A daffodil curve

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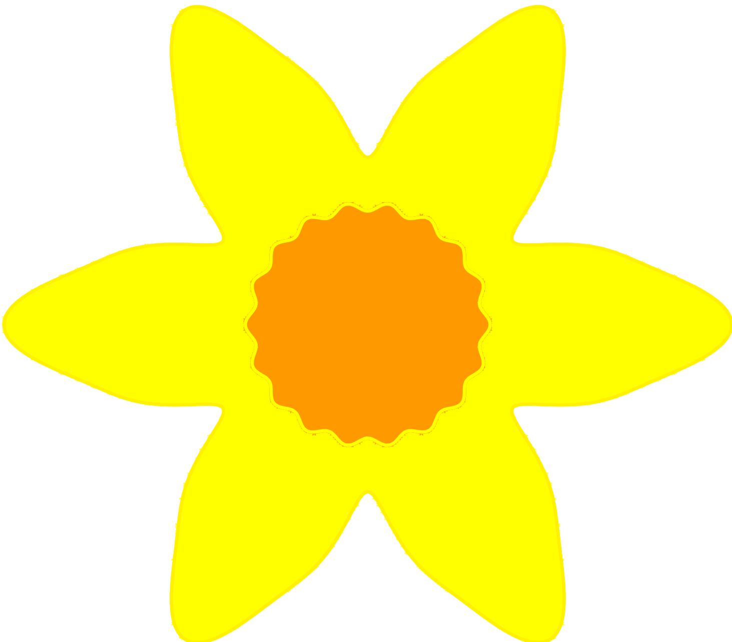
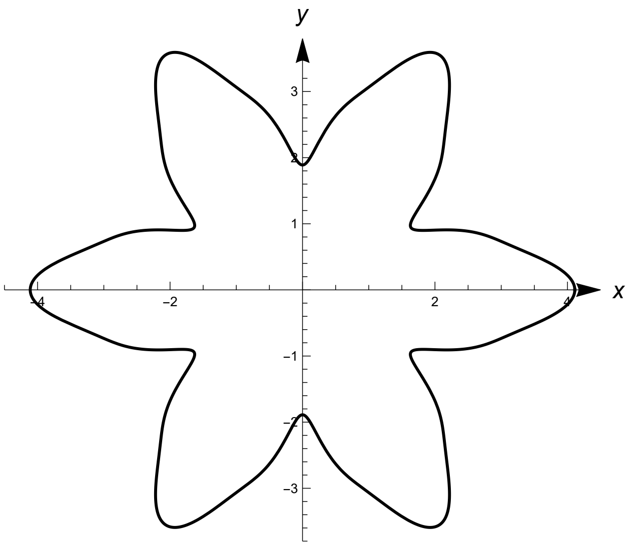


The daffodil on the front page is bounded by the curve C_1 defined by the polar equation

$$r(\varphi) = 3 + \cos(6\varphi) + \frac{\cos(18\varphi)}{9}$$

shown below.



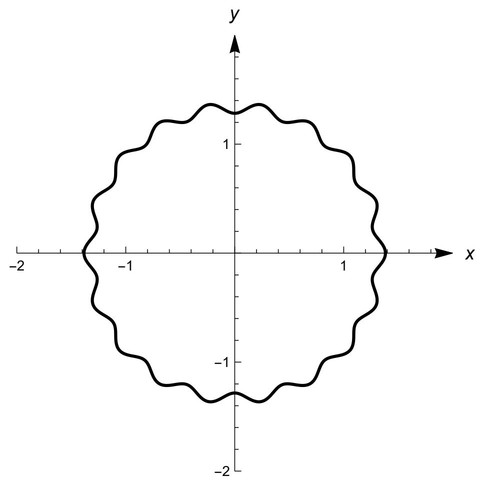
The coefficients were chosen because $\cos(\varphi)$ and $\cos(3\varphi)/9$ are the first two terms in the Fourier series of a triangle wave with period 2π, $\cos(6\varphi)$ and $\cos(18\varphi)/9$ are the first two terms in the Fourier series of a triangle wave with period $\pi/3$.

It follows from standard properties of sine that r is an even function of φ and it is periodic with period $\pi/3$, therefore C is symmetric about the x axis and it also has 6-fold rotational symmetry about the origin.

The centre of the daffodil is bounded by the curve C_2 with polar equation

$$r(\varphi) = \frac{4}{3} + \frac{\cos(18\varphi)}{20}$$

shown below



Let $R_1(\varphi) = 3 + \cos(6\varphi) + \dfrac{\cos(18\varphi)}{9}$ and $R_2(\varphi) = \dfrac{4}{3} + \dfrac{\cos(18\varphi)}{20}$.

