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On the American Option Problem

GORAN PESKIR*

We show how the change-of-variable formula with local time on curves derived recently in [17] can be used to prove that the optimal stopping boundary for the American put option can be characterized as the unique solution of a nonlinear integral equation arising from the early exercise premium representation. This settles the question raised in [15] (dating back to [13]).

1. Introduction

According to the theory of modern finance (see e.g. [10]) the arbitrage-free price of the American put option with a strike price K coincides with the value function V of the optimal stopping problem with the gain function $G = (K-x)^+$. The optimal stopping time in this problem is the first time when the price process (geometric Brownian motion) falls below the value of a time-dependent boundary b. When the option's expiration date T is finite, the mathematical problem of finding V and b is inherently two-dimensional and therefore analytically more difficult (for infinite T the problem is one-dimensional and b is constant).

The first mathematical analysis of the problem is due to McKean [13] who considered a 'discounted' American call with the gain function $G = e^{-\beta t}(x-K)^+$ and derived a free-boundary problem for V and b. He further expressed V in terms of b so that b itself solves a countable system of nonlinear integral equations (p. 39 in [13]). The approach of expressing V in terms of b was in line with the ideas coming from earlier work of Kolodner [12] on free-boundary problems in mathematical physics (such as Stefan's ice melting problem). The existence and uniqueness of a solution to the system for b derived by McKean was left open in [13].

McKean's work was taken further by van Moerbeke [21] who derived a single non-linear integral equation for b (pp. 145-146 in [21]). The connection to the physical problem is obtained by introducing the auxiliary function $\widetilde{V} = \partial(V-G)/\partial t$ so that the 'smooth-fit condition' from the optimal stopping problem translates into the 'condition of heat balance' (i.e. the law of conservation of energy) in the physical problem. A motivation for the latter may be seen from the fact that in the mathematical physics literature at the time it was realized that the existence and local uniqueness of a solution to such nonlinear integral equations can be proved by applying the contraction principle (fixed point theorem) firstly for a small time interval and then extending it to any interval of time by induction (see [14] and [5]). Applying this method van Moerbeke has proved the existence and local uniqueness of a solution to the integral equations of a general optimal stopping problem (see Sections 3.1 and 3.2 in [21]) while the proof of the same claim in the context of the discounted American call [13] is 'merely indicated' (see Section 4.4 in [21]). One of the technical difficulties

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in this context is that the derivative b' of the optimal boundary b is not bounded at the initial point T as used in the general proof (cf. Sections 3.1 and 3.2 in [21]).

The fixed point method usually results in a long and technical proof with an indecisive end where the 'details are' often sketched or 'omitted'. Another consequence of the approach is the fact that the integral equations in [13] and [21] involve both b and its derivative b', so that either the fixed point method results in proving that b is differentiable, or this needs to be proved a priori if the existence is claimed simply by identifying b with the boundary of the set where $V = G$. The latter proof, however, appears difficult to give directly, so that if one is only interested in the actual values of b which indicate when to stop, it seems that the differentiability of b plays a minor role. Finally, since it is not obvious to see (and it was never explicitly addressed) how the 'condition of heat balance' relates to the economic mechanism of 'no-arbitrage' behind the American option, one is led to the conclusion that the integral equations derived by McKean and van Moerbeke, being motivated purely by the mathematical tractability arising from the work in mathematical physics, are perhaps more complicated then needed from the standpoint of optimal stopping.

This was to be confirmed in the beginning of the 1990's when Kim [11], Jacka [8] and Carr et al. [2] have independently arrived at a nonlinear integral equation for b that is closely linked to *the early exercise premium representation* of V having a clear economic meaning (see Section 1 in [2] and Corollary 3.1 in [15]). In fact, the equation is obtained by inserting $x = b(t)$ in this representation, and for this reason it will be called *the free-boundary equation* (see (2.21) below). The early exercise premium representation for V follows transparently from the free-boundary formulation (given that the smooth-fit condition holds) and moreover corresponds to the decomposition of the superharmonic function V into its harmonic and its potential part (the latter being the basic principle of optimal stopping established in the works of Snell [19] and Dynkin [4]). These facts will be reviewed in more detail in Section 2 below.

It follows from the preceding that the optimal stopping boundary b satisfies the free-boundary equation, however, as pointed out by Myneni [15] (p. 17) 'the uniqueness and regularity of the stopping boundary from this integral equation remain open'. Theorem 4.3 of Jacka [8] states that if the equation holds for all $x \le c(t)$ with a candidate function c (which in effect corresponds to c satisfying a countable system of nonlinear integral equations) then this c must coincide with the optimal stopping boundary b. In order to determine the boundary explicitly (by numerical calculation) one is therefore faced with a complicated task of making sure that all these equations are satisfied by a candidate (in practice this fact is often ignored).

