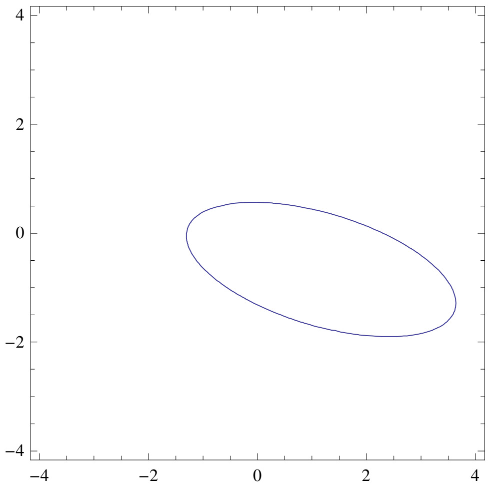
Classification of conics

Example 1. The conic $x^2 + 2xy + 4y^2 - x + 3y - 3 = 0$

The graph below shows this curve, it is an ellipse.

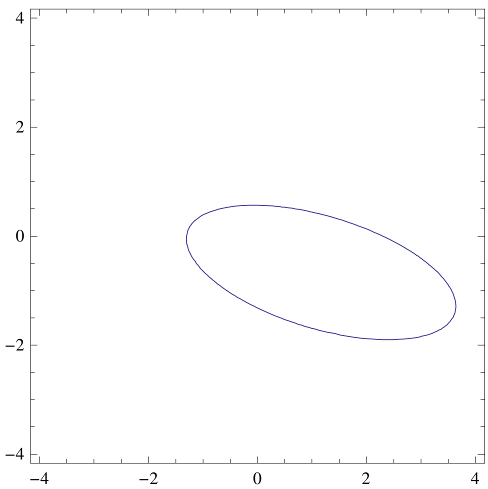
```
F = x^2 + 2 x y + 4 y^2 - x + 3 y - 3;
ContourPlot[F == 0, {x, -4, 4}, {y, -4, 4}, ImageSize → {250, 250}]
```



Step 1. As the coefficient of $x^2$is not 0, we do not need to do anything, just keep the current co-ordinates.

```
F1 = F /. {x → x1, y → y1} // Expand
ContourPlot[F1 == 0, {x1, -4, 4}, {y1, -4, 4}, ImageSize → {250, 250}]
```

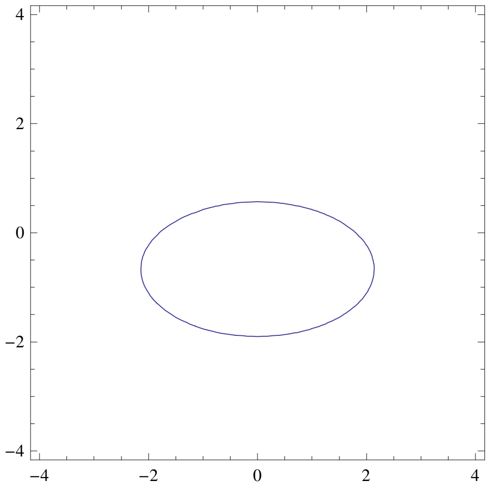
$-3 - x_1 + x_1^2 + 3y_1 + 2x_1y_1 + 4y_1^2$



Step 2. Now we complete the square by introducing a new variable $x_2 = x_1 + y_1 - 1/2$, then $x_2{}^2 = \frac{1}{4} - x_1 + x_1^2 - y_1 + 2x_1y_1 + y_1^2$, so it contains all the terms involving x_1. Geometrically this means that after this change of co-ordinates, the curve will be symmetric about the y-axis.

```
Solve[{x2 == x1 + y1 - 1 / 2, y2 == y1}, {x1, y1}];
F2 = F1 /. %[[1]] // Simplify // Expand
ContourPlot[F2 == 0, {x2, -4, 4}, {y2, -4, 4}, ImageSize → {250, 250}]
```

