3  Tangent spaces, dimension and singularities

Let $H \subset \mathbb{A}^n(K)$ be a hypersurface defined by the equation $f(x_1, x_2, \ldots, x_n) = 0$, where $f \in K[x_1, x_2, \ldots, x_n]$, and and let $P \in H$. One of the possible ways of defining that the line $\{P + tv \mid t \in K\}$ $(v = (v_1, v_2, \ldots, v_n) \in K^n \setminus \{0\})$ is tangent to $H$ at $P$ is to require that $f(P + tv)$, now simply a function of $t$, has a stationary point at $t = 0$.

The Taylor expansion of $f(P + tv)$ about $t = 0$ is

$$f(P + tv) = f(P) + t \left( \frac{df(P + tv)}{dt} \right)_{P} + (\text{terms of order } \geq 2 \text{ in } t)$$

$$= f(P) + t \sum_{j=1}^{n} \left( \frac{\partial f}{\partial x_j} \right)_{P} v_j + (\text{terms of order } \geq 2 \text{ in } t).$$

(derivatives of polynomials can be defined purely formally over any field without using the concept of limits and Taylor expansion works for polynomials over any field.)

Thus $f(P + tv)$ has a stationary point at $t = 0$ if and only if $\sum_{j=1}^{n} \left( \frac{\partial f}{\partial x_j} \right)_{P} v_j = 0$. By using the gradient $\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$, the condition $\sum_{j=1}^{n} \left( \frac{\partial f}{\partial x_j} \right)_{P} v_j = 0$ can be re-written as $(\nabla f)_P \cdot v = 0$.

**Definition.** Let $V \subseteq \mathbb{A}^n(K)$ be an affine algebraic variety and let $P \in V$. A line $\ell$ given in parametric form as $\{P + tv \mid t \in K\}$, $(v \in K^n \setminus \{0\})$, is said to be tangent to $V$ at $P$ if and only if $(\nabla f)_P \cdot v = 0$ for every $f \in \mathcal{I}(V)$.

**Definition.** The tangent space to $V$ at $P$, denoted by $T_P V$ is the union of the tangent lines to $V$ at $P$ and the point $P$.

**Theorem 3.1** Let $V \subseteq \mathbb{A}^n$ be an affine algebraic variety. Let $f_1, f_2, \ldots, f_r$ be a set of generators for $\mathcal{I}(V)$. Let $P \in V$ and let $v = (v_1, v_2, \ldots, v_n) \in K^n \setminus \{0\}$. The line $\ell = \{P + tv \mid t \in K\}$ is tangent to $V$ at $P$ if and only if $J_P v = 0$, where $J_P v$ is the Jacobian of $\{f_1, f_2, \ldots, f_r\}$ at $P$.
where $J$ is the Jacobian matrix,

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \frac{\partial f_r}{\partial x_2} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}$$

and $J_P$ is $J$ evaluated at $P$.

In particular, $T_P V$ is an affine subspace of dimension $n - \text{rank} J_P$.

**Proof.** Assume that $\ell$ is tangent to $V$ at $P$. Then $(\nabla f)_P. v = 0$ for every $f \in \mathcal{I}(V)$, in particular, $(\nabla f_i)_P. v = 0$ for every $i$, $1 \leq i \leq r$. $(\nabla f_i)_P. v$ is exactly the $i$th component of $J_P v$, therefore $J_P v = 0$.

Assume now that $J_P v = 0$. Let $f \in \mathcal{I}(V)$. Then $f = \sum_{i=1}^r g_i f_i$ for some $g_i \in K[x_1, x_2, \ldots, x_n]$, $1 \leq i \leq r$. Now

$$\nabla f = \sum_{i=1}^r \nabla (g_i f_i) = \sum_{i=1}^r ((\nabla g_i) f_i + g_i \nabla f_i),$$

therefore

$$(\nabla f)_P = \sum_{i=1}^r ((\nabla g_i)_P f_i(P) + g_i(P)(\nabla f_i)_P) = \sum_{i=1}^r g_i(P)(\nabla f_i)_P.$$

In the last step we used that $f_i(P) = 0$ for every $i$, $1 \leq i \leq r$. Hence $(\nabla f)_P. v = \sum_{i=1}^r g_i(P)(\nabla f_i)_P. v$, but $(\nabla f_i)_P. v = 0$, because it is the $i$th component of $J_P v = 0$, therefore $(\nabla f)_P. v = 0$.