The main purpose of the present paper is to show that the question of Myneni can be answered affirmatively and that the free-boundary equation alone does indeed characterize the optimal stopping boundary b. The key argument in the proof is based on the change-of-variable formula with local time on curves derived recently in [17]. The same method of proof can be applied to more general continuous Markov processes (diffusions) in problems of optimal stopping with finite horizon. For example, in this way it is also possible to settle the somewhat more complicated problem of the Russian option with finite horizon (see [18]).

For more information on the American option problem we refer to the survey paper by Myneni [15] and Sections 2.5-2.8 in the book by Karatzas and Shreve [10] where further references can also be found. For a numerical discussion of the free-boundary equation and possible improvements along these lines see the recent paper by Hou et al [7].

2. Preliminaries

In this section we will introduce the setting and notation of the American put problem and provide some commentary. In the end of the section we will recall a change-of-variable formula with local times on curves that will be used in the proof of the main result in the next section.

1. The *arbitrage-free price* of the American put option at time $t \in [0,T]$ is given by:

$$V(t,x) = \sup_{0 \le \tau \le T-t} E_{t,x}\Big(e^{-r\tau}(K - X_{t+\tau})^+\Big) \tag{2.1}$$

where τ is a stopping time of the geometric Brownian motion $X = (X_{t+s})_{s \ge 0}$ solving:

$$dX_{t+s} = rX_{t+s}\, ds + \sigma X_{t+s}\, dB_s \tag{2.2}$$

with $X_t = x$ under $P_{t,x}$. [Throughout $B = (B_s)_{s \ge 0}$ denotes the standard Brownian motion process started at zero.] We recall that T is the expiration date (maturity), $r > 0$ is the interest rate, $K > 0$ is the strike (exercise) price, and $\sigma > 0$ is the volatility coefficient. The strong solution of (2.2) under $P_{t,x}$ is given by:

$$X_{t+s} = x\, \exp\Big(\sigma B_s + (r - \sigma^2/2)\, s\Big) \tag{2.3}$$

whenever $t > 0$ and $x > 0$ are given and fixed. The process X is strong Markov (diffusion) with the infinitesimal generator given by:

$$I\!\!L_X = rx \frac{\partial}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x} \; . \tag{2.4}$$

We refer to [10] for more information on the derivation and economic meaning of (2.1).

2. From earlier work (going back to [13]) it is known (and can readily be verified) that V from (2.1) solves the following *free-boundary problem* of parabolic type:

$$V_t + I\!\!L_X V = rV \quad \text{in} \quad C \tag{2.5}$$

$$V(t,x) = (K-x)^+ \quad \text{for} \quad x = b(t) \quad \text{or} \quad t = T \tag{2.6}$$

$$V_x(t,x) = -1 \quad \text{for} \quad x = b(t) \quad (\textit{smooth fit}) \tag{2.7}$$

$$V(t,x) > (K-x)^+ \quad \text{in} \quad C \tag{2.8}$$

$$V(t,x) = (K-x)^+ \quad \text{in} \quad D \tag{2.9}$$

where the continuation region C and the stopping region $S = \bar{D}$ are defined by:

$$C = \{\, (t,x) \in [0,T\rangle \times I\!\!R_+ \mid x > b(t)\,\} \tag{2.10}$$

$$D = \{\, (t,x) \in [0,T\rangle \times I\!\!R_+ \mid x < b(t)\,\} \tag{2.11}$$

and $b : [0,T] \rightarrow I\!\!R$ is the optimal stopping boundary, i.e. the stopping time:

$$\tau_b = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \le b(t+s)\,\} \tag{2.12}$$

is optimal in (2.1) (the supremum is attained at this stopping time).

The following properties of V and b are known to be valid and can readily be verified (see the survey paper by Myneni [15] for references to proofs):

(2.13) V is continuous on $[0,T] \times I\!R_+$

(2.14) V is $C^{1,2}$ on C (and $C^{1,2}$ on $\bar{D}$)

(2.15) $x \mapsto V(t,x)$ is decreasing and convex with $V_x(t,x) \in [-1,0]$

(2.16) $t \mapsto V(t,x)$ is decreasing with $V(T,x) = (K-x)^+$

(2.17) $t \mapsto b(t)$ is increasing and continuous with $0 < b(0+) < K$ and $b(T-) = K$.

Note also that (2.7) means that $x \mapsto V(t,x)$ is C^1 at $b(t)$. With reference to [13] and [21] it is claimed in [15] (and used in some other papers too) that b is C^1 but I could not find a complete proof in either of these papers (nor anywhere else). If it is known that b is C^1 then the proof below shows that C in (2.14) can be replaced by $\bar{C}$, implying also that $s \mapsto V(s, b(t))$ is C^1 at t . For both, in fact, it is sufficient to know that b is (locally) Lipschitz, but it seems that this fact is no easier to establish directly, and I do not know of any transparent proof.

