

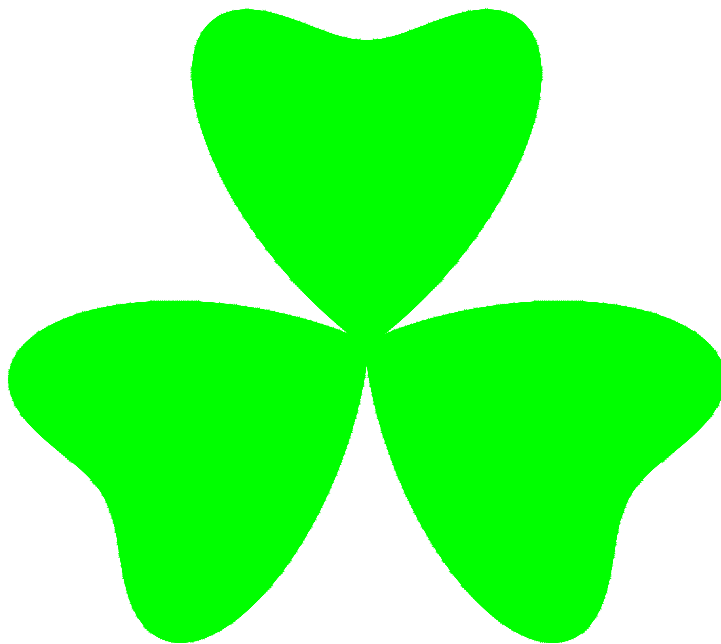
An algebro-geometric cloverleaf

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Abstract

We define a curve in polar co-ordinates which has the shape of a three-leaved clover. We calculate a rational parametrisation of the curve, find its Cartesian equation and prove that it is a rational curve. The methods used are also applicable to other curves given by trigonometric parametrisation in polar or Cartesian co-ordinates.

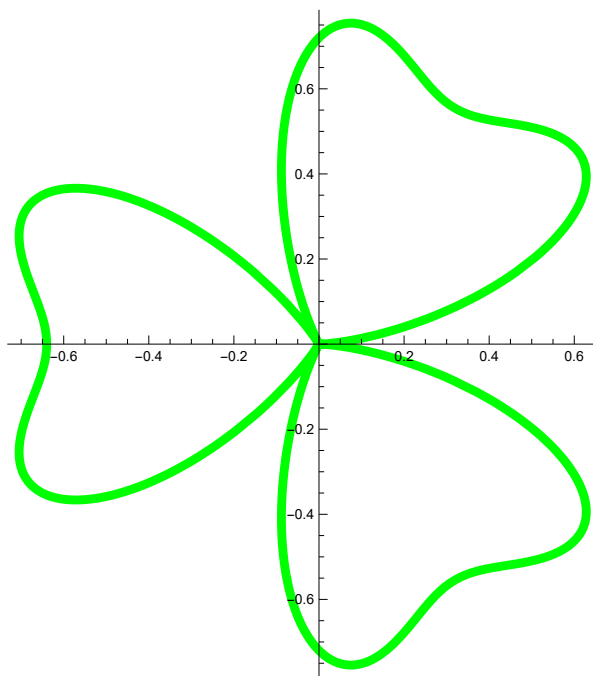


1 The polar equation

The cloverleaf on the front page is bounded by the curve C defined by the polar equation

$$r(\varphi) = \left(\sin \frac{3\varphi}{2} + \frac{\sin \frac{9\varphi}{2}}{5} \right)^2, \quad (0 \leq \varphi \leq 2\pi),$$

shown below, rotated through $\pi/2$ clockwise.



It follows from standard properties of sine that r is an even function of φ and it is periodic with period $2\pi/3$, therefore C is symmetric about the x axis and it also has 3-fold rotational symmetry about the origin.