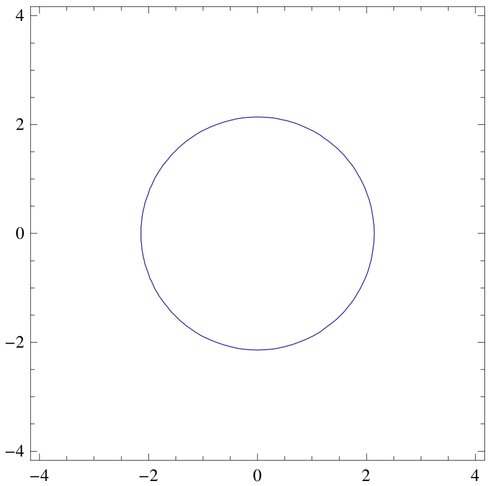
$-\frac{13}{4} + x_2^2 + 4y_2 + 3y_2^2$



Step 3. Now we complete the square by introducing a a new variable $y_3 = \sqrt{3}\ (y_2 + 2/3)$, then $y_3{}^2 = \frac{4}{3} + 4\ y_2 + 3\ y_2^2$, so it contains all the terms involving y_2. Geometrically this means that after this change of co-ordinates, the curve will also be symmetric about the x-axis.

```
Solve[{x3 == x2, y3 == √3 (y2 + 2 / 3)}, {x2, y2}];
F3 = F2 /. %[[1]] // Simplify // Expand
ContourPlot[F3 == 0, {x3, -4, 4}, {y3, -4, 4}, ImageSize → {250, 250}]
```

$-\frac{55}{12} + x_3^2 + y_3^2$

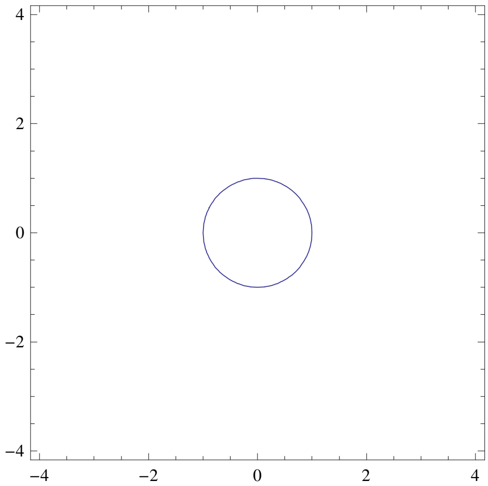


Step 4. Now we divide both $x_3$and y_3 by $\sqrt{55/12}$ and then divide the whole equation by 55/12 to obtain one of the standard forms.

```
Solve[{x4 == x3 / √55 / 12 , y4 == y3 / √55 / 12 }, {x3, y3}];
F4 = F3 /. %[[1]] // Simplify
F4 = F4 / (55 / 12)
ContourPlot[F4 == 0, {x4, -4, 4}, {y4, -4, 4}, ImageSize → {250, 250}]
```

$\frac{55}{12}\left(-1 + x_4^2 + y_4^2\right)$

$-1 + x_4^2 + y_4^2$

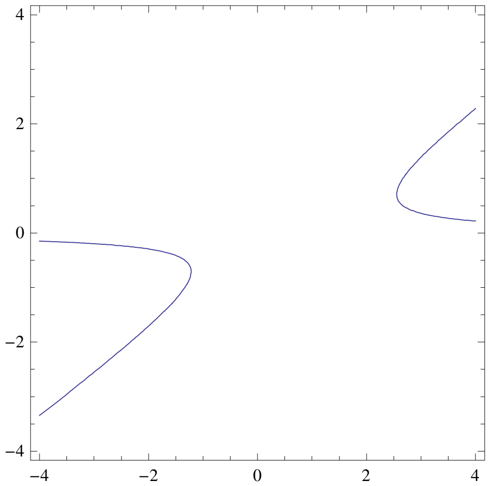


Every ellipse is affine equivalent to the circle $x^2 + y^2 - 1 = 0$ over the real numbers.