3. The *superharmonic characterization* of the value function V (due to Snell [19] and Dynkin [4]) implies that $e^{-rs} V(t+s, X_{t+s})$ is the smallest supermartingale dominating $e^{-rs}(K - X_{t+s})^+$ on $[0, T-t]$, i.e. that $V(t,x)$ is the smallest superharmonic function (relative to $\partial/\partial t + I\!L_X - rI$) dominating $(K-x)^+$ on $[0,T] \times I\!R_+$. The two requirements (i.e. smallest and superharmonic) manifest themselves in the single analytic condition of smooth-fit (2.7).

The derivation of the smooth-fit condition given in Myneni [15] upon integrating the second formula on p. 15 and obtaining the third one seems to violate the Leibnitz-Newton formula unless $x \mapsto V(t,x)$ is smooth at $b(t)$ so that there is nothing to prove. Myneni writes that this proof is essentially from McKean [13]. A closer inspection of his argument on p. 38 in [13] reveals the same difficulty. Alternative derivations of the smooth-fit principle for Brownian motion and diffusions are given in Grigelionis & Shiryaev [6] and Chernoff [3] by a Taylor expansion of V at $(t, b(t))$ and in Bather [1] and van Moerbeke [21] by a Taylor expansion of G at $(t, b(t))$. The latter approach seems more satisfactory generally since V is not known a priori. Jacka [9] (Corollary 7) develops a different approach which he applies in [8] (Proposition 2.8) to verify (2.7).

4. Once we know that V satisfying (2.7) is 'sufficiently regular' (cf. footnote 14 in [2] when $t \mapsto V(t,x)$ is known to be C^1 for all x), we can apply Itô's formula to $e^{-rs} V(t{+}s, X_{t+s})$ in its standard form and take the $P_{t,x}$-expectation on both sides in the resulting identity. The martingale term then vanishes by the optional sampling theorem using the final part of (2.15) above, so that by (2.5) and (2.6)+(2.9) upon setting $s = T-t$ (being the key advantage of the finite horizon) one obtains the *early exercise premium representation* of the value function:

$$
\begin{aligned}
(2.18) \qquad V(t,x) &= e^{-r(T-t)} E_{t,x}\big(G(X_T)\big) - \int_0^{T-t} e^{-ru} E_{t,x}\Big(H(t+u, X_{t+u})\, I\big(X_{t+u} \le b(t{+}u)\big)\Big)\, du \\
&= e^{-r(T-t)} E_{t,x}\big(G(X_T)\big) + rK \int_0^{T-t} e^{-ru} P_{t,x}\big(X_{t+u} \le b(t{+}u)\big)\, du
\end{aligned}
$$

for all $(t,x) \in [0,T]\times I\!\!R_+$ where we set $G(x) = (K-x)^+$ and $H = G_t + I\!\!L_X G - rG$ so that $H = -rK$ for $x < b(t)$.

Since $V(t,x) = G(x) = (K-x)^+$ in $\bar{D}$ by (2.6)+(2.9), we see that (2.18) reads:

$$K - x = e^{-r(T-t)}\, E_{t,x}(K - X_T)^+ + rK \int_0^{T-t} e^{-ru} P_{t,x}\big(X_{t+u} \le b(t+u)\big)\, du \tag{2.19}$$

for all $x \in \langle 0, b(t)]$ and all $t \in [0,T]$. Theorem 4.3 of Jacka [8] states that if $c : [0,T] \to I\!\!R$ is a 'left-continuous' solution of (2.19) for all $x \in \langle 0, c(t)]$ satisfying $0 < c(t) < K$ for all $t \in \langle 0, T\rangle$, then c is the optimal stopping boundary b . Since (2.19) is a different equation for each new $x \in \langle 0, c(t)]$, we see that this assumption in effect corresponds to c solving a countable system of nonlinear integral equations (obtained by letting x in $\langle 0, c(t)]$ run through rationals for instance). From the standpoint of numerical calculation it is therefore of interest to reduce the number of these equations.

5. A natural candidate equation is obtained by inserting $x = b(t)$ in (2.19). This leads to *the free-boundary equation* (cf. [11], [8], [2]):

$$K - b(t) = e^{-r(T-t)}\, E_{t,b(t)}(K - X_T)^+ + rK \int_0^{T-t} e^{-ru} P_{t,b(t)}\big(X_{t+u} \le b(t+u)\big)\, du \tag{2.20}$$

which upon using (2.3) more explicitly reads as follows:

$$\begin{aligned} K - b(t) = {} & e^{-r(T-t)} \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{b(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)\right) dz \\ & + rK \int_0^{T-t} e^{-ru}\, \Phi\left(\frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{b(t+u)}{b(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)u\right)\right) du \end{aligned} \tag{2.21}$$

for all $t \in [0,T]$ where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^x e^{-z^2/2}\, dz$ for $x \in I\!\!R$. It is a *nonlinear Volterra integral equation of the second kind* (see [20]).