The set $W = \{ v \in K^n \mid J_P v = 0 \}$ is a linear subspace, therefore $T_P V = P + W$ is an affine subspace. Its dimension is $\dim T_P V = \dim W = n - \text{rank} J_P$ by the Rank-Nullity Formula.

**Example:** Let $V \subset A^2(\mathbb{C})$ be the cuspidal cubic curve defined by the equation $y^2 - x^3 = 0$. It can be checked that $y^2 - x^3$ is irreducible, so $\langle y^2 - x^3 \rangle$ is a prime ideal, therefore it is radical and then the Nullstellensatz implies that that $\mathcal{I}(V) = \langle y^2 - x^3 \rangle$. Therefore $J = (-3x^2, 2y)$. 

34
At $P = (1, 1)$, $J_P = (-3, 2)$, therefore the line $\{P + tv \mid t \in \mathbb{C}\}$ is tangent to $V$ at $(1, 1)$ if and only if $-3v_1 + 2v_2 = 0$. Hence $T_{(1,1)}V$ is the line of slope $3/2$ through $P$ with equation $y = 3x/2 - 1/2$. It agrees with the tangent line obtained by, for example, implicit differentiation.

At $Q = (0, 0)$, however, something strange happens. $J_Q = (0, 0)$, therefore every line through $(0, 0)$ is a tangent line and $T_{(0,0)}V = \mathbb{A}^2$.

Let $v = (\alpha, \beta) \neq (0, 0)$ be the direction vector of a line through $Q = (0, 0)$. Then $Q + tv = (t\alpha, t\beta)$ and if we substitute $x = t\alpha$, $y = t\beta$ into $y^2 - x^3$ we obtain $t^2\beta^2 - t^3\alpha^3 = t^2(\beta^2 - t\alpha^3)$, which has a zero of multiplicity at least 2 at $t = 0$, so any line through $(0, 0)$ is a tangent line by our definition. If $\beta = 0$, then $t^2(\beta^2 - t\alpha^3)$ has a zero of multiplicity 3 at $t = 0$, so the line $y = 0$ has a higher order tangency to the curve than a general line through $(0, 0)$.

**Proposition–Definition 3.2** Let $V \subseteq \mathbb{A}^n$ be an irreducible affine algebraic variety. If $V \neq \emptyset$, there exists a proper subvariety $Z$ of $V$ ($Z \neq V$, but $Z$ may be $\emptyset$) and an integer $d \geq 0$ such that $\dim T_PV = d$ for every $P \in V \setminus Z$ and $\dim T_PV > d$ for $P \in Z$. (In other words, $\dim T_PV$ is constant on a non-empty Zariski open subset of $V$.) The *dimension* of $V$, denoted by $\dim V$, is defined to be $d$. The points of $V \setminus Z$, where $\dim T_PV = \dim V$, are called *non-singular*, while the points of $Z$, where $\dim T_PV > \dim V$, are called *singular* and the set of singular points of $V$ is denoted by $\text{Sing} V$. (Sometimes it is convenient to define $\dim \emptyset = -1$ and $\text{Sing} \emptyset = \emptyset$.)

**Proof.** Let $V_m = \{P \in V \mid \dim T_PV \geq m\}$, where $m \geq 0$ is an integer. By Theorem 3.1, $V_m = \{P \in V \mid \text{rank} J_P \leq n - m\}$. It is a theorem in linear algebra that a matrix has rank $\leq k$ if and only if all of its $(k + 1) \times (k + 1)$ minors vanish. Therefore

$$V_m = V \cap \mathcal{V}(\{(n - m + 1) \times (n - m + 1) \text{ minors of } J\}).$$
This shows that $V_m$ is the affine algebraic variety defined by the ideal generated by $\mathcal{I}(V)$ and by the $(n-m+1) \times (n-m+1)$ minors of $J$. We have $V = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_n \supseteq V_{n+1} = \emptyset$. Choose $d$ maximal such that $V_d = V$. Then $\dim T_P V = d$ for $P \in V \setminus V_{d+1}$ and $\dim T_P V > d$ for $P \in V_{d+1}$, so $\dim V = d$ and $\text{Sing } V = V_{d+1}$.