Example 2. The conic $3xy - 4y^2 - 2y - 2 = 0$

The graph below shows this curve, it is a hyperbola.

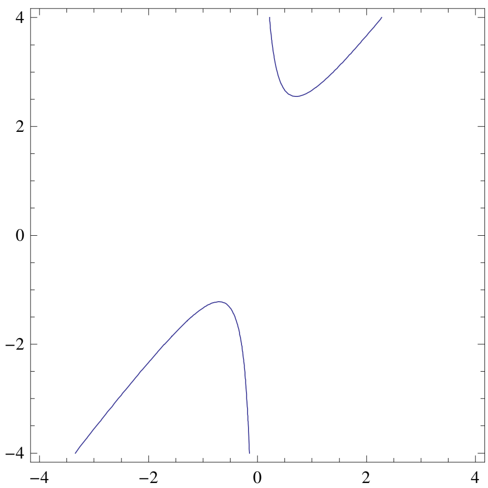
```
G = 3 x y - 4 y^2 - 2 y - 2;
ContourPlot[G == 0, {x, -4, 4}, {y, -4, 4}, ImageSize → {250, 250}]
```



Step 1. As the coefficient of $x^2$is 0, but the coefficient of $y^2$is not, we swap x and y, this corresponds to a reflection in the line $y = x$.

```
G1 = G /. {x → y1, y → x1} // Expand
ContourPlot[G1 == 0, {x1, -4, 4}, {y1, -4, 4}, ImageSize → {250, 250}]
```

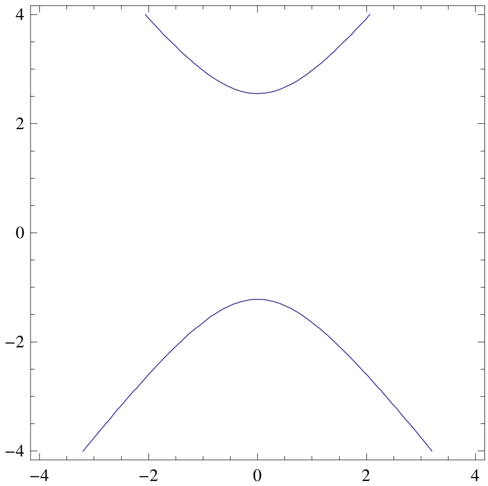
$-2 - 2x_1 - 4x_1^2 + 3x_1y_1$



Step 2. Now we complete the square by introducing a a new variable $x_2 = 2(x_1 - (3/8)y_1 + 1/4)$, then $x_2{}^2 = -\frac{1}{4} - 2x_1 - 4x_1^2 + \frac{3y_1}{4} + 3x_1y_1 - \frac{9y_1^2}{16}$, so it contains all the terms involving x_1. Geometrically this means that after this change of co-ordinates, the curve will be symmetric about the y-axis.

```
Solve[{x2 == 2 (x1 - (3 / 8) y1 + 1 / 4), y2 == y1}, {x1, y1}];
G2 = G1 /. %[[1]] // Simplify // Expand
ContourPlot[G2 == 0, {x2, -4, 4}, {y2, -4, 4}, ImageSize → {250, 250}]
```

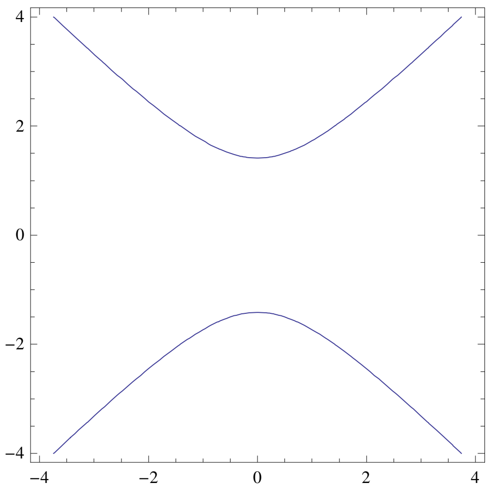
$$-\frac{7}{4} - x_2^2 - \frac{3y_2}{4} + \frac{9y_2^2}{16}$$