Knowing that the optimal stopping boundary b solves (2.21), it is an equally important question to determine if this solution is unique. As stated in the introduction, this was firstly pointed out by Myneni [15] (p. 17). This attempt is in line with McKean [13] (p. 33) who wrote that 'another inviting unsolved problem is to discuss the integral equation for the free-boundary of section 6', concluding the paper (p. 39) with the words 'even the existence and uniqueness of solutions is still unproved'. McKean's integral equations [13] (p. 39) are more complicated (involving b' as well) than the equation (2.19). Thus the simplification of his equations to the equations (2.19) and finally the equation (2.21) may be viewed as a step to the solution of the problem.

The main purpose of the present paper is to show that the uniqueness problem of (2.21) can be answered affirmatively by means of the change-of-variable formula with local time on curves derived recently in [17]. For further reference we will recall this formula here.

6. Let $X = (X_t)_{t\ge 0}$ be a continuous semimartingale, let $c : I\!\!R_+ \to I\!\!R$ be a continuous function of bounded variation, and let $F : I\!\!R_+ \times I\!\!R \to I\!\!R$ be a continuous function satisfying:

$$F \text{ is } C^{1,2} \text{ on } \bar{C}_1 \tag{2.22}$$

(2.23) $$F \text{ is } C^{1,2} \text{ on } \bar{C}_2$$

where C_1 and C_2 are given as follows:

(2.24) $$C_1 = \{ (t,x) \in I\!R_+ \times I\!R \mid x > c(t) \}$$

(2.25) $$C_2 = \{ (t,x) \in I\!R_+ \times I\!R \mid x < c(t) \} .$$

Then the following change-of-variable formula holds (cf. [17]):

(2.26) $$F(t,X_t) = F(0,X_0) + \int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\,ds$$
$$+ \int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,dX_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)\; I(X_s \neq c(s))\; d\langle X,X\rangle_s$$
$$+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\; I(X_s = c(s))\; d\ell_s^c(X)$$

where $\ell_s^c(X)$ is the local time of X at the curve c given by:

(2.27) $$\ell_s^c(X) = I\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^s I(c(r)-\varepsilon < X_r < c(r)+\varepsilon)\, d\langle X,X\rangle_r$$

and $d\ell_s^c(X)$ refers to the integration with respect to the continuous increasing function $s \mapsto \ell_s^c(X)$.

Moreover, if X solves the stochastic differential equation:

(2.28) $$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$$

where μ and σ are locally bounded and $\sigma \geq 0$, then the following condition:

(2.29) $$P\big(X_s = c(s)\big) = 0 \quad \text{for} \quad s \in \langle 0,t]$$

implies that the first two integrals in (2.26) can be simplified to read:

(2.30) $$F(t,X_t) = F(0,X_0) + \int_0^t (F_t + I\!L_X F)(s,X_s)\, I(X_s \neq c(s))\, ds$$
$$+ \int_0^t F_x(s,X_s)\, \sigma(X_s)\, I(X_s \neq c(s))\; dB_s$$
$$+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\; I(X_s = c(s))\; d\ell_s^c(X)$$

where $I\!L_X F = \mu F_x + (\sigma^2/2) F_{xx}$ is the action of the infinitesimal generator $I\!L_X$ on F .

In view of V being 'sufficiently regular' stated prior to (2.18) above, as well as our discussion following (2.17) above, the key point to be noted is that (2.30) remains valid even if the condition (2.22) is relaxed. To state the extension which is sufficient for present purposes, let us assume that X solves (2.28) and satisfies (2.29), where $c : I\!R_+ \rightarrow I\!R$ is a continuous function of bounded variation, and let $F : I\!R_+ \times I\!R \rightarrow I\!R$ be a continuous function satisfying the following conditions instead of (2.22)+(2.23) above:

(2.31) F is $C^{1,2}$ on $C_1 \cup C_2$

(2.32) $F_t + I\!\!L_X F$ is locally bounded

(2.33) $x \mapsto F(t,x)$ is convex

(2.34) $t \mapsto F_x(t, b(t)\pm)$ is continuous.

Then it can be proved that the change-of-variable formula (2.30) still holds (cf. [17]). In this case, even if F_t is to diverge when the boundary c is approached within C_1, this deficiency is counterbalanced by a similar behaviour of F_{xx} through (2.32), and consequently the first integral in (2.30) is still well-defined and finite. [When we say in (2.32) that $F_t + I\!\!L_X F$ is locally bounded, we mean that $F_t + I\!\!L_X F$ is bounded on $K \cap (C_1 \cup C_2)$ for each compact set in $I\!\!R_+ \times I\!\!R$]. The condition (2.33) can further be relaxed to the form where $F_{xx} = F_1 + F_2$ on $C_1 \cup C_2$ where F_1 is non-negative and F_2 is continuous on $I\!\!R_+ \times I\!\!R$. This will be referred to below as the relaxed form of (2.31)-(2.34). For more details on this and other extensions see [17].

3. The result and proof

In this section we adopt the setting and notation of the American put problem from the previous section. The main result of the paper may now be stated as follows (see also Remark 3.2 below).

