**Ertel’s potential vorticity theorem**

after H. Ertel, Ein Neuer Hydrodynamischer Wirbelsatz, Met. Z., 59, 277-281, 1942.

Start from the Navier-Stokes equation:

$\frac{∂V}{∂t}+\left(V.∇\right)V= -\frac{1}{ρ}∇p+g-2Ω×V+F$………….. 1

Substitute from the vector identity $\left(V.∇\right)V= ½∇\left(V^{2}\right)- V×(∇×V)$

$\frac{∂V}{∂t}+½∇\left(V^{2}\right)- V×(∇× V)= -\frac{1}{ρ}∇p+g-2Ω×V+F$

$\frac{∂V}{∂t}+½∇\left(V^{2}\right)- V×(2Ω+∇×V)= -\frac{1}{ρ}∇p+g+F$………………….. 2

Take the curl of this equation, writing **ξ** for (2**Ω** + **∇**$×$**V**), the absolute vorticity:

$\frac{∂ξ}{∂t}- ∇×\left(V×ξ\right)= \frac{1}{ρ^{2}}∇ϱ×∇p+ ∇×F$……………… 3

Now use the expansion: $∇×\left(V×ξ\right)=V∇.ξ+\left(ξ. ∇\right)V-\left(V.∇\right)ξ- ξ∇.V$

with ∇.**ξ = 0** (div curl ≡ 0 and ∇.**Ω** is very small – it is not zero because although **Ω** is constant the local coordinate system is not).

$\frac{∂ξ}{∂t}- \left(ξ. ∇\right)V+\left(V.∇\right)ξ+ ξ∇.V= \frac{1}{ρ^{2}}∇ϱ×∇p+ ∇×F$

Collecting terms and applying the continuity equation:

$\frac{dξ}{dt}- \left(ξ. ∇\right)V- \frac{ξ}{ρ}\frac{dρ}{dt}= \frac{1}{ρ^{2}}∇ϱ×∇p+ ∇×F$

$ρ\frac{d}{dt}\left(\frac{ξ}{ρ}\right)- \left(ξ. ∇\right)V= \frac{1}{ρ^{2}}∇ϱ×∇p+ ∇×F$…………. 4

Now multiply by **∇**q where q is any scalar field:

$ρ∇q.\frac{d}{dt}\left(\frac{ξ}{ρ}\right)- ∇q.\left(ξ. ∇\right)V= \frac{1}{ρ^{2}}∇q.∇ϱ×∇p+ ∇q.∇×F$

$ρ\frac{d}{dt}\left(\frac{ξ.∇q}{ρ}\right)-ξ.\frac{d}{dt}∇q- ∇q.\left(ξ. ∇\right)V= \frac{1}{ρ^{2}}∇q.∇ϱ×∇p+ ∇q.∇×F$

Use $ξ.\frac{d}{dt}∇q+ ∇q.\left(ξ. ∇\right)V= ξ.\frac{∂}{∂t}∇q+ ξ.\left(V.∇\right)∇q+ ∇q.\left(ξ. ∇\right)V= ξ.∇\frac{∂q}{∂t}+ ξ.∇\left(V.∇\right)q= ξ.∇\frac{dq}{dt} $

$ρ\frac{d}{dt}\left(\frac{ξ.∇q}{ρ}\right)-ξ.∇\frac{dq}{dt}= \frac{1}{ρ^{2}}∇q.∇ϱ×∇p+ ∇q.∇×F$

Finally, if q is a function of p and ρ, as θ is, the first term on the RHS vanishes, leaving:

$$\frac{d}{dt}\left(\frac{ξ.∇θ}{ρ}\right)= \frac{1}{ρ}ξ.∇\frac{dθ}{dt}+ \frac{1}{ρ}∇θ.∇×F$$

The quantity (**ξ.∇**θ)/ρ = $\frac{1}{ρ}\left(∇×V+2Ω\right).∇θ$ is the *Ertel potential vorticity.* The equation may also be written in flux form (Haynes and McIntyre, 1990) as: $\frac{d}{dt}\left(\frac{ξ.∇θ}{ρ}\right)= \frac{1}{ρ}∇.N where N= ξ\frac{dθ}{dt}+F×∇θ$