The j-invariant of an elliptic curve

The j-invariant of an elliptic curve with affine equation $y^2 = x^3 + px + q$ is defined to be

$$j = 1728\frac{4p^3}{4p^3 + 27q^2}.$$

p and q are not uniquely defined, if we change co-ordinates so that $y' = \alpha^3 y$ and $x' = \alpha^2 x$ for some $\alpha \neq 0$, then the new co-efficients are $p' = \alpha^4 p$ and $q' = \alpha^6 q$, so we still get the same value of j.

Another common form of the equation of the elliptic curve is the Legendre form $y^2 = x(x-1)(x-\lambda)$, where $\lambda \neq 0, 1$. j can be expressed in terms of λ as

$$j = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \qquad (*)$$

λ is not uniquely defined, depending on which two roots of the cubic are chosen to be mapped to 0 and 1, it can take 6 values, λ, $1-\lambda$, $1/\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda-1)$ and $(\lambda-1)/\lambda$, but they all give the same value of j.

It is not clear from either of these definition that j is really an invariant of the curve E, since by choosing a different inflection point we could get possibly different Weierstraß or Legendre forms.

There is another definition, which is less useful for calculations, but which shows that j is really an invariant of the curve. If E is an arbitrary elliptic curve in $\mathbb{P}^2$, it does not have to have an equation in a special form, then from any point E one can draw four tangent lines to E. Let λ be the cross ratio of the slopes of these line, and then (*) gives the j-invariant. This gives the following theorem:

Theorem 6.5. *j is an invariant of the elliptic curve, it does not depend on the choice of inflection point if the equation is written in the Weierstraß or Legendre form. Over an algebraically closed field, two elliptic curves are isomorphic if and only if they have the same j-invariant.*