Theorem 3.1

The optimal stopping boundary in the American put problem (2.1) can be characterized as the unique solution of the free-boundary equation (2.21) in the class of continuous increasing functions $c : [0,T] \rightarrow I\!\!R$ *satisfying* $0 < c(t) < K$ *for all* $0 < t < T$.

Proof. The fact that the optimal stopping boundary b solves (2.20) i.e. (2.21) is well-known (cf. Section 2). A detail worth mentioning in this regard (see (3.8) below) is that (2.18) follows from (2.30) with $F(t+s, X_{t+s}) = e^{-rs} V(t+s, X_{t+s})$ without knowing a priori that $t \mapsto V(t,x)$ is C^1 at $b(t)$ as required under the condition of 'sufficiently regular' recalled prior to (2.18) above. This approach is more direct since the sufficient conditions (2.31)-(2.34) for (2.30) are easier verified than sufficient conditions [such as b is C^1 or (locally) Lipschitz] for $t \mapsto V(t,x)$ to be C^1 at $b(t)$. This is also more in the spirit of the free-boundary equation (2.21) where differentiability or a Lipschitz property of b play no role in the formulation.

The main emphasis of the theorem is thus on the claim of uniqueness. Let us therefore assume that a continuous increasing $c : [0,T] \rightarrow I\!\!R$ solving (2.21) is given such that $0 < c(t) < K$ for all $0 < t < T$, and let us show that this c must then coincide with the optimal stopping boundary b. The proof of this implication will be presented in the nine steps as follows.

1. In view of (2.18) and with the aid of calculations similar to those leading from (2.20) to (2.21), let us introduce the function:

$$U^c(t,x) = e^{-r(T-t)} E_{t,x}\big(G(X_T)\big) + rK \int_0^{T-t} e^{-ru} P_{t,x}\big(X_{t+u} \le c(t+u)\big)\, du \tag{3.1}$$

$$= e^{-r(T-t)}\, U_1^c(t,x) + rK\, U_2^c(t,x)$$

where U_1^c and U_2^c are defined as follows:

$$U_1^c(t,x) = \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{x}\right) - \gamma\,(T-t)\right)\right) dz \tag{3.2}$$

$$U_2^c(t,x) = \int_t^T e^{-r(v-t)}\,\Phi\left(\frac{1}{\sigma\sqrt{v-t}}\left(\log\left(\frac{c(v)}{x}\right) - \gamma\,(v-t)\right)\right) dv \tag{3.3}$$

for all $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$ upon setting $\gamma = r - \sigma^2/2$ and substituting $v = t+u$.

Denoting $\varphi = \Phi'$ we then have:

$$\frac{\partial U_1^c}{\partial x}(t,x) = -\frac{1}{\sigma x\sqrt{T-t}} \int_0^K \varphi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{x}\right) - \gamma\,(T-t)\right)\right) dz \tag{3.4}$$

$$\frac{\partial U_2^c}{\partial x}(t,x) = -\frac{1}{\sigma x} \int_t^T \frac{e^{-r(v-t)}}{\sqrt{v-t}}\,\varphi\left(\frac{1}{\sigma\sqrt{v-t}}\left(\log\left(\frac{c(v)}{x}\right) - \gamma\,(v-t)\right)\right) dv \tag{3.5}$$

for all $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$ where the interchange of differentiation and integration is justified by standard means. From (3.4) and (3.5) we see that $\partial U_1^c/\partial x$ and $\partial U_2^c/\partial x$ are continuous on $[0,T\rangle \times \langle 0,\infty\rangle$, which in view of (3.1) implies that U_x^c is continuous on $[0,T\rangle \times \langle 0,\infty\rangle$.

2. In accordance with (2.18) define a function $V^c : [0,T\rangle \times \langle 0,\infty\rangle \rightarrow I\!R$ by setting $V^c(t,x) = U^c(t,x)$ for $x > c(t)$ and $V^c(t,x) = G(x)$ for $x \le c(t)$ when $0 \le t < T$. Note that since c solves (2.21) we have that V^c is continuous on $[0,T\rangle \times \langle 0,\infty\rangle$, i.e. $V^c(t,x) = U^c(t,x) = G(x)$ for $x = c(t)$ when $0 \le t < T$. Let C_1 and C_2 be defined by means of c as in (2.24) and (2.25) with $[0,T\rangle$ instead of $I\!R_+$, respectively.

Standard arguments based on the Markov property (or a direct verification) show that V^c i.e. U^c is $C^{1,2}$ on C_1 and that:

$$V_t^c + I\!L_X V^c = rV^c \quad \text{in} \quad C_1\,. \tag{3.6}$$

Moreover, since U_x^c is continuous on $[0,T\rangle \times \langle 0,\infty\rangle$ we see that V_x^c is continuous on $\bar{C}_1$. Finally, since $0 < c(t) < K$ for $0 < t < T$ we see that V^c i.e. G is $C^{1,2}$ on $\bar{C}_2$.

