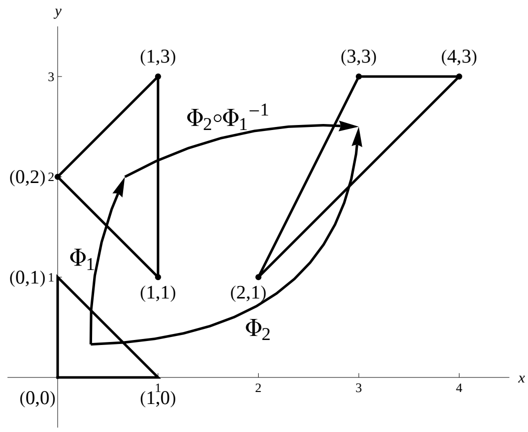
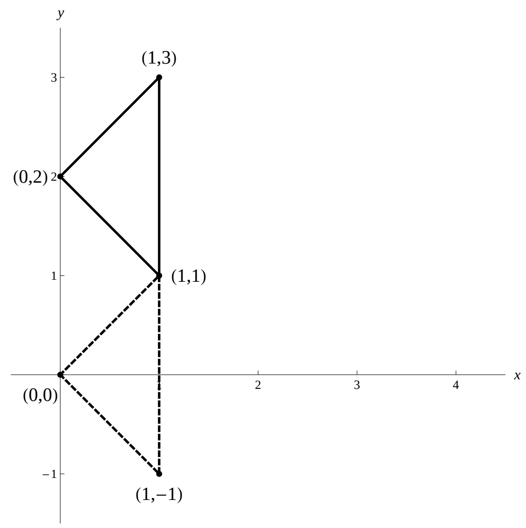
Affine equivalence of triangles

Problem: Find an affine map Φ that maps the triangle Δ_1 with vertices $(1,1)$, $(1,3)$, $(0,2)$ to the triangle Δ_2 with vertices $(4,3)$, $(3,3)$, $(2,1)$.



Solution: Let Φ_1 be the affine tranformation that maps the triangle Δ_0 with vertices $(1,0)$, $(0,1)$ and $(0,0)$ to Δ_1 and let Φ_2 be the affine tranformation that maps Δ_0 to Δ_2. The affine map we want is $\Phi_2 \circ \Phi_1^{-1}$.



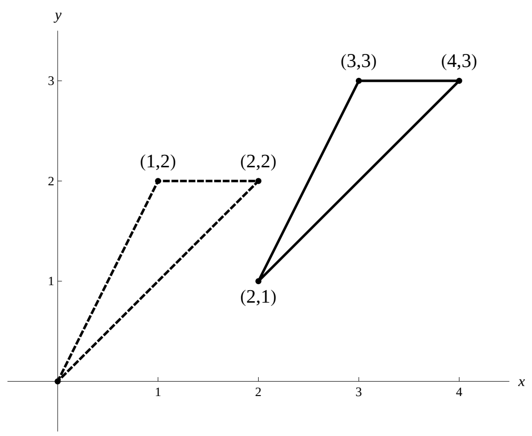
To find Φ_1, we translate Δ_1 by $(0,-2)$ so that the vertex $(0,2)$ goes to the origin. The other vertices of the translated triangle are $(1,-1)$ and $(1,1)$. If we form a matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, whose column vectors are the co-ordinates of these vertices, then the linear transformation represented by this matrix maps Δ_0 to the translated copy of Δ_1, by adding back $(0,2)$ we obtain Φ_1. Therefore

$$\Phi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

By solving $\Phi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ for x and y in terms of u and v, we get $x = \frac{u-v}{2}+1$, $y = \frac{u+v}{2} - 1$, Therefore

$$\Phi_1^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

It is clear from this form that Φ_1^{-1} is also an affine map.



Similarly, to find Φ_2, we translate Δ_2 by $(-2,-1)$ so that the vertex $(2,1)$ goes to the origin. The other vertices of the translated triangle are $(2,2)$ and $(1,2)$. Hence

$$\Phi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore the affine transformation that maps Δ_1 to Δ_2 is

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = (\Phi_2 \circ \Phi_1^{-1}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$