$\cos(n\varphi)$ can be written as a polynomial in $\cos\varphi$ $\cos(n\varphi) = T_n(\cos\varphi)$, where T_n is the nth Chebyshev polynomial of the first kind. We have

$$T_6(z) = 32z^6 - 48z^4 + 18z^2 - 1$$

and

$$\begin{aligned} T_{18}(z) = {} & 131072z^{18} - 589824z^{16} + 1105920z^{14} - 1118208z^{12} \\ & + 658944z^{10} - 228096z^8 + 44352z^6 - 4320z^4 + 162z^2 - 1. \end{aligned}$$

If $r(\varphi)$ is polynomial in $\sin\varphi$ and $\cos\varphi$, then so are $x = r(\varphi)\cos(\varphi)$ and $y = r(\varphi)\sin\varphi$, too.

We shall show two methods for finding the Cartesian equation of C_1 and C_2.

Method 1
This method is applicable whenever x and y can be written as rational functions in $\cos\varphi$ and $\sin\varphi$.

By substituting $\cos\varphi = x/\sqrt{x^2+y^2}$, $\sin\varphi = y/\sqrt{x^2+y^2}$ into the expression for x or y we obtain an equation involving x, y and $\sqrt{x^2+y^2}$. After clearing the denominators and replacing even powers of $\sqrt{x^2+y^2}$ by appropriate powers of x^2+y^2, we obtain an equation of the form $u(x,y)+v(x,y)\sqrt{x^2+y^2} = 0$, where $u(x,y)$ and $v(x,y)$ are polynomials in x and y. The curve satisfies the equation $(u(x,y))^2 - (x^2+y^2)(v(x,y))^2 = 0$.

If we apply this to $R_1(\varphi)$ and use the expression for x, which only involves

$\cos\varphi$, then we get

$$\begin{aligned}u(x,y) = x(&37x^{18} + 9x^{16}y^2 + 3492x^{14}y^4 - 17340x^{12}y^6 + 46566x^{10}y^8\\ &- 39762x^8y^{10} + 21876x^6y^{12} - 1548x^4y^{14} + 477x^2y^{16} + 17y^{18})\end{aligned}$$

and $v(x,y) = -9x(x^2+y^2)^9$, which, after dividing by x^2, gives the equation

$$\begin{aligned}0 = &-81x^{38} - 1539x^{36}y^2 + 1369x^{36} - 13851x^{34}y^4 + 666x^{34}y^2 - 78489x^{32}y^6\\ &+ 258489x^{32}y^4 - 313956x^{30}y^8 - 1220304x^{30}y^6 - 941868x^{28}y^{10}\\ &+ 15327828x^{28}y^8 - 2197692x^{26}y^{12} - 123206760x^{26}y^{10} - 4081428x^{24}y^{14}\\ &+ 626795652x^{24}y^{12} - 6122142x^{22}y^{16} - 1892327472x^{22}y^{14}\\ &- 7482618x^{20}y^{18} + 3700127934x^{20}y^{16} - 7482618x^{18}y^{20}\\ &- 4472575652x^{18}y^{18} - 6122142x^{16}y^{22} + 3675388590x^{16}y^{20}\\ &- 4081428x^{14}y^{24} - 1900258992x^{14}y^{22} - 2197692x^{12}y^{26}\\ &+ 645496932x^{12}y^{24} - 941868x^{10}y^{28} - 104077800x^{10}y^{26} - 313956x^8y^{30}\\ &+ 21914100x^8y^{28} - 78489x^6y^{32} - 733008x^6y^{30} - 13851x^4y^{34}\\ &+ 174897x^4y^{32} - 1539x^2y^{36} + 16218x^2y^{34} - 81y^{38} + 289y^{36}\end{aligned}$$

This shows that C_1 is an affine algebraic variety. (More precisely, this only shows that C_1 is contained in this affine algebraic variety by the above equation, but it can be proved that C_1 is equal the whole affine algebraic variety defined over $\mathbb{R}$.)