**Example:** If $V$ is an affine subspace of $\mathbb{A}^n$, $T_P V = V$ at every point $P \in V$, so the dimension of $V$ as a variety agrees with its dimension as an affine space and it has no singular points. In particular, $\mathbb{A}^n$ is non-singular and has dimension $n$.

Worked examples can be found in the separate handout at [https://personalpages.manchester.ac.uk/staff/gabor.megyesi/teaching/MATH32062/singularities.pdf](https://personalpages.manchester.ac.uk/staff/gabor.megyesi/teaching/MATH32062/singularities.pdf). Herwig Hauser’s gallery at [http://homepage.univie.ac.at/herwig.hauser/bildergalerie/gallery.html](http://homepage.univie.ac.at/herwig.hauser/bildergalerie/gallery.html) has many nice examples of singular surfaces.

The combination of the preceding results gives the following procedure for finding the dimension and singular points of an irreducible variety $V \subseteq \mathbb{A}^n(K)$.

1. Choose a set of generators $f_1, f_2, \ldots, f_r$ for $\mathcal{I}(V)$. (In all the problems you will have to solve, the field will be algebraically closed and the defining ideal of $V$ will be a radical ideal, so it will be equal to $\mathcal{I}(V)$.)
2. Calculate the Jacobian matrix $J$.
3. For $k = 0, 1, 2, \ldots$, find the points where all the $(k+1) \times (k+1)$ minors of $J$ vanish and also $f_1 = f_2 = \ldots = f_r = 0$. These are the points where $\text{rank } J_P \leq k$ and therefore $\dim T_P V \geq n-k$. Do this until you find the smallest $k$ such that $\dim T_P V \geq n-k$ for every point $P \in V$.
4. Then $\dim V = n-k$ and $\text{Sing } V$ is the set of points where $\dim T_P V > n-k$, i.e., the points of $V$ where all the $k \times k$ minors of $J$ vanish.

**Definition.** Let $V$ be an arbitrary (not necessarily irreducible) affine algebraic variety. The dimension of $V$, $\dim V$, is the maximum of the dimensions of the irreducible components of $V$.

The **local dimension** of $V$ at $P \in V$ is defined to be the maximum of the dimension of the irreducible components of $V$ containing $P$.

$P$ is called **non-singular** if and only if $\dim T_P V$ is equal to the local dimension of $V$ at $P$.

$P$ is called **singular** if and only if $\dim T_P V$ is greater than the local dimension of $V$ at $P$. The set of singular points of $V$ is denoted by $\text{Sing } V$, just as in the irreducible case.

**Fact:** $\text{Sing } V$ consists of the singular points of the individual irreducible components and of intersection points of different components.
This means that in general, one needs to decompose the variety into its irreducible components in order to find its dimension and its singular points, however, by the proposition below and by problem sheet 5, question 3, hypersurfaces over an algebraically closed field are an exception.

**Proposition 3.3** Let $K$ be an algebraically closed field. Let $H \subset \mathbb{A}^n(K)$ be a hypersurface, that is, $H = \mathcal{V}(f)$ for some non-constant polynomial $f \in K[x_1, x_2, \ldots, x_n]$, then $\dim H = n - 1$.

**Proof.** $f$ can be factorised as $f = f_1^{\alpha_1}f_2^{\alpha_2} \cdots f_r^{\alpha_r}$, where the $f_i$, $1 \leq i \leq r$, are irreducible polynomials such that $f_j$ is not a scalar multiple of $f_j$ if $i \neq j$, and $\alpha_i$, $1 \leq i \leq r$, are positive integers. The irreducible components are $\mathcal{V}(f_i)$, $1 \leq i \leq r$, so if we prove the proposition for $f$ irreducible, then it will imply the general case, because each component of $H$ has dimension $n - 1$.

Let’s assume that $f$ is irreducible, then $H$ is also irreducible and $\mathcal{I}(H) = \langle f \rangle$.