By using the trigonometric identities $\sin^2 \alpha = (1 - \cos(2\alpha))/2$ and $\sin \alpha \sin \beta = (\cos(\alpha - \beta) - \cos(\alpha + \beta))/2$ we obtain

$$\begin{aligned} r(\varphi) &= \left(\sin \frac{3\varphi}{2} + \frac{\sin \frac{9\varphi}{2}}{5} \right)^2 = \sin^2 \left(\frac{3\varphi}{2} \right) + \frac{2}{5} \sin \frac{3\varphi}{2} \sin \frac{9\varphi}{2} + \frac{1}{25} \sin^2 \left(\frac{9\varphi}{2} \right) \\ &= \frac{1 - \cos(3\varphi)}{2} + \frac{\cos(3\varphi) - \cos(6\varphi)}{10} + \frac{1 - \cos(9\varphi)}{50}. \end{aligned}$$

We can use the double angle identity $\cos(2\alpha) = 2\cos^2\alpha - 1$ and the triple angle identity $\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$ to express $\cos(3\varphi)$, $\cos(6\varphi)$ and $\cos(9\varphi)$ in terms $\cos\varphi$ and thus we obtain

$$r(\varphi) = \frac{2(1 - \cos\varphi)(2\cos\varphi + 1)^2(4\cos^3\varphi - 3\cos\varphi + 3)^2}{25}.$$

2 Rational parametrisation

This method works for any curve given by a parametrisation in which x and y are rational functions in $\sin\varphi$ and $\cos\varphi$.

$x = r(\varphi)\cos(\varphi)$ and $y = r(\varphi)\sin\varphi$, so they are polynomials in $\sin\varphi$ and $\cos\varphi$. $\sin\varphi = \frac{2\tan(\varphi/2)}{1 + \tan^2(\varphi/2)}$ and $\cos\varphi = \frac{1 - \tan^2(\varphi/2)}{1 + \tan^2(\varphi/2)}$, these are rational functions in $\tan(\varphi/2)$. x and y are polynomials in $\sin\varphi$ and $\cos\varphi$, $\sin\varphi$ and $\cos\varphi$ are rational functions in $\tan(\varphi/2)$, therefore we obtain a parametrization $(x, y) = (p(t), q(t))$, where $t = \tan(\varphi/2)$ and p, q are rational functions.

The explicit expressions are

$$p(t) = \frac{16t^2(1 - t^2)(t^2 - 3)^2(t^6 + 12t^4 - 3t^2 + 2)^2}{25(t^2 + 1)^{10}}$$

and

$$q(t) = \frac{32t^3(t^2 - 3)^2(t^6 + 12t^4 - 3t^2 + 2)^2}{25(t^2 + 1)^{10}}.$$

$\psi : \mathbb{A}^1 \dashrightarrow \mathbb{A}^2$, $\psi(t) = (p(t), q(t))$ is clearly a rational map whose image is contained in the curve C , and in fact the image is the whole of the C except the point $(-16/25, 0)$ corresponding to $\varphi = \pi$, but we have not proved that C is an affine algebraic variety, so we cannot call ψ a rational map $\mathbb{A}^1 \dashrightarrow C$ yet.

3 Calculating the Cartesian equation of the curve

Method 1

This method is applicable whenever x and y can be written as rational functions in $\cos\varphi$ and $\sin\varphi$.

By substituting $\cos\varphi = x/\sqrt{x^2 + y^2}$, $\sin\varphi = y/\sqrt{x^2 + y^2}$ into the expression for x or y we obtain an equation involving x , y and $\sqrt{x^2 + y^2}$. After

clearing the denominators and replacing even powers of $\sqrt{x^2 + y^2}$ by appropriate powers of $(x^2 + y^2)^k$, we obtain an equation of the form $u(x, y) + v(x, y)\sqrt{x^2 + y^2} = 0$, where $u(x, y)$ and $v(x, y)$ are polynomials in x and y . The curve satisfies the equation $(u(x, y))^2 - (x^2 + y^2)(v(x, y))^2 = 0$.

If we use the polynomial for x , which only involves $\cos \varphi$, then we get

$$u(x, y) = x(25x^{10} + 8x^9 + 125x^8y^2 - 18x^7y^2 + 250x^6y^4 + 18x^5y^4 + 250x^4y^6 - 102x^3y^6 + 125x^2y^8 - 18xy^8 + 25y^{10})$$

