Examples of projective morphisms and rational maps

All examples are over $\mathbb{C}$.

1. Let $V = \mathbb{P}^1$ with homogeneous co-ordinates $(S : T)$. Let $(X : Y : Z)$ be the homogeneous co-ordinates on $\mathbb{P}^2$ and let $W = \mathcal{V}(\langle XZ - Y^2\rangle) \subset \mathbb{P}^2$. $XZ - Y^2$ is irreducible, therefore the ideal $\langle XZ - Y^2\rangle$ is prime so it is radical and by the Projective Nullstellensatz it is the homogeneous ideal of W. W is the projective closure of a parabola in $\mathbb{A}^2$.

Let $\Phi : V \dashrightarrow W$, $\Phi(S : T) = (S^2 : ST : T^2)$ and $\Psi : W \dashrightarrow V$, $\Psi(X : Y : Z) = (X : Y) = (Y : Z)$. As $XZ = Y^2$ on W, $(X : Y)$ and $(Y : Z)$ represent the same rational map.

Both Φ and Ψ are defined by homogeneous polynomials of the same degree, so they are rational maps. The image is Φ is contained in W, since $(S^2)(T^2) - (ST)^2 = 0$, so Φ is a rational map $V \dashrightarrow W$, not just $V \dashrightarrow \mathbb{P}^2$.

As S, T do not vanish simultaneously, at least one of S^2 and T^2 is non-zero, so $\Phi(S : T)$ is defined at every point $(S : T) \in V$, Φ is a morphism.

X, Y and Z cannot all be 0 simultaneously, so at least one of the forms $(X : Y)$ and $(Y : Z)$ is defined at every point of W, therefore Ψ is also a morphism.

Moreover, if $S \neq 0$, then $(\Psi \circ \Phi)(S : T) = \Psi(S^2 : ST : T^2) = (S^2 : ST) = (S : T)$ while if $T \neq 0$, then $(\Psi \circ \Phi)(S : T) = \Psi(S^2 : ST : T^2) = (ST : T^2) = (S : T)$ so $\Psi \circ \Phi = I_V$.

If $X \neq 0$, then $(\Phi \circ \Psi)(X : Y : Z) = \Phi(X : Y) = (X^2 : XY : Y^2)$. However, $Y^2 = XZ$ on W, therefore $(X^2 : XY : Y^2) = (X^2 : XY : XZ) = (X : Y : Z)$. Similarly, if $Z \neq 0$, then $(\Phi \circ \Psi)(X : Y : Z) = \Phi(Y : Z) = (Y^2 : YZ : Z^2) = (XZ : YZ : Z^2) = (X : Y : Z)$, so $\Phi \circ \Psi = I_W$.

Therefore Φ and Ψ are inverses of each other, so the projective line and the conic curve W are isomorphic.

2. Let $V = \mathbb{P}^1$ with homogeneous co-ordinates $(S : T)$. Let $(X : Y : Z : U)$ be the homogeneous co-ordinates on $\mathbb{P}^3$ and let $W = \mathcal{V}(\langle XZ - Y^2, YU - Z^2, XU - YZ\rangle)$. The ideal $\langle XZ - Y^2, YU - Z^2, XU - YZ\rangle$ is prime so it is the homogeneous ideal of W. W is called the *twisted cubic*.

Let $\Phi : V \dashrightarrow W$, $\Phi(S : T) = (S^3 : S^2T : ST^2 : T^3)$ and $\Psi : W \dashrightarrow V$, $\Psi(X : Y : Z : U) = (X : Y) = (Z : U)$. $XU = YZ$ on W, therefore $(X : Y)$ and $(Z : U)$ represent the same rational map.

Both Φ and Ψ are defined by homogeneous polynomials of the same degree, so they are rational maps. To verify that the image of Φ is contained in W, we need to substitute $X = S^3$, $Y = S^2T$, $Z = ST^2$ and $U = T^3$ into each of the generators of the ideal of W. We find that $(S^3)(ST^2) - (S^2T)^2 = (S^2T)(T^3) - (ST^2)^2 = (S^3)(T^3) - (S^2T)(ST^2) = 0$, so Φ is a rational map $V \dashrightarrow W$, not just $V \dashrightarrow \mathbb{P}^3$.

As S, T do not vanish simultaneously, at least one of S^3 and T^3 is non-zero, so $\Phi(S:T)$ is defined at every point $(S:T) \in V$, Φ is a morphism.

X, Y, Z and U cannot all be 0 simultaneously, so at least one of the forms $(X:Y)$ and $(Z:U)$ is defined at every point of W, therefore Ψ is also a morphism.

Moreover, $(\Psi \circ \Phi)(S:T) = \Psi(S^3 : S^2T : ST^2 : T^3) = (S^3 : S^2T) = (S:T)$ if $S \neq 0$ and $(\Psi \circ \Phi)(S:T) = \Psi(S^3 : S^2T : ST^2 : T^3) = (ST^2 : T^3) = (S:T)$ if $T \neq 0$, so $\Psi \circ \Phi = I_V$.