We have $J = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$. $\text{rank } J_P$ can only be 0 or 1 and if $\text{rank } J_P = 1$ for some $P \in H$, then $\dim H = n - 1$.

Assume to the contrary that $\text{rank } J_P = 0$ for every $P \in H$. Then $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \in \mathcal{I}(H) = \langle f \rangle$, that is, all the partial derivatives are multiples of $f$.

As their degree is smaller than the degree of $f$, the only way this can happen is if they are all 0. This is not possible in characteristic 0 since then $f$ would be a constant, which has been excluded.

In characteristic $p$ it is possible for all the partial derivatives to be 0, it happens if and only if the exponent of every variable in every term is a multiple of $p$. ($\frac{dx^n}{dx} = nx^{n-1} = 0$ if and only if $p$ divides $n$.) Then

$$f = \sum_{i_1, i_2, \ldots, i_n} \alpha_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = \left( \sum_{i_1, i_2, \ldots, i_n} \sqrt[p]{\alpha_{i_1, i_2, \ldots, i_n}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \right)^p,$$

so $f$ is reducible, contradicting our assumption that it was irreducible.

**Remark.** Let $K$ be an algebraically closed field. If $V_1, V_2 \subseteq \mathbb{A}^n(K)$ are affine algebraic varieties of dimension $d_1$ and $d_2$, respectively, then every irreducible component of $V_1 \cap V_2$ has dimension at least $d_1 + d_2 - n$, but this is hard to prove with our definition of dimension. In particular, this implies that if $V = \mathcal{V}(f_1, f_2, \ldots, f_r) \subseteq \mathbb{A}^n(K)$, then every component of $V$ has dimension at least $n - r$. If $f_1$ and $f_2$ are non-constant polynomials without a non-trivial common factor, then every component of $\mathcal{V}(f_1, f_2)$ has dimension exactly $n - 2$. 37
Theorem 3.4 Let $V \subseteq \mathbb{A}^n$ be an affine algebraic variety and let $P \in V$. Let $m_P = \{ F \in K[V] \mid F(P) = 0 \}$ be the maximal ideal of $K[V]$ corresponding to $P$. Then $\dim T_P V = \dim(m_P/m_P^2)$. (The quotient $K[V]/m_P^2$ is a finite dimensional vector space, $m_P/m_P^2$ is a subspace in it.)

Proof. Let’s introduce the notation $\overline{f} = f + \mathcal{I}(V)$ for $f \in K[V]$. Let $P = (a_1, a_2, \ldots, a_n)$ and let $F \in m_P$. $F = \overline{f}$ for a suitable $f \in K[V]$ and $F(P) = f(P) = 0$. We noted in the proof of the Nullstellensatz (Theorem 1.7) that $f(P) = 0$ if and only if $f \in \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle$, therefore there exist polynomials $g_i \in K[x_1, x_2, \ldots, x_n]$ such that $f = \sum_{i=1}^{n} (x_i - a_i)g_i$ in $K[x_1, x_2, \ldots, x_n]$. Hence $\overline{f} = \sum_{i=1}^{n} (\overline{x_i} - \overline{a_i})\overline{g_i}$ in $K[V]$. $\overline{x_i} - \overline{a_i} \in m_P$ for every $i$, $1 \leq i \leq n$, obviously, therefore $m_P = \langle \overline{x_1} - \overline{a_1}, \overline{x_2} - \overline{a_2}, \ldots, \overline{x_n} - \overline{a_n} \rangle \triangleleft K[V]$. In the rest of the proof we shall assume $P = (0, 0, \ldots, 0)$ for simplicity, then we have $m_P = \langle \overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \rangle \triangleleft K[V]$.

We can also assume that $T_P V$ is the subspace $x_1 = x_2 = \ldots = x_k = 0$ of $K^n$, where $k = n - \dim T_P V$, this can be achieved by a linear change of co-ordinates. This means that the reduced row echelon form of $J_P$ is

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

with 1s on the diagonal in the first $k$ rows and all the other entries are 0. The rows of the reduced row echelon form are linear combinations of the rows of $J_P$. By applying the same linear combinations to the generators of $I(V)$ we obtain a set of generators $f_1, f_2, \ldots, f_r$ for $\mathcal{I}(V)$ such that the Jacobian of this set of generators is the reduced row echelon form above, therefore $f_i = x_i + (\text{terms of degree } \geq 2)$ for $1 \leq i \leq k$, and $f_i$ only contains terms of degree $\geq 2$ for $k + 1 \leq i \leq r$.

