})$$

for $n \geq 1$ and $j \geq 1$, then it is easily verified that the same proof yields the existence of a sequence of random functions $\{\ \hat{\theta}_n \mid n \geq 1\ \}$ satisfying:

(4.16) $\quad \hat{\theta}_n \rightrightarrows K$ on Ω

(4.17) $\quad \mathcal{C}\{\hat{\theta}_n(\omega)\} \subset K$, for all $\omega \in \Omega$

(4.18) $\quad \bar{H}_0(\omega,K) = \liminf_{n\to\infty} h_n(\omega,\hat{\theta}_n(\omega)) \leq \limsup_{n\to\infty} h_n(\omega,\hat{\theta}_n(\omega)) = \bar{H}(\omega,K)$, for all $\omega \notin N$

(4.19) $\quad \hat{\theta}_n$ is $\mathcal{S}_n$-measurable for all $n \geq 1$

where $N \in \mathcal{F}$ is a P-null set.

Remark 4.3 It is very accustomed in the context of statistical models mentioned in the introduction to imbed the topological space Θ_0 into a compact space Θ and extend functions from $\mathcal{H}$ by putting $h_n(\omega,\theta) = -\infty$, $\forall \omega \in \Omega$, $\forall \theta \in \Theta \setminus \Theta_0$ and $\forall n \geq 1$. Let $A, B \subset \bar{\Theta}_0$ be given sets, where $\bar{\Theta}_0$ denotes the closure of Θ_0 in Θ . Suppose that the space $\bar{\Theta}_0$ and the family $\bar{\mathcal{H}} = (\{\ h_n(\omega,\theta)\ , \mathcal{S}_n \mid n \geq 1\ \} \mid \theta \in \bar{\Theta}_0)$ satisfy one of the conditions (4.1)-(4.4), and suppose that $\bar{\mathcal{H}}$ is measurable. Then by Theorem 4.1 there exists a sequence $\{\ \hat{\theta}_n \mid n \geq 1\ \}$ satisfying (4.5)-(4.10) with $K = \bar{A}$ and $L = \bar{B}$. Note that every function $\hat{\theta}_n$ is $\bar{\Theta}_0$-valued in general, and a slight modification in the proof of Theorem 4.1 by putting:

$$G_j = G_j^* \cap \Theta_0 \qquad \text{and} \qquad H_j = H_j^* \cap \Theta_0$$

for all $j \geq 1$, where $\mathcal{G}(K) = \{\ G_j^* \mid j \geq 1\ \}$ and $\mathcal{G}(L) = \{\ H_j^* \mid j \geq 1\ \}$ are decreasing sequences of open sets in $\bar{\Theta}_0$ associated to K and L in the sense of Lemma 3.1, yields the existence of a Θ_0-valued sequence $\{\ \hat{\theta}_n \mid n \geq 1\ \}$ satisfying (4.5)-(4.10). Note, however, that $\bar{\Theta}_0$ satisfying (4.1) is a compact analytic space and hence metrizable.

Let us turn to some applications of Theorem 4.1. For this suppose that $\mathcal{H} = (\{\ h_n(\omega,\theta)\ , \mathcal{S}_n \mid n \geq 1\ \} \mid \theta \in \Theta_0)$ is a family of reversed submartingales defined on the probability space $(\Omega, \mathcal{F}, P)$, and indexed by the topological space Θ_0 which admits a compactification Θ . Let us extend h_n on $\Omega \times \Theta$ by putting $h_n(\omega,\theta) = -\infty$, $\forall \omega \in \Omega$, $\forall \theta \in \Theta \setminus \Theta_0$ and $\forall n \geq 1$. From the general theory of reversed submartingales we know that $h_n(\theta)$ converges P-almost surely to a random variable $h_\infty(\theta)$ as $n \to \infty$, for all $\theta \in \Theta_0$. Therefore *the information function*:

$$I(\omega,\theta) = \lim_{n\to\infty} h_n(\omega,\theta)$$

associated to $\mathcal{H}$ may be well-defined for all $\theta \in \Theta_0$ and all $\omega \in \Omega \setminus N_\theta$, where N_θ is an exceptional P-null set for which the limit on the right-hand side above does not exists. A sequence of functions $\{ \hat{\theta}_n \mid n \geq 1 \}$ from Ω into Θ may be called *a sequence of approximating maximums associated to* $\mathcal{H}$, if there exists a function $q : \Omega \rightarrow \mathbf{N}$ and a P-null set $N \in \mathcal{F}$ satisfying:

(4.20) $$\hat{\theta}_n(\omega) \in \Theta_0 \ , \quad \forall \omega \in \Omega \setminus N \ , \ \forall n \geq q(\omega)$$

(4.21) $$\liminf_{n\rightarrow\infty} \ h_n(\omega, \hat{\theta}_n(\omega)) \ \geq \ \sup_{\theta\in\Theta_0} \ I(\omega,\theta) \ , \ \forall \omega \in \Omega \setminus N$$

provided that the right-hand side in the inequality (4.21) is well-defined. Respecting the statistical nature lying behind we may do it as follows:

$$I(\omega,\theta) = \liminf_{n\rightarrow\infty} \ h_n(\omega,\theta)$$

for all $\omega \in \Omega$ and all $\theta \in \Theta$. Then it is easily verified that there always exists a sequence of functions $\{ \hat{\theta}_n \mid n \geq 1 \}$ from Ω into Θ satisfying (4.20) and (4.21). Moreover, let us define:

$$\hat{M} = \hat{M}(\mathcal{H}) = \{ \ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}(\omega,\theta) \geq \beta(\omega) \ \ P\text{-}a.s. \ \}$$

$$\hat{L} = \hat{L}(\mathcal{H}) = \{ \ \theta \in \bar{\Theta}_0 \ \mid \ \bar{H}_0(\omega,\theta) \geq \beta(\omega) \ \ P\text{-}a.s. \ \}$$

where $\beta(\omega) = \beta_{\mathcal{H}}(\omega) = \sup_{\theta\in\Theta_0} I(\omega,\theta)$ for $\omega \in \Omega$. (These sets turn out to be the most important for consistency of statisitical models mentioned in the introduction.) Then we have the following consequence of Theorem 4.1, Remark 4.2 and Remark 4.3.

Corollary 4.4

Suppose that the space $\bar{\Theta}_0$ *and the family* $\bar{\mathcal{H}} = (\{ h_n(\omega,\theta) \, , \mathcal{S}_n \mid n \geq 1 \} \mid \theta \in \bar{\Theta}_0)$ *satisfy one of the conditions (4.1)-(4.4), and suppose that* $\bar{\mathcal{H}}$ *is measurable. Then we have:*

(1): For every $\theta \in \hat{M}(\mathcal{H})$ *there exists a sequence of approximating maximums* $\{ \hat{\theta}_n \mid n \geq 1 \}$ *associated to* $\mathcal{H}$ *satisfying:*

(4.22) $\hat{\theta}_n$ *is* $\mathcal{S}_n$*-measurable for all* $n \geq 1$

(4.23) $\theta \in \mathcal{C}\{\hat{\theta}_n(\omega)\}$, *for all* $\omega \notin N$

where $N \in \mathcal{F}$ *is a* P*-null set.*

(2): For every $\theta \in \hat{L}(\mathcal{H})$ *there exists a sequence of approximating maximums* $\{ \hat{\theta}_n \mid n \geq 1 \}$ *associated to* $\mathcal{H}$ *satisfying:*

(4.24) $\hat{\theta}_n$ *is* $\mathcal{S}_n$*-measurable for all* $n \geq 1$

(4.25) $\hat{\theta}_n \rightrightarrows \{\theta\}$ *on* Ω

(4.26) $\bar{H}_0(\omega,\theta) = \liminf_{n\rightarrow\infty} h_n(\omega,\hat{\theta}_n(\omega)) \leq \limsup_{n\rightarrow\infty} h_n(\omega,\hat{\theta}_n(\omega)) = \bar{H}(\omega,\theta)$, *for all* $\omega \notin N$

where $N \in \mathcal{F}$ *is a* P*-null set.*

Proof. (1): By definition of the information function I , we directly find:

$$\bar{H}_0(\omega,\bar{\Theta}_0) = \liminf_{n\rightarrow\infty} h_n^*(\omega,\bar{\Theta}_0) = \liminf_{n\rightarrow\infty} h_n^*(\omega,\Theta_0) \geq \sup_{\theta\in\Theta_0} \ I(\omega,\theta) = \beta(\omega)$$

for all $\omega \in \Omega$. Now if $\theta \in \hat{M}(\mathcal{H})$, then there exists a P-null set $N \in \mathcal{F}$ such that:

$$\bar{H}(\omega,\theta) = \bar{H}(\omega,\{\theta\}) \geq \beta(\omega)$$

for all $\omega \in \Omega \setminus N$. From these two facts we get:

$$\bar{H}_0(\omega, \bar{\Theta}_0) \wedge \bar{H}(\omega, \{\theta\}) \geq \beta(\omega)$$

for all $\omega \in \Omega \setminus N$. Hence the proof follows straightforward by applying Theorem 4.1 with Remark 4.3 to the compact sets $K = \bar{\Theta}_0$ and $L = \{\theta\}$.

(2): Straightforward by Theorem 4.1, Remark 4.3, and the last part of Remark 4.2 with the compact sets $K = L = \{\theta\}$, for $\theta \in \hat{L}(\mathcal{H})$ being given. □

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