If $X \neq 0$, then $(\Phi \circ \Psi)(X:Y:Z:U) = \Phi(X:Y) = (X^3 : X^2Y : XY^2 : Y^3)$. However, $XY^2 = X^2Z$ and $Y^3 = XYZ$ on W since $XZ - Y^2 \in I(W)$, and $XYZ = X^2U$ since $XU - YZ \in I(W)$, therefore $(X^3 : X^2Y : XY^2 : Y^3) = (X^3 : X^2Y : X^2Z : X^2U) = (X:Y:Z:U)$. Similarly, $(\Phi \circ \Psi)(X:Y:Z:U) = (X:Y:Z:U)$ if $U \neq 0$. X and U cannot vanish simultaneously on W, because that would force $Y = Z = 0$, so $\Phi \circ \Psi = I_W$.

Therefore Φ and Ψ are inverses of each other, so the projective line and the twisted cubic are isomorphic.

3. In the first example of the handout on rational maps of affine varieties, `http://www.maths.manchester.ac.uk/~gm/teaching/MATH32062/rational.pdf`, we had the mutually inverse rational maps $\varphi(t) = \left(\frac{t^2-1}{t^2+1}, -\frac{2t}{t^2+1}\right)$ and $\psi(x,y) = y/(x-1)$ between the $\mathbb{A}^1$ and the circle $x^2 + y^2 - 1 = 0$ in $\mathbb{A}^2$.

We shall consider the projective versions of these maps. Let $(S:T)$ be homogeneous co-ordinates on $\mathbb{P}^1$ and let $(X:Y:Z)$ be the homogeneous co-ordinates on $\mathbb{P}^2$. We identify $\mathbb{A}^1$ with the affine piece $S \neq 0$ of $\mathbb{P}^1$, so that $t = T/S$ and we identify $\mathbb{A}^2$ with the affine piece $Z \neq 0$ of $\mathbb{P}^2$, so that $x = X/Z$, $y = Y/Z$. The projective closure of the circle is the projective variety $W \subset \mathbb{P}^2$ defined by the equation $(X/Z)^2 + (Y/Z)^2 - 1 = 0$ or $X^2 + Y^2 - Z^2 = 0$.

By subsituting T/S for t in φ and by using the identification of $(x,y) \in \mathbb{A}^2$ with $(x:y:1) \in \mathbb{P}^2$, we get the map

$$\Phi(S:T) = \left(\frac{(T/S)^2-1}{(T/S)^2+1} : -\frac{2(T/S)}{(T/S)^2+1} : 1\right).$$

By multiplying by $(T/S)^2 + 1$ and then by S^2, we get

$$\Phi(S:T) = ((T/S)^2 - 1 : -2(T/S) : (T/S)^2 + 1) = (T^2 - S^2 : -2ST : S^2 + T^2).$$

Now Φ is clearly recognisable as a rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. Furthermore $(T^2 - S^2)^2 + (-2ST)^2 - (S^2 + T^2)^2 = 0$, therefore Φ is a rational map $\mathbb{P}^1 \dashrightarrow W$.

$T^2-S^2=-2ST=S^2+T^2=0$ implies $S=T=0$, therefore Φ is a *morphism*, unlike φ, which was not defined at $t=\pm i$. The points $t=\pm i\in\mathbb{A}^1$ correspond to $(1:\pm i)\in\mathbb{P}^1$, which are mapped to "points at infinity" by Φ, this is why the affine rational map φ is not defined there.

We obtain the projective version of ψ by substituting $x=X/Z$ and $y=Y/Z$ into it, therefore

$$\Psi(X:Y:Z)=(1:(Y/Z)/((X/Z)-1))=((X/Z)-1:(Y/Z))=(X-Z:Y).$$

Ψ is clearly a rational map $W\dashrightarrow\mathbb{P}^1$. $(X-Z:Y)$ gives $(0:0)$ at $(1:0:1)$, but $(X-Z:Y)=(-Y:X+Z)$ on W since $(X+Z)(X-Z)=X^2-Z^2=-Y^2$. $X-Z=Y=X+Z=0$ implies $X=Y=Z=0$, therefore at least one of the reprentations $(X-Z:Y)$, $(-Y:X+Z)$ evaluates to something other than $(0:0)$, so Ψ is defined at every point of W, Ψ is a *morphism*. ψ is not defined at $(1,0)\in\mathbb{A}^2$, corresponding to $(1:0:1)\in\mathbb{P}^2$, but $\Psi(1:0:1)=(0:2)$. Again the image is the "point at infinity", this is why ψ is not defined.