Let’s define a function $D : m_P \rightarrow K^{n-k}$ in the following way. Let $F \in m_P$. Then $F = \overline{f}$ for some $f \in \langle x_1, x_2, \ldots, x_n \rangle \triangleleft K[x_1, x_2, \ldots, x_n]$. $f$ can be written as $f = \sum_{i=1}^{n} \alpha_i x_i + (\text{terms of degree } \geq 2)$ and we set $D(F) = (\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n)$.

We shall prove that $D$ is a surjective linear map with kernel $m_P^2$, then it will follow that $m_P/m_P^2 \cong K^{n-k}$, so $\dim(m_P/m_P^2) = \dim K^{n-k} = n - k$. 

38
Step 1. First we need to show that the vector \((\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n)\) does not depend on the choice of \(f\), so that \(D\) is indeed a function. Let \(g \in K[x_1, x_2, \ldots, x_n]\) be another polynomial such that \(F = \overline{g}\). Then \(f - g \in I(V)\), so there exist \(h_i \in K[x_1, x_2, \ldots, x_n], 1 \leq i \leq r\), such that \(f - g = \sum h_i f_i\). As \(f_i = x_i + \text{(terms of degree } \geq 2)\) for \(1 \leq i \leq k\), and \(f_i\) only contains terms of degree \(\geq 2\) for \(k+1 \leq i \leq r\), the sum \(\sum h_i f_i\) has no linear terms in \(x_{k+1}, x_{k+2}, \ldots, x_n\), so the coefficients \(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\) are uniquely determined and therefore \(D(F)\) is indeed a function.

Step 2. \(D\) is clearly linear, \(D(F + G) = D(F) + D(G)\) for any \(F, G \in m_P\) and \(D(\lambda F) = \lambda D(F)\) for any \(F \in m_P\) and any \(\lambda \in K\).

Step 3. \(D\) is surjective, since \(D(\sum_{i=k+1}^{n} \alpha_i x_i) = (\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n)\) for any \(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n \in K\).

Step 4. \(\ker D = \{F \in m_P \mid D(F) = 0\}\) by definition. We shall prove that \(\ker D = m_P^2\). Let \(F = \overline{f}, f = \sum_{i=1}^{n} \alpha_i x_i + \text{(terms of degree } \geq 2)\) as in the definition of \(D\). Then \(F \in \ker D\) if and only if \(\alpha_{k+1} = \alpha_{k+2} = \ldots = \alpha_n\). This implies that \(f - \sum_{i=1}^{k} \alpha_i f_i\) only contains terms of degree \(\geq 2\), i.e., it is an element of the ideal \(\langle \{x_i x_j \mid 1 \leq i \leq j \leq n\} \rangle \subseteq K[x_1, x_2, \ldots, x_n]\). This is equivalent to the existence of polynomials \(g_{ij} \in K[x_1, x_2, \ldots, x_n], 1 \leq i \leq j \leq n\), such that

\[
 f - \sum_{i=1}^{k} \alpha_i f_i = \sum_{1 \leq i \leq j \leq n} g_{ij} x_i x_j
\]
in \(K[x_1, x_2, \ldots, x_n]\). Then in \(K[V]\), we have

\[
 \overline{f} = \sum_{1 \leq i \leq j \leq n} g_{ij} x_i x_j,
\]
i.e., \(F = \overline{f} \in \langle \{x_i x_j \mid 1 \leq i \leq j \leq n\} \rangle = m_P^2 \trianglelefteq K[V]\). This shows that \(\ker D \subseteq m_P^2\).