Step 3. Now we complete the square by introducing a a new variable $y_3 = $ **(3 / 4) (y₂ - 2 / 3)**, then $y_3{}^2 = \frac{1}{4} - \frac{3 y_2}{4} + \frac{9 y_2^2}{16}$, so it contains all the terms involving y_2. Geometrically this means that after this change of co-ordinates, the curve will also be symmetric about the x-axis.

```
Solve[{x3 == x2, y3 == (3 / 4) (y2 - 2 / 3)}, {x2, y2}];
G3 = G2 /. %[[1]] // Simplify // Expand
ContourPlot[G3 == 0, {x3, -4, 4}, {y3, -4, 4}, ImageSize → {250, 250}]
```

$-2 - x_3^2 + y_3^2$

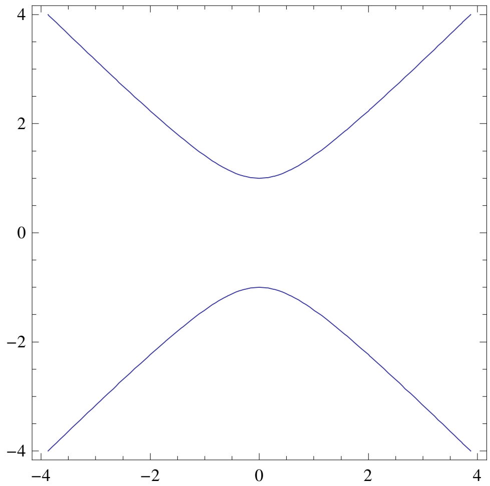


Step 4. Now we divide both $x_3$and $y_3$by $\sqrt{2}$ and then divide the whole equation by -2 to obtain one of the standard forms.

```
Solve[{x4 == x3 / √2, y4 == y3 / √2}, {x3, y3}];
G4 = G3 /. %[[1]] // Simplify
G4 = G4 / (-2)
ContourPlot[G4 == 0, {x4, -4, 4}, {y4, -4, 4}, ImageSize → {250, 250}]
```

$-2 \left(1 + x_4^2 - y_4^2\right)$

$1 + x_4^2 - y_4^2$

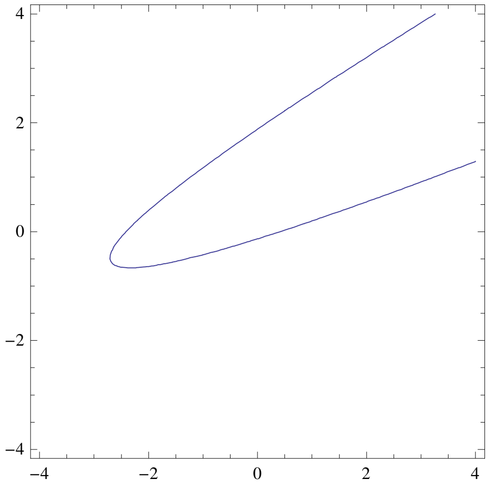


Every hyperbola is affine equivalent to $x^2 - y^2 + 1 = 0$.

Example 3. The conic $x^2 - 4xy + 4y^2 + 2x - 7y - 1 = 0$

The graph below shows this curve, but from this picture it is not clear whether the curve is an ellipse, a parabola or a hyperbola.