3. Summarizing the preceding conclusions one can easily verify that with $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$ given and fixed the function $F : [0,T-t\rangle \times \langle 0,\infty\rangle \rightarrow I\!R$ defined by:

$$F(s,y) = e^{-rs} V^c(t+s, x\cdot y) \tag{3.7}$$

satisfies (2.31)-(2.34) (in the relaxed form) so that (2.30) can be applied. In this way we get:

$$\begin{aligned} e^{-rs} V^c(t+s, X_{t+s}) &= V^c(t,x) \\ &+ \int_0^s e^{-ru}\Big(V_t^c + I\!L_X V^c - rV^c\Big)(t+u, X_{t+u}) I(X_{t+u} \ne c(t+u))\, du \\ &+ M_s^c + \frac{1}{2}\int_0^s e^{-ru} \Delta_x V_x^c(t+u, c(t+u))\, d\ell_u^c(X) \end{aligned} \tag{3.8}$$

where $M_s^c = \int_0^s e^{-ru} V_x^c(t+u, X_{t+u})\, \sigma X_{t+u}\, I(X_{t+u} \ne c(t+u))\, dB_u$ and we set $\Delta_x V_x^c(v, c(v)) =$

$V_x^c(v,c(v)+) - V_x^c(v,c(v)-)$ for $t \le v \le T$. Moreover, it is easily seen from (3.4) and (3.5) that $(M_s^c)_{0 \le s \le T-t}$ is a martingale under $P_{t,x}$ so that $E_{t,x}(M_s^c) = 0$ for each $0 \le s \le T-t$.

4. Setting $s = T-t$ in (3.8) and then taking the $P_{t,x}$-expectation, using that $V^c(T,x) = G(x)$ for all $x > 0$ and that V^c satisfies (3.6) in C_1, we get:

$$\begin{aligned}(3.9)\qquad e^{-r(T-t)} E_{t,x}\big(G(X_T)\big) &= V^c(t,x)\\ &+ \int_0^{T-t} e^{-ru} E_{t,x}\big(H(t+u, X_{t+u})\, I(X_{t+u} \le c(t+u))\big)\, du\\ &+ \frac{1}{2} \int_0^{T-t} e^{-ru} \Delta_x V_x^c(t+u, c(t+u))\, d_u E_{t,x}(\ell_u^c(X))\end{aligned}$$

for all $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$ where $H = G_t + I\!\!L_X G - rG = -rK$ for $x \le c(t)$. From (3.9) we thus see that:

$$\begin{aligned}(3.10)\qquad V^c(t,x) = e^{-r(T-t)} E_{t,x}\big(G(X_T)\big) &+ rK \int_0^{T-t} e^{-ru} P_{t,x}(X_{t+u} \le c(t+u))\, du\\ &- \frac{1}{2} \int_0^{T-t} e^{-ru} \Delta_x V_x^c(t+u, c(t+u))\, d_u E_{t,x}(\ell_u^c(X))\end{aligned}$$

for all $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$. Comparing (3.10) with (3.1), and recalling the definition of V^c in terms of U^c and G, we get:

$$(3.11)\qquad \int_0^{T-t} e^{-ru} \Delta_x V_x^c(t+u, c(t+u))\, d_u E_{t,x}(\ell_u^c(X)) = 2\big(U^c(t,x) - G(x)\big)\, I(x \le c(t))$$

for all $0 \le t < T$ and $x > 0$, where $I(x \le c(t))$ equals 1 if $x \le c(t)$ and 0 if $x > c(t)$.

5. From (3.11) we see that if we are to prove that:

$$(3.12)\qquad x \mapsto V^c(t,x) \ \text{ is } \ C^1 \ \text{ at } \ c(t)$$

for each $0 \le t < T$ given and fixed, then it will follow that:

$$(3.13)\qquad U^c(t,x) = G(x) \ \text{ for all } \ 0 < x \le c(t)\ .$$

On the other hand, if we know that (3.13) holds, then using the general fact:

$$(3.14)\qquad \frac{\partial}{\partial x}\Big(U^c(t,x) - G(x)\Big)\Big|_{x=c(t)} = V_x^c(t,c(t)+) - V_x^c(t,c(t)-) = \Delta_x V_x^c(t,c(t))$$

for all $0 \le t < T$, we see that (3.12) holds too (since U_x^c is continuous). The equivalence of (3.12) and (3.13) just explained then suggests that instead of dealing with the equation (3.11) in order to derive (3.12) above (which was the content of an earlier proof) we may rather concentrate on establishing (3.13) directly. [To appreciate the simplicity and power of the probabilistic argument to be given shortly below one may differentiate (3.11) with respect to x, compute the left-hand side explicitly (taking care of a jump relation), and then try to prove the uniqueness of the zero solution to the resulting (weakly singular) Volterra integral equation using any of the known analytic methods (see e.g. [20]).]