Conversely, let’s now assume that \(F \in m_P^2\). Then

\[
 F = \sum_{1 \leq i \leq j \leq n} G_{ij} x_i x_j
\]
for suitable $G_{ij} \in K[V]$, $1 \leq i \leq j \leq n$. Now let $g_{ij} \in K[x_1, x_2, \ldots, x_n]$, $1 \leq i \leq j \leq n$, be such that $g_{ij} = G_{ij}$, then we can take $f = \sum_{1 \leq i \leq j \leq n} g_{ij} x_i x_j$ and it satisfies $\bar{f} = F$. As $f$ has no linear terms, we have $D(F) = (0, 0, \ldots, 0)$, therefore $F \in \ker D$. This shows $m_P^2 \subseteq \ker D$, by combining it with the previous result we obtain $\ker D = m_P^2$.

We have proved that $D$ is a surjective linear map $m_P \to K^{n-k}$ with kernel $m_P^2$, therefore $m_P/m_P^2 \cong K^{n-k}$, in particular $\dim(m_P/m_P^2) = \dim K^{n-k} = n - k = \dim T_P V$.

**Corollary 3.5** Let $\varphi : V \to W$ be an isomorphism between affine algebraic varieties. Then $\dim T_P V = \dim T_{\varphi(P)} W$ for every $P \in V$, therefore $\dim V = \dim W$ and $\varphi(Sing V) = Sing W$. In particular, a singular variety cannot be isomorphic to a non-singular variety. □

**Proof.** By Theorem 2.1, $\varphi$ induces a morphism $\varphi^* : K[W] \to K[V]$ and by Corollary 2.2, if $\varphi$ is an isomorphism of varieties then $\varphi^*$ is an isomorphism of rings. It easy to check $\varphi^*$ also gives a bijection between $m_P \triangleleft K[V]$ and $m_{\varphi(P)} \triangleleft K[W]$ and also between $m_P^2 \triangleleft K[V]$ and $m_{\varphi(P)}^2 \triangleleft K[W]$. Hence $m_P/m_P^2$ and $m_{\varphi(P)}/m_{\varphi(P)}^2$ are isomorphic vector spaces, therefore by Theorem 3.4, $\dim T_P V = \dim m_P/m_P^2 = \dim m_{\varphi(P)}/m_{\varphi(P)}^2 = \dim T_{\varphi(P)} W$.

The other assertions follow from this, since dimension and singularity are defined in terms of the dimensions of the tangent spaces at the points of the variety. □

**Examples:**
1. The affine line $\mathbb{A}^1(\mathbb{C})$ and the cuspidal cubic curve defined by the equation $y^2 - x^3 = 0$ in $\mathbb{A}^2(\mathbb{C})$ are not isomorphic, since the former has no singular point, while the latter has a singular point at $(0, 0)$.
2. Let $W_1 = \mathcal{V}(\langle xy, xz, yz \rangle) \subset \mathbb{A}^3(\mathbb{C})$ and let $W_2 = \mathcal{V}(\langle xy(x+y) \rangle) \subset \mathbb{A}^2(\mathbb{C})$. 

\begin{center}
\includegraphics[width=0.4\textwidth]{example1.png} \hspace{1cm} \includegraphics[width=0.4\textwidth]{example2.png}
\end{center}
The irreducible components of $W_1$ are the co-ordinate axes, $\mathcal{V}(\langle x, y \rangle)$, $\mathcal{V}(\langle x, z \rangle)$ and $\mathcal{V}(\langle y, z \rangle)$. The irreducible components of $W_2$ are the lines $\mathcal{V}(\langle x \rangle)$, $\mathcal{V}(\langle y \rangle)$ and $\mathcal{V}(\langle x + y \rangle)$. Both varieties consist of three lines meeting at a point, the origin, but we shall prove that $W_1$ and $W_2$ are not isomorphic.

Let’s calculate $T_{(0,0,0)}W_1$ and $T_{(0,0)}W_2$. $(xy, xz, yz)$ and $(xy(x+y))$ are radical ideals, so they are equal to $\mathcal{I}(W_1)$ and $\mathcal{I}(W_2)$, respectively.

The Jacobian of $W_1$ is $J(W_1) = \begin{pmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$. At $(0,0,0)$, $J_{(0,0,0)}(W_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so it has rank 0 and therefore $\dim T_{(0,0,0)}W_1 = 3$. $(T_{(0,0,0)}W_1 = \mathbb{A}^3$ can be proved even without calculating the Jacobian. Any $f \in \mathcal{I}(W_1)$ is identically 0 on the three co-ordinate axes, therefore these axes are tangent lines. If we know that $T_{(0,0,0)}W_1$ is an affine subspace, then it follows that $T_{(0,0,0)}W_1 = \mathbb{A}^3$ as no other affine subspace contains all three co-ordinate axes.)