If we apply this method to $R_2(\varphi)$, we obtain the equation

$$\begin{aligned}0 = &-3600x^{38} - 68400x^{36}y^2 + 6889x^{36} - 615600x^{34}y^4 + 43326x^{34}y^2\\ &- 3488400x^{32}y^6 + 2070081x^{32}y^4 - 13953600x^{30}y^8 - 1834032x^{30}y^6\\ &- 41860800x^{28}y^{10} + 143344980x^{28}y^8 - 97675200x^{26}y^{12}\\ &- 1127536056x^{26}y^{10} - 181396800x^{24}y^{14} + 5754812388x^{24}y^{12}\\ &- 272095200x^{22}y^{16} - 16736442192x^{22}y^{14} - 332560800x^{20}y^{18}\\ &+ 33353463006x^{20}y^{16} - 332560800x^{18}y^{20} - 40526722060x^{18}y^{18}\\ &- 272095200x^{16}y^{22} + 32977882206x^{16}y^{20} - 181396800x^{14}y^{24}\\ &- 17022598992x^{14}y^{22} - 97675200x^{12}y^{26} + 5748073188x^{12}y^{24}\\ &- 41860800x^{10}y^{28} - 1050398136x^{10}y^{26} - 13953600x^8y^{30}\\ &+ 168193620x^8y^{28} - 3488400x^6y^{32} - 5243952x^6y^{30} - 615600x^4y^{34}\\ &+ 419841x^4y^{32} - 68400x^2y^{36} + 181566x^2y^{34} - 3600y^{38} + 5929y^{36}\end{aligned}$$

for C_2.

Method 2
This method works for any curve parametrised by rational functions.

First we have to find a parametrisation of the curve by rational functions, which can be done easily for any curve given by a trigonometric parametrisation in which x are y are rational functions in $\sin\varphi$ and $\cos\varphi$. We can express $\sin\varphi$ and $\cos\varphi$ in terms of $\tan(\varphi/2)$ as $\sin\varphi = \dfrac{2\tan(\varphi/2)}{1+\tan^2(\varphi/2)}$ and $\cos\varphi = \dfrac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)}$, these are rational functions in $\tan(\varphi/2)$. If x and y are rational functions of $\sin\varphi$ and $\cos\varphi$ (including the possibility that they are polynomials), we obtain a parametrization $(x, y) = (p(t), q(t))$, where $t = \tan(\varphi/2)$ and p, q are rational functions.

If we apply this to C_1, so $x = R_1(\varphi)\cos\varphi$ and $y = R_1(\varphi)\sin\varphi$, then we get $p(t) = (1-t^2)s(t)/(9(t^2+1)^{19})$ and $q(t) = 2ts(t)/(9(t^2+1)^{19})$, where

$$\begin{aligned}
s(t) = {} & 37t^{36} - 630t^{34} + 60957t^{32} - 1917840t^{30} + 30415428t^{28} \\
& - 253758312t^{26} + 1252379412t^{24} - 3795626736t^{22} \\
& + 7308192870t^{20} - 9075033924t^{18} + 7308192870t^{16} \\
& - 3795626736t^{14} + 1252379412t^{12} - 253758312t^{10} \\
& + 30415428t^{8} - 1917840t^{6} + 60957t^{4} - 630t^{2} + 37.
\end{aligned}$$

$\psi(t) = (p(t), q(t))$ is a rational parametrisation of C_1, and the image is the whole of C_1 except the point $(-37/9, 0)$ corresponding to $\varphi = \pi$.

By multiplying $x = p(t)$, $y = q(t)$ by $9(t^2+1)^{19}$, the common denominator of p and q, we obtain $9(t^2+1)^{19}x = P(t)$, $9(t^2+1)^{19}y = Q(t)$, where P and Q are the numerators of p and q, resp. If we calculate the reduced Gröbner basis of the ideal $\langle 25(t^2+1)^{10}x - P(t), 25(t^2+1)^{10}y - Q(t)\rangle$ with respect to a suitable monomial ordering, e. g., the lexicographical ordering with $t \succ x \succ y$, the first element of the Gröbner basis will be the same polynomial that obtained by the first method. (If you are not taking MATH32012 or if you have not covered Gröbner bases yet, a Gröbner basis a generating set of an ideal with some nice properties.)

The Gröbner basis calculated previously also includes several elements which have degree 1 in t, the simplest one has the form $tH(x, y) - G(x, y)$, where

$G(x,y) = y(x^2+y^2)^9$ and

$$
\begin{aligned}
H(x,y) = {} & 9x^{19} + 37x^{18} + 81x^{17}y^2 + 9x^{16}y^2 + 324x^{15}y^4 + 3492x^{14}y^4 \\
& + 756x^{13}y^6 - 17340x^{12}y^6 + 1134x^{11}y^8 + 46566x^{10}y^8 \\
& + 1134x^9y^{10} - 39762x^8y^{10} + 756x^7y^{12} + 21876x^6y^{12} + 324x^5y^{14} \\
& - 1548x^4y^{14} + 81x^3y^{16} + 477x^2y^{16} + 9xy^{18} + 17y^{18}.
\end{aligned}
$$

Hence we can express t as a rational function in x and y, $t = \dfrac{G(x,y)}{H(x,y)}$. Let $\chi(x,y) = \dfrac{G(x,y)}{H(x,y)}$. It can be verified by direct calculation that $(\chi \circ \psi)(t) = t$ and $(\psi \circ \chi)(x,y) = (x,y)$ wherever the compositions are defined.

A similar calculation can be done for C_2.