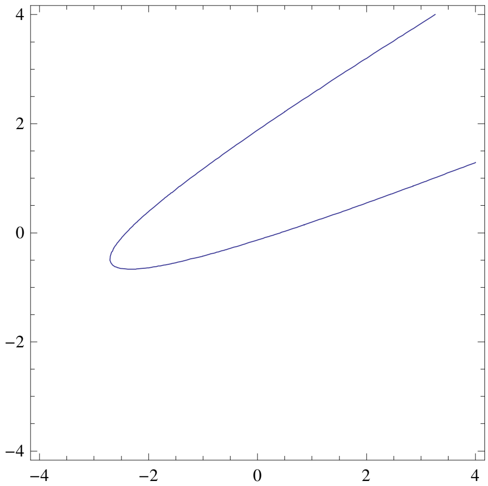
```
H = x^2 - 4 x y + 4 y^2 + 2 x - 7 y - 1;
ContourPlot[H == 0, {x, -4, 4}, {y, -4, 4}, ImageSize → {250, 250}]
```



Step 1. As the coefficient of $x^2$is not 0, we do not need to do anything, just keep the current co-ordinates.

```
H1 = H /. {x → x1, y → y1} // Expand
ContourPlot[H1 == 0, {x1, -4, 4}, {y1, -4, 4}, ImageSize → {250, 250}]
```

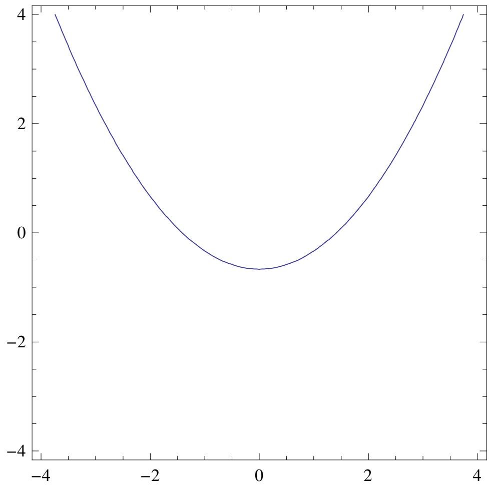
$-1 + 2\,x_1 + x_1^2 - 7\,y_1 - 4\,x_1\,y_1 + 4\,y_1^2$



Step 2. Now we complete the square by introducing a a new variable $x_2 = x_1 - 2\,y_1 + 1$, then $x_2{}^2 = 1 + 2\,x_1 + x_1^2 - 4\,y_1 - 4\,x_1\,y_1 + 4\,y_1^2$, so it contains all the terms involving x_1. Geometrically this means that after this change of co-ordinates, the curve will be symmetric about the y-axis.

```
Solve[{x2 == x1 - 2 y1 + 1, y2 == y1}, {x1, y1}];
H2 = H1 /. %[[1]] // Simplify // Expand
ContourPlot[H2 == 0, {x2, -4, 4}, {y2, -4, 4}, ImageSize → {250, 250}]
```

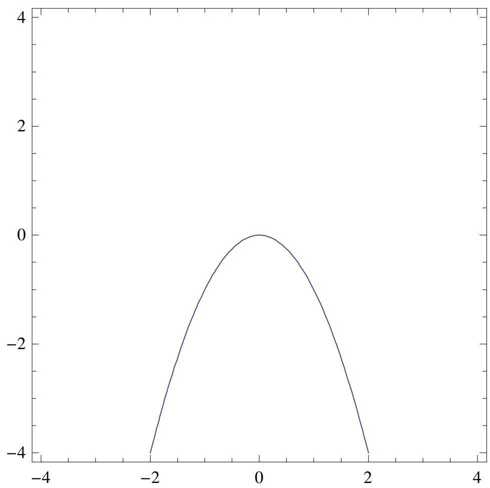
$-2 + x_2^2 - 3\,y_2$



Step 3. It is now clear that the curve is a parabola. We introduce a new variable $y_3 = -3\, y_2 - 2$ to absorb all the linear terms. Geometrically this means that after this change of co-ordinates, the apex of the parabola will be at the origin.

```
Solve[{x3 == x2, y3 == -3 y2 - 2}, {x2, y2}];
H3 = H2 /. %[[1]] // Simplify // Expand
ContourPlot[H3 == 0, {x3, -4, 4}, {y3, -4, 4}, ImageSize → {250, 250}]
```

$x_3^2 + y_3$



The equation is already in a standard form, no further steps are needed.
Every parabola is affine equivalent to $x^2 + y = 0$ over the real numbers.