and

$$v(x, y) = 2x(x^2 + y^2)(4x^6 + 57x^4y^2 - 18x^2y^4 + 9y^6),$$

which, after factoring out x^2 , gives the equation

$$\begin{aligned} &625x^{20} + 400x^{19} + 6250x^{18}y^2 + 1100x^{17}y^2 + 28125x^{16}y^4 - 2304x^{16}y^2 \\ &+ 400x^{15}y^4 + 75000x^{14}y^6 - 17472x^{14}y^4 - 5600x^{13}y^6 + 131250x^{12}y^8 \\ &- 37156x^{12}y^6 - 24400x^{11}y^8 + 157500x^{10}y^{10} - 17016x^{10}y^8 \\ &- 50600x^9y^{10} + 131250x^8y^{12} - 6588x^8y^{10} - 56400x^7y^{12} + 75000x^6y^{14} \\ &+ 5040x^6y^{12} - 33600x^5y^{14} + 28125x^4y^{16} + 1188x^4y^{14} - 9600x^3y^{16} \\ &+ 6250x^2y^{18} + 648x^2y^{16} - 900xy^{18} + 625y^{20} - 324y^{18} = 0 \end{aligned}$$

This shows that C is an affine algebraic variety and by substituting $p(t)$ for x and $q(t)$ for y into this polynomial, it can be verified directly that ψ is a rational map $\mathbb{A}^1 \dashrightarrow C$.

Method 2

This method works for any curve parametrised by rational functions.

x and y can be written as rational functions $x = p(t)$, $y = q(t)$, where $t = \tan(\varphi/2)$. By multiplying them by $25(t^2 + 1)^{10}$, the common denominator of p and q , we obtain $25(t^2 + 1)^{10}x = P(t)$, $25(t^2 + 1)^{10}y = Q(t)$, where P and Q are the numerators of p and q , resp. If we calculate the reduced Gröbner basis of the ideal $\langle 25(t^2 + 1)^{10}x - P(t), 25(t^2 + 1)^{10}y - Q(t) \rangle$ with respect to a suitable monomial ordering, e. g., the lexicographical ordering with $t > x > y$, the first element of the Gröbner basis will be the polynomial obtained above.

4 Finding the inverse of ψ

The Gröbner basis calculated previously also includes several elements which have degree 1 in t , the simplest one is

$$t(8x^8y + 122x^6y^3 + 78x^4y^5 - 18x^2y^7 + 18y^9) + (-25x^{10} - 125y^2x^8 + 140y^2x^7 - 250y^4x^6 + 60y^4x^5 - 250y^6x^4 + 84y^6x^3 - 125y^8x^2 + 36y^8x - 25y^{10})$$

and hence we can express t as a rational function in x and y , $t = \frac{G(x, y)}{H(x, y)}$, where

$$G(x, y) = 25x^{10} + 125x^8y^2 - 140x^7y^2 + 250x^6y^4 - 60x^5y^4 + 250x^4y^6 - 84x^3y^6 + 125x^2y^8 - 36xy^8 + 25y^{10}$$

and

$$H(x, y) = 8x^8y + 122x^6y^3 + 78x^4y^5 - 18x^2y^7 + 18y^9.$$

$\chi(x, y) = \frac{G(x, y)}{H(x, y)}$ is a rational map $C \dashrightarrow \mathbb{A}^1$ and it can be verified by direct calculation that $\chi \circ \psi = I_{\mathbb{A}^1}$ and $\psi \circ \chi = I_C$, so ψ and χ are inverses of each other, therefore C is a rational curve.

5 Summary

1. If a curve is defined by a parametric equation such that the x and y co-ordinates are polynomials in $\cos \varphi$ and $\sin \varphi$, then the curve can be parametrised by rational functions of $t = \tan(\varphi/2)$. This includes as special case curves defined by a polar equation where $r(\varphi)$ is a polynomial in $\cos \varphi$ and $\sin \varphi$, but also the Lissajous curves.
2. Any plane curve parametrised by rational functions of a variable t is an affine algebraic variety (the parametrisation may miss finitely many points of the variety). The defining equation of the curve can be found by calculating a Gröbner basis as in Method 2 in Section 3.
3. If the general point of the curve has a single preimage under the parametrisation by rational functions, that is, the parametrisation is injective outside a finite set of points, then the Gröbner basis has an element which has degree 1 in t , so t can be expressed a rational function in x and y . This gives a rational

map from the curve to the affine line, which is the inverse of the rational map given by the parametrisation of the curve, therefore the curve is rational.

4. Even if the parametrisation is not “almost injective”, Lüroth’s Theorem (<http://mathworld.wolfram.com/LuerothsTheorem.html>) states that any plane curve given by a parametrisation involving two non-constant rational functions is rational.