Le us verify that Φ and Ψ are inverses of each other. If $S\neq 0$,

$$\begin{aligned}(\Psi\circ\Phi)(S:T) &= \Psi(T^2-S^2:-2ST:S^2+T^2)\\ &= ((T^2-S^2)-(S^2+T^2):-2ST)=(-2S^2:-2ST)=(S:T),\end{aligned}$$

while if $S=0$, we obtain $(\Psi\circ\Phi)(S:T)=(S:T)$ by using the other representation of Ψ.

$$(\Phi\circ\Psi)(X:Y:Z)=\Phi(X-Z:Y)=(Y^2-(X-Z)^2:-2(X-Z)Y:Y^2+(X-Z)^2).$$

On W, $Y^2=Z^2-X^2=-(X-Z)(X+Z)$, hence $Y^2-(X-Z)^2=-(X-Z)(X+Z)-(X-Z)^2=-2X(X-Z)$ and $Y^2+(X-Z)^2=-(X-Z)(X+Z)+(X-Z)^2=-2Z(X-Z)$. Thus

$$(\Phi\circ\Psi)(X:Y:Z)=(-2X(X-Z):-2Y(X-Z):-2Z(X-Z))=(X:Y:Z).$$

Using the other representation gives the same result.

4. Let $(X:Y:Z:W)$ be homogeneous co-ordinates on $\mathbb{P}^3$, $(S:T:U)$ homogeneous co-ordinates on $\mathbb{P}^2$. Let Q be the variety $Q=\mathcal{V}(\langle XY-ZW\rangle)\subset\mathbb{P}^3$. Let $\Phi:Q\dashrightarrow\mathbb{P}^2$ be the rational map $\Phi(X:Y:Z:W)=(X:Y:Z)$.

Φ is not a morphism because it is not defined at $(0:0:0:1)$. If it were a morphism, then its restriction to any subvariety would be a morphism, too. Q contains the lines $X=Z=0$ and $Y=Z=0$, and the points of these lines other than $(0:0:0:1)$ are mapped to $(0:1:0)$ and $(1:0:0)$, resp. So if Φ were a morphism, it would have to map $(0:0:0:1)$ both to $(1:0:0)$ and to $(0:1:0)$, which is impossible.

Φ has an inverse rational map. If $\Phi(X : Y : Z : W) = (S : T : U)$, then $(X : Y : Z) = (S : T : U)$ as points in $\mathbb{P}^2$, and then W can be calculated from $XY - ZW = 0$, so the inverse is $\Psi(S : T : U) = (S : T : U : ST/U)$. Multiply it by U to make the components polynomials, $\Psi(S : T : U) = (SU : TU : U^2 : ST)$. The image of Ψ is clearly in Q, since $(SU)(TU) - (U^2)(ST) = 0$.

Φ and Ψ are indeed inverses of each other because $(\Psi \circ \Phi)(X : Y : Z : W) = \Psi(X : Y : Z) = (XZ : YZ : Z^2 : XY) = (XZ : YZ : Z^2 : ZW) = (X : Y : Z : W)$, and $(\Phi \circ \Psi)(S : T : U) = \Phi(SU : TU : U^2 : ST) = (SU : TU : U^2) = (S : T : U)$.

Ψ is not a morphism either, because it is not defined at $(1 : 0 : 0)$ and at $(0 : 1 : 0)$ in $\mathbb{P}^2$. $(SU : TU : U^2 : ST)$ evaluates to $(0 : 0 : 0 : 0)$ at these points. In general, in order to determine whether a rational map is defined at a point, we need to consider all its possible representations, but $K[\mathbb{P}^2] = K[S, T, U]$ is a unique factorisation domain, and there is no non-constant polynomial dividing SU, TU, U^2, ST, so any other representation of Ψ would be just a multiple of $(SU : TU : U^2 : ST)$ by some polynomial, and would still give $(0 : 0 : 0 : 0)$ at $(1 : 0 : 0)$ and at $(0 : 1 : 0)$.

The conclusion is that Φ and Ψ give a birational equivalence between Q and $\mathbb{P}^2$. It can be proved by other methods that Q and $\mathbb{P}^2$ are not isomorphic.

Let C be the curve $C = (\langle XY - ZW, X^2 + Z^2 - YW\rangle) \subset Q$. We claim that $\Phi|_C$ extends to a morphism on C. The only issue whether it is defined at $(0 : 0 : 0 : 1)$. By using the elements of defining ideal of C, we obtain that $YW^2 = (X^2 + Z^2)W = X^2W + Z^2W = X^2W + XYZ$ and $ZW^2 = XYW$ in $K[C]$, therefore $(X : Y : Z) = (XW^2 : YW^2 : ZW^2) = (XW^2 : X^2W + XYZ : XYW) = (W^2 : XW + YZ : YW)$ on C. This means that $(W^2 : XW + YZ : YW)$ also represents $\Phi|_C$, and this form evaluates to $(1 : 0 : 0)$ at $(0 : 0 : 0 : 1)$, so $\Phi|_C$ is defined at $(0 : 0 : 01)$ and $\Phi|_C$ is a morphism.

It can be verified easily that the image of $\Phi|_C$ is contained in the curve $\mathcal{V}(\langle S^2U + U^3 - ST^2\rangle) \subset \mathbb{P}^2$.