6. To derive (3.13) first note that standard arguments based on the Markov property (or a direct verification) show that U^c is $C^{1,2}$ on C_2 and that:

$$U^c_t + \mathbb{L}_X U^c - rU^c = -rK \quad \text{in} \quad C_2 \; . \tag{3.15}$$

Since F in (3.7) with U^c instead of V^c is continuous and satisfies (2.31)-(2.34) (in the relaxed form), we see that (2.30) can be applied just as in (3.8), and this yields:

$$e^{-rs} U^c(t+s, X_{t+s}) = U^c(t,x) - rK \int_0^s e^{-ru} I(X_{t+u} \le c(t+u))\, du + \widetilde{M}^c_s \tag{3.16}$$

upon using (3.6) and (3.15) as well as that $\Delta_x U^c_x(t+u, c(t+u)) = 0$ for all $0 \le u \le s$ since U^c_x is continuous. In (3.16) we have $\widetilde{M}^c_s = \int_0^s e^{-ru} U^c_x(t+u, X_{t+u})\, \sigma X_{t+u}\, I(X_{t+u} \neq c(t+u))\, dB_u$ and $(\widetilde{M}^c_s)_{0 \le s \le T-t}$ is a martingale under $P_{t,x}$.

Next note that (2.30) applied to F in (3.7) with G instead of V^c yields:

$$e^{-rs} G(X_{t+s}) = G(x) - rK \int_0^s e^{-ru} I(X_{t+u} < K)\, du + M^K_s + \frac{1}{2} \int_0^s e^{-ru}\, d\ell^K_u(X) \tag{3.17}$$

upon using that $G_t + \mathbb{L}_X G - rG$ equals $-rK$ on $\langle 0, K \rangle$ and 0 on $\langle K, \infty \rangle$ as well as that $\Delta_x G_x(t+u, K) = 1$ for $0 \le u \le s$. In (3.17) we have $M^K_s = \int_0^s e^{-ru} G'(X_{t+u})\, \sigma X_{t+u}\, I(X_{t+u} \neq K)\, dB_u = -\int_0^s e^{-ru} \sigma X_{t+u}\, I(X_{t+u} < K)\, dB_u$ and $(M^K_s)_{0 \le s \le T-t}$ is a martingale under $P_{t,x}$.

For $0 < x \le c(t)$ consider the stopping time:

$$\sigma_c = \inf \{\, 0 \le s \le T-t \mid X_{t+s} \ge c(t+s) \,\} \; . \tag{3.18}$$

Then using that $U^c(t, c(t)) = G(c(t))$ for all $0 \le t < T$ since c solves (2.21), and that $U^c(T,x) = G(x)$ for all $x > 0$ by (3.1), we see that $U^c(t+\sigma_c, X_{t+\sigma_c}) = G(X_{t+\sigma_c})$. Hence from (3.16) and (3.17) using the optional sampling theorem we find:

$$\begin{aligned} U^c(t,x) &= E_{t,x}\Big(e^{-r\sigma_c} U^c(t+\sigma_c, X_{t+\sigma_c}) \Big) + rK\, E_{t,x}\Big(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} \le c(t+u))\, du \Big) \\ &= E_{t,x}\Big(e^{-r\sigma_c} G(X_{t+\sigma_c}) \Big) + rK\, E_{t,x}\Big(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} \le c(t+u))\, du \Big) \\ &= G(x) - rK\, E_{t,x}\Big(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} < K)\, du \Big) \\ &\qquad + rK\, E_{t,x}\Big(\int_0^{\sigma_c} e^{-ru} I(X_{t+u} \le c(t+u))\, du \Big) = G(x) \end{aligned} \tag{3.19}$$

since $X_{t+u} < K$ and $X_{t+u} \le c(t+u)$ for all $0 \le u < \sigma_c$. This establishes (3.13) and thus (3.12) holds as well as explained above.

7. Consider the stopping time:

$$\tau_c = \inf \{\, 0 \le s \le T-t \mid X_{t+s} \le c(t+s) \,\} \; . \tag{3.20}$$

Note that (3.8) using (3.6) and (3.12) reads:

$$e^{-rs}V^c(t+s,X_{t+s}) = V^c(t,x) + \int_0^s e^{-ru} H(t+u,X_{t+u})\, I(X_{t+u} \le c(t+u))\, du + M_s^c \tag{3.21}$$

where $H = G_t + I\!\!L_X G - rG = -rK$ for $x \le c(t)$ and $(M_s^c)_{0\le s\le T-t}$ is a martingale under $P_{t,x}$. Thus $E_{t,x}(M_{\tau_c}^c) = 0$, so that after inserting τ_c in place of s in (3.21), it follows upon taking the $P_{t,x}$-expectation that:

$$V^c(t,x) \,=\, E_{t,x}\Big(e^{-r\tau_c}(K - X_{t+\tau_c})^+\Big) \tag{3.22}$$

for all $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$ where we use that $V^c(t,x) = G(x) = (K-x)^+$ for $x \le c(t)$ or $t = T$. Comparing (3.22) with (2.1) we see that:

$$V^c(t,x) \,\le\, V(t,x) \tag{3.23}$$

for all $(t,x) \in [0,T\rangle \times \langle 0,\infty\rangle$.