The Jacobian of $W_2$ is $J(W_2) = \begin{pmatrix} 2xy + y^2 & x^2 + 2xy \\ 0 & 0 \end{pmatrix}$. At $(0,0)$, $J_{(0,0)}(W_2) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, so it has rank 0 and therefore $\dim T_{(0,0)}W_2 = 2$.

If there existed an isomorphism $\varphi : W_1 \to W_2$, then we would have $\varphi(0,0,0) = (0,0)$, as $(0,0,0)$ and $(0,0)$ are uniquely characterised as the intersection points of the components in $W_1$ and $W_2$, respectively. However, isomorphisms preserve the dimension of the tangent space and $\dim T_{(0,0,0)}W_1 = 3 \neq \dim T_{(0,0)}W_2 = 2$, therefore there does not exist an isomorphism between $W_1$ and $W_2$.

The rest of the material in this section is not examinable.

For any vector space $X$ over $K$, the dual space $X^*$ is defined to be the vector space of all linear maps $X \to K$ with the obvious operations. For any linear map $\alpha : X \to Y$, there is a dual map $\alpha^* : Y^* \to X^*$. What Theorem 3.4 really proves, without mentioning dual spaces explicitly, is that $T_PV \cong (m_P/m_P^2)^*$. While any two finite dimensional vector spaces of the same dimension are isomorphic, the isomorphism between $T_PV$ and $(m_P/m_P^2)^*$ is natural, it can be defined without reference to bases.

Given an arbitrary morphism $\varphi : V \to W$, by Theorem 2.1 we have a morphism of rings $\varphi^* : K[W] \to K[V]$, which also induces a linear map $m_{\varphi}(P)/m_{\varphi}^2(P) \to m_P/m_P^2$. Taking duals, we obtain the differential map $d\varphi_P : T_PV \cong (m_P/m_P^2)^* \to T_{\varphi(P)}W \cong (m_{\varphi}(P)/m_{\varphi}^2(P))^*$. $d\varphi_P$ is similar to the differential map between tangent spaces in differential geometry.

Isomorphisms can be characterised by using the differential map, $\varphi$ is an isomor-
phism if and only if it is bijective and $d\varphi_P$ is an isomorphism for every $P \in V$.

The dimension of a variety can be defined in other ways as well, which are more suited for certain purposes, but are less useful for practical calculations.

1. $\dim V$ is equal to the maximum $d$ for which there exists a chain of irreducible subvarieties $V_0 \subset V_1 \subset \ldots V_d \subseteq V$. Moreover, if $V$ is irreducible, any such chain that cannot be extended by inserting further subvarieties has the same length. From this definition it is clear that the dimension of a proper subvariety of an irreducible variety is smaller than the dimension of the whole variety, which is hard to prove by using tangent spaces.

2. If $V$ is irreducible, then $\dim V$ is transcendence degree of $K(V)$ over $K$, i.e., the maximal number of elements $x_1, x_2, \ldots, x_d \in K(V)$ which do not satisfy any non-zero polynomial with coefficients in $K$. As a consequence of an algebraic result called the Noether Normalisation Lemma, any variety is birationally equivalent to a hypersurface, therefore $\dim V = d$ if and only if $V$ is birationally equivalent to a hypersurface in $\mathbb{A}^{d+1}$. This interpretation of the dimension shows that the dimension of irreducible varieties is invariant under birational equivalence, not just under isomorphism.

**Fact:** For any irreducible variety $V$ in characteristic 0, there exist a non-singular variety $\tilde{V}$ and a birational morphism $\varphi : \tilde{V} \to V$ such that $\varphi$ is an isomorphism outside $\text{Sing} V$. This is called the desingularisation of $V$. The existence of a desingularisation is a very hard theorem, for which the Japanese mathematician Heisuke Hironaka received the Fields Medal in 1970. The problem is still open in prime characteristic in dimension $\geq 3$. 

42