8. Let us now show that $c \ge b$ on $[0,T]$. For this, recall that by the same arguments as for V^c we also have:

$$e^{-rs}V(t+s,X_{t+s}) = V(t,x) + \int_0^s e^{-ru} H(t+u,X_{t+u})\, I(X_{t+u} \le b(t+u))\, du + M_s^b \tag{3.24}$$

where $H = G_t + I\!\!L_X G - rG = -rK$ for $x \le b(t)$ and $(M_s^b)_{0\le s\le T-t}$ is a martingale under $P_{t,x}$. Fix $(t,x) \in \langle 0,T\rangle \times \langle 0,\infty\rangle$ such that $x < b(t) \wedge c(t)$ and consider the stopping time:

$$\sigma_b = \inf\,\{\, 0 \,\le\, s \,\le\, T-t \;|\; X_{t+s} \,\ge\, b(t+s)\,\}\ . \tag{3.25}$$

Inserting σ_b in place of s in (3.21) and (3.24) and taking the $P_{t,x}$-expectation, we get:

$$E_{t,x}\big(e^{-r\sigma_b}V^c(t+\sigma_b,X_{t+\sigma_b})\big) = G(x) - rK\, E_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru} I(X_{t+u} \le c(t+u))\, du\bigg) \tag{3.26}$$

$$E_{t,x}\big(e^{-r\sigma_b}V(t+\sigma_b,X_{t+\sigma_b})\big) \,=\, G(x) \,-\, rK\, E_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru}\, du\bigg)\ . \tag{3.27}$$

Hence by (3.23) we see that:

$$E_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru} I(X_{t+u} \le c(t+u))\, du\bigg) \,\ge\, E_{t,x}\bigg(\int_0^{\sigma_b} e^{-ru}\, du\bigg) \tag{3.28}$$

from where it follows by the continuity of c and b that $c(t) \ge b(t)$ for all $0 \le t \le T$.

9. Finally, let us show that c must be equal to b. For this, assume that there is $t \in \langle 0,T\rangle$ such that $c(t) > b(t)$, and pick $x \in \langle b(t), c(t)\rangle$. Under $P_{t,x}$ consider the stopping time τ_b from (2.12). Inserting τ_b in place of s in (3.21) and (3.24) and taking the $P_{t,x}$-expectation, we get:

$$E_{t,x}\big(e^{-r\tau_b}G(X_{t+\tau_b})\big) = V^c(t,x) - rK\, E_{t,x}\bigg(\int_0^{\tau_b} e^{-ru} I(X_{t+u} \le c(t+u))\, du\bigg) \tag{3.29}$$

$$E_{t,x}\big(e^{-r\tau_b}G(X_{t+\tau_b})\big) \,=\, V(t,x)\ . \tag{3.30}$$

Hence by (3.23) we see that:

$$E_{t,x}\left(\int_0^{\tau_b} e^{-ru} I(X_{t+u} \le c(t+u))\, du\right) \le 0 \tag{3.31}$$

from where it follows by the continuity of c and b that such a point x cannot exist. Thus c must be equal to b, and the proof is complete. □

Remark 3.2

The fact that U^c defined in (3.1) must be equal to G below c when c solves (2.21) is truly remarkable. The proof of this fact given above (Subsections 2-6) follows the way which led to its discovery. A shorter but somewhat less revealing proof can also be obtained by introducing U^c as in (3.1) and then verifying directly (using the Markov property only) that:

$$e^{-rs} U^c(t+s, X_{t+s}) + rK \int_0^s e^{-ru} I(X_{t+u} \le c(t+u))\, du \tag{3.32}$$

is a martingale under $P_{t,x}$ for $0 \le s \le T-t$. In this way it is possible to circumvent the material from Subsections 2-4 and carry out the rest of the proof starting with (3.17) onward. Moreover, it may be noted that the martingale property of (3.32) does not require that c is increasing (but only measurable). This shows that the claim of uniqueness in Theorem 3.1 holds in the class of continuous (or left-continuous) functions $c : [0,T] \to I\!R$ such that $0 < c(t) < K$ for all $0 < t < T$. It also identifies some limitations of the approach based on the local time-space formula (2.30) as initially undertaken (where c needs to be of bounded variation).

Remark 3.3

Note that in Theorem 3.1 above we do not assume that the solution starts (ends) at a particular point. The equation (2.21) is highly nonlinear and seems to be out of the scope of any existing theory on nonlinear integral equations (the kernel having four arguments). Similar equations arise in the first-passage problem for Brownian motion (cf. [16]).

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Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk