An Efficient Method to Reduce the Numerical Dispersion in the LOD-FDTD Method based on the (2, 4) Stencil

Journal: Transactions on Antennas and Propagation

Manuscript ID: AP0905-0486.R1

Proposed Manuscript Type: Paper

Date Submitted by the Author: 09-Oct-2009

Complete List of Authors: Liu, Qi-Feng; Center for microwave and RF technologies, School of Electronic Information and Electrical Engineering, YIN, WEN-YAN; Center for Microwave and RF Technologies, Department of Electronic Engineering Chen, Zhizhang; Dalhousie University, Director of Microwave and Wireless Research Laboratory Liu, Peiguo; National University of Defense Technology, Department of No.4

Key Words: Numerical dispersion, Finite difference methods
An Efficient Method to Reduce the Numerical Dispersion in the LOD-FDTD Method based on the (2, 4) Stencil

Qi-Feng Liu, Wen-Yan Yin, Senior Member, IEEE, Zhizhang (David) Chen, Senior Member, IEEE, and Pei-Guo Liu

Abstract—This paper presents a parameter-optimized (2, 4) stencil based locally-one-dimensional (LOD) finite-difference time-domain (FDTD) with much reduced numerical dispersion errors. The method is first proved to be unconditionally stable. Then by using different optimization schemes, the method is optimized to satisfy different accuracy requirements, such as minimum dispersion errors in the axial directions, in the diagonal direction, and in specified angles. The variation of the performance of the parameter-optimized LOD-FDTD with different time steps and frequencies is also studied. It is found that the parameter optimization can significantly reduce numerical dispersion errors, bringing them down to the level of the conventional FDTD but with the time step exceeding the CFL limit and without much additional computational cost. Furthermore, the optimized parameters are not sensitive to frequencies; in particular, the optimized parameters obtained at a higher frequency still present low numerical dispersion errors at a lower frequency.

Index Terms—alternating direction implicit, courant-Friedrich-Levy limit, finite-difference time-domain method, locally one-dimensional, numerical dispersion, unconditionally stability, numerical dispersion, and phase velocity error.

I. INTRODUCTION

The finite-difference time domain (FDTD) has found wide applications due to its simplicity and flexibility. However, for electrically large and high-Q structures, the FDTD computational expenditure is still high due to the Courant-Friedrich-Levy (CFL) stability condition that limits the size of time step. To alleviate this problem, the unconditionally stable alternating direction implicit finite-difference time-domain (ADI-FDTD) was presented in [1] and [2]. In it, the CFL limit is removed and the computations involve solution of a tri-diagonal matrix at each time step. More recently, other unconditionally stable methods such as the split-step [3]-[4] and the locally one-dimensional (LOD) FDTD methods [5]-[7] have been proposed. The LOD approach, often termed as the LOD-FDTD method, applied local splitting of the operators and was claimed to be more CPU time efficient than the ADI-FDTD method. To account for open-region problems, the LOD-FDTD method has also been incorporated with the split-field and the convolution PML [8] and [9], respectively. The LOD FDTD has been mostly implemented in two dimensions except that in [10], [11] where the three-dimensional LOD-FDTD method was shown; however, it requires input processing for every simulation and output processing at every time step for every field component at each time step.

In our previous work, we proposed an arbitrary-order LOD-FDTD algorithm with its stability and numerical dispersion analyses [12]. It is found that the second-order LOD-FDTD has the same level of numerical dispersion error as that of the unconditionally stable alternating direction implicit finite-difference time-domain (ADI-FDTD) method but with higher computational efficiency.

On the other hand, all the above implicit methods, either LOD-FDTD or ADI-FDTD, suffer from the large numerical dispersion errors when time steps increase. To further reduce the dispersion errors, a parameter optimized method has been proposed for the ADI-FDTD method based on the (2, 2) stencil [13] (the method is then termed as (2, 2) PO-ADI-FDTD). It minimized the dispersion errors of the ADI-FDTD methods for different incident angles and different time step sizes. [14] applied the method to the LOD-FDTD, and implemented a parameter optimized method to reduce the numerical dispersion errors of the (2, 2) stencil-based LOD-FDTD (note that there exist some errors in [14] including equation (4) and (5) there). [16] is an extension of the work in [13][15] and applied the optimized method to the ADI-FDTD method based on the (2, 4) stencil (the method is then termed as (2,4) PO-ADI-FDTD). By comparisons, it can be shown that the (2, 4) PO-ADI-FDTD [15] further reduces the dispersion error compared to the (2, 2) PO-ADI-FDTD method [13]. In addition, it also has smaller dispersion errors than the standard ADI-FDTD method based on the (2, 4) stencil. In [17],

http://mc.manuscriptcentral.com/tap ieee
artificial anisotropy was introduced to reduce the numerical dispersion in the traditional FDTD method, and the similar approach was then applied to ADI-FDTD method [18].

This paper applies the parameter optimized method to the fourth-order LOD-FDTD based on the (2, 4) stencil to further reduce the dispersion errors of LOD-FDTD. In addition, the application is in three dimensions rather than two dimensions like those in [14]-[16].

In the following sections, we will first present in details a three-dimensional parameter optimized LOD FDTD method based on the (2,4) stencil as well as its stability and dispersion analysis. Then we will show the results that minimize dispersion errors based on different requirements. Finally, we will present the parameter optimization in terms of time steps and frequency.

II. THE PARAMETER-OPTIMIZED THREE-DIMENSIONAL ADI-FDTD BASED ON (2, 4) STENCIL

Let us consider the wave propagating in a lossless homogenous medium with permittivity $\varepsilon$ and permeability $\mu$. The second-order finite-difference is applied to approximate the time derivative while the central finite difference scheme with four stencils is used to replace the spatial differential operators in Maxwell’s equations. This forms the three-dimensional (2, 4) stencil-based LOD-FDTD method.

The updating procedure of the three-dimensional (2, 4)-stencil LOD-FDTD method is performed in two sub-procedures. The first sub-procedure is represented by (1a)-(1c) for time marching form $n$ to $n+1/2$. The second sub-procedure is represented by (2a)-(2c) for time marching form $n+1/2$ to $n+1$. They are formulated as follows.

Sub-step #1: advancement form $n$ to $n+1/2$ time step:

$$E_x^{n+1}_{i,j,k} = E_x^n_{i,j,k} + \frac{\Delta t}{2\varepsilon} \delta_t (H_y^n_{i,j,k} + H_z^n_{i,j,k} + H_x^n_{i,j,k})$$  \hspace{1cm} (3a)$$

$$E_y^{n+1}_{i,j,k} = E_y^n_{i,j,k} + \frac{\Delta t}{2\varepsilon} \delta_t (H_x^n_{i,j,k} + H_z^n_{i,j,k} + H_y^n_{i,j,k})$$  \hspace{1cm} (3b)$$

$$E_z^{n+1}_{i,j,k} = E_z^n_{i,j,k} + \frac{\Delta t}{2\varepsilon} \delta_t (H_x^n_{i,j,k} + H_y^n_{i,j,k} + H_z^n_{i,j,k})$$  \hspace{1cm} (3c)$$

$$H_x^{n+1}_{i,j,k} = H_x^n_{i,j,k} + \frac{\Delta t}{2\mu} \delta_t (E_y^n_{i,j,k} + E_z^n_{i,j,k} + E_x^n_{i,j,k})$$  \hspace{1cm} (4a)$$

$$H_y^{n+1}_{i,j,k} = H_y^n_{i,j,k} + \frac{\Delta t}{2\mu} \delta_t (E_x^n_{i,j,k} + E_z^n_{i,j,k} + E_y^n_{i,j,k})$$  \hspace{1cm} (4b)$$

$$H_z^{n+1}_{i,j,k} = H_z^n_{i,j,k} + \frac{\Delta t}{2\mu} \delta_t (E_x^n_{i,j,k} + E_y^n_{i,j,k} + E_z^n_{i,j,k})$$  \hspace{1cm} (4c)$$

Here $\delta$ is the approximation of a spatial differential operator with a fourth-order central finite-difference in the $p$ direction. For example, $\delta_p$ can be the difference operator defined as:

$$\delta_p (V_{i,j,k}) = c_p \frac{V_{i,j,k+1/2} - V_{i,j,k-1/2}}{\Delta x} + (1-c_p) \frac{V_{i,j,k+1/2} - V_{i,j,k-1/2}}{3\Delta x}$$  \hspace{1cm} (5a)$$

$$\delta_p (V_{i,j,k}) = c_p \frac{V_{i,j,k+1/2} - V_{i,j,k-1/2}}{\Delta y} + (1-c_p) \frac{V_{i,j,k+1/2} - V_{i,j,k-1/2}}{3\Delta y}$$  \hspace{1cm} (5b)$$

$$\delta_p (V_{i,j,k}) = c_p \frac{V_{i,j,k+1/2} - V_{i,j,k-1/2}}{\Delta z} + (1-c_p) \frac{V_{i,j,k+1/2} - V_{i,j,k-1/2}}{3\Delta z}$$  \hspace{1cm} (5c)$$

where $V = E_y$, $H_y, \eta = x, \ y, \ z$; $c_x$, $c_y$ and $c_z$ are the parameters to be optimized.

With (5a)-(5c), equations (1a)-(4c) represent the finite difference approximations to the first-order spatial derivative of electric and magnetic fields. Their accuracy is generally of the 2nd order, but becomes of the 4th order when $c_x = c_y = c_z = 9/8$, as explained in [19].

In general, these parameters can be chosen and optimized in such a way that they improve the accuracy of the numerical method or achieve other accuracy-related objectives. To facilitate the set up of these objective functions, we perform the stability and dispersion analysis in the following sections.
III. STABILITY AND DISPERSION ANALYSIS

The von Neumann method has been used as a standard approach to stability analysis of an unconditionally stable FDTD method where eigenvalues of the amplification matrix in spectral domain are evaluated [1][2]; if all the eigenvalues are not larger than unity in magnitudes, the method is stable. In this section, we will use the method to prove the unconditional stability of the (2, 4) LOD method described in this paper.

Assume that $k_x$, $k_y$, and $k_z$ are the spatial frequencies along the $x$, $y$, and $z$ directions, respectively. The field components in the spectral domain at the $n^\text{th}$ time step are then:

$$\phi_{x,y,z}^n = V^n e^{-j(k_x x + k_y y + k_z z)}$$

Here $V^n = [E_x^n, E_y^n, E_z^n, H_x^n, H_y^n, H_z^n]^T$ is the magnitude of the field components with $E_z$ being the spectral-domain $E_z$ at the $n^\text{th}$ time step and $h_z$ being the magnitude of the spectral-domain $H_z$ at the $n^\text{th}$ time step. $\xi = x, y, z$.

By substituting (6) into (1a)-(4c), the following equations can be obtained:

$$V^{n+1} = \Lambda V^n$$

where

$$\Lambda = M_{1z}^{-1} M_{z2} M_{2z}^{-1} M_{z1}$$

is the amplification matrix with

$$M_{1z} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -jR_y \\
0 & 1 & 0 & -jR_x & 0 & 0 \\
0 & 0 & 1 & 0 & -jR_y & 0 \\
0 & -jQ_z & 0 & 1 & 0 & 0 \\
0 & 0 & -jQ_z & 0 & 1 & 0 \\
-jQ_z & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

and

$$M_{z2} = \begin{bmatrix}
1 & 0 & 0 & 0 & -jR_y & 0 \\
0 & 1 & 0 & 0 & 0 & -jR_x \\
0 & 0 & 1 & 0 & -jR_y & 0 \\
0 & -jQ_z & 0 & 1 & 0 & 0 \\
-jQ_z & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$M_{2z} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & jR_y \\
0 & 1 & 0 & 0 & 0 & jR_x \\
0 & 0 & 1 & 0 & 0 & jR_y \\
0 & jQ_z & 0 & 1 & 0 & 0 \\
0 & 0 & jQ_z & 0 & 1 & 0 \\
jQ_z & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$M_{z1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

With the help of MAPLE, the eigenvalues of $\Lambda$ can be found as

$$\lambda_1 = \lambda_2 = 1$$

$$\lambda_3 = \lambda_4 = \frac{B + \sqrt{-D^2}}{A}$$

$$\lambda_5 = \lambda_6 = \frac{B - \sqrt{-D^2}}{A}$$

where

$$A = 1 + A_1 + A_2 + A_3$$

$$B = 1 + A_1 - A_2 - A_3$$

$$D^2 = A^2 - B^2 = 4(A_2 + A_1 + B_1 + B_2)$$

with

$$A_1 = Q_y R_y Q_y R_y$$

$$A_2 = Q_y R_y + Q_y R_y + Q_y R_y$$

$$A_3 = Q_y R_y Q_y R_y + Q_y R_y Q_y R_y + Q_y R_y Q_y R_y$$
\[ B_i = Q_i R_i R_i Q_i R_i R_i + Q_i R_i R_i Q_i R_i R_i + Q_i R_i R_i Q_i R_i R_i \]
\[ B_i = Q_i R_i R_i Q_i R_i R_i + Q_i R_i R_i Q_i R_i R_i + Q_i R_i R_i Q_i R_i R_i \]

Since \( P_i (\xi = x, y, z) \) are real, \( B \) and \( D \) are both positive and \( D^2 = A^2 - B^2 \geq 0 \), the magnitudes of all the eigenvalues expressed by (11) are then unity. Therefore, the (2, 4)-stencil based LOD-FDTD method presented in Section II is unconditionally stable.

The numerical dispersion of the proposed method can be found by following the procedure described in [20]. For a time-harmonic signal,

\[ V^{n+1} = e^{j\omega t} V^n \]

Then (7) becomes

\[ (e^{j\omega t} I - M_{12}^{-1} M_{21}^{-1} M_{11}^{-1} M_{11}) V^n = 0 \]

To ensure a non-trivial solution, the determinant of (14) should be zero, which leads to the numerical dispersion relationship:

\[ |e^{j\omega t} I - M_{12}^{-1} M_{21}^{-1} M_{11}^{-1} M_{11}| = 0 \]

After some manipulations, (15) can be simplified as

\[ \sin^2(\omega \Delta t) = \frac{4(A_1 + A_2 + B_1 + B_2)}{A^2} \]

(16) is the numerical dispersion relationship of the (2,4)-stencil based LOD-FDTD method. Now suppose that a plane wave propagates in the directions of \( \phi \) and \( \theta \) in the spherical coordinates and the numerical wave number is \( k_{num} \). Then,

\[ k_x = k_{num} \cos \phi \sin \theta \]
\[ k_y = k_{num} \sin \phi \sin \theta \]
\[ k_z = k_{num} \cos \theta \]

where \( k_x, k_y \) and \( k_z \) are the numerical spatial frequencies along the \( x, y \), and \( z \) directions, respectively. By substituting (17) into (16) with a known propagation direction defined by \( (\phi, \theta) \), one can solve for numerical wave number \( k_{num} \). By comparing \( k_{num} \) with the analytical wave number \( k_o \), numerical dispersion error can be found. Note that if \( \theta = 90^\circ \) is taken, the dispersion is reduced to the one for the two dimensional parameter-optimized LOD-FDTD.

IV. PARAMETER OPTIMIZATIONS

In this section, several optimization criteria are described and used to find parameters \( c_x, c_y \) and \( c_z \). Before the descriptions, several notations are introduced for clarity.

The CFL Number (CFLN)

CFLN is introduced as the ratio of the time step \( \Delta t \) to the CFL limit \( \Delta t_{CFL} \); i.e.:

\[ CFLN = \frac{\Delta t}{\Delta t_{CFL}} \]

Normalized numerical phase velocity \( u(\theta, \phi) \)

From electromagnetic theory, the numerical phase velocity in propagation angle \( (\theta, \phi) \) is equal to \( v_p = \frac{\omega}{k_{num}} \), where \( \omega = 2\pi f \) being the angular frequency and \( f \) being the frequency. As a result, the numerical phase velocity normalized to speed of light \( c \) in free space, can be expressed as

\[ u(\theta, \phi) = \frac{v_p}{c} = \frac{\omega}{ck_{num}} \]

Numerical phase velocity error

With the above definitions, the phase velocity error, denoted as \( e_{ph}(\theta, \phi) \) at a propagation angle \( (\theta, \phi) \) can be defined as:

\[ e_{ph}(\theta, \phi) = \left| \frac{v_p - c}{c} \right| \times 100\% \]
\[ = u(\theta, \phi) - 1 \times 100\% = |\frac{k_o - k_{num}}{k_o}| \times 100\% \]

Note that \( c = \frac{\omega}{k_o} \) is the speed of light with \( k_o \) being the analytical wave number.

The averaged phase velocity error over \( M \times N \) pre-selected angles \( (\theta_o, \phi_o) \) is then:

\[ e_{ave} = \frac{1}{MN} \sum_{n=1}^{M} \sum_{m=1}^{N} e_{ph}(\theta_o, \phi_o) \]

The averaged phase velocity function (20) can be used for forming different optimization objectives, such as minimization of dispersion errors in the axial directions, in the diagonal direction, and in a range of specified angles. They lead to different parameter values for minimization of the numerical dispersion errors as described below.

A. Minimization of Dispersion Errors in the Axial Directions

In some applications, minimization of the dispersion errors along three axial directions is desired. In other words, by using (20), the following objective function can be set up for the minimization:

\[ e_{ave} = \frac{1}{3} \left[ e_{ph}(\theta = 90^\circ, \phi = 0^\circ) + e_{ph}(\theta = 90^\circ, \phi = 90^\circ) + e_{ph}(\theta = 0^\circ, \phi) \right] \]
Suppose that \( CFLN = 2 \) and \( \Delta x = \Delta y = \Delta z = \frac{\lambda}{20} \) with \( \lambda \) being the wavelength associated with \( \omega \) (i.e. \( \lambda = \frac{c}{\omega} = \frac{2\pi c}{\omega} \)). By applying the well-known conjugate gradient technique to minimize (21), the following optimized parameters are obtained:

\[
c_x = c_y = c_z = 1.4684 \quad (22)
\]

Substitution of (22) into (16) finds the numerical wave number \( k_{\text{num}} \), which in turn presents the numerical phase velocity error as defined by (19).

Fig. 1 shows the phase velocity error with and without the above parameter optimization. Without the optimization, the maximum phase velocity error is about 1%. After the optimization, it is reduced to 0.1%, more than 10 times smaller.

The dispersion error curves of the conventional explicit FDTD based on the (2, 4) stencil with \( CFLN = 1 \) are also plotted in Fig. 1 for comparison. The parameter-optimized implicit LOD-FDTD method is shown to even have smaller dispersion errors than the conventional FDTD.

The minimization of the dispersion error can also be set for three sets of specified propagation angles. After the optimization, the phase velocity error is about 0.6%; with the optimization, it is reduced to about 0.01%, 60 times smaller. In other words, the error reduction is quite significant. The dispersion curves of the conventional (2, 4) FDTD with \( CFLN = 1 \) are also plotted in Fig. 2 for comparison. As can be seen, the LOD-FDTD with the parameter optimization has smaller errors that the conventional FDTD method around the intended diagonal directions.

### B. Minimization of Dispersion Error in the Diagonal Direction

The minimization of the dispersion error can also be set along the diagonal direction \( (\theta = 45^\circ, \phi = 45^\circ) \). That is, the objective function below is minimized again with the conjugate gradient technique:

\[
e_{\text{ave}} = e_{\phi \theta} (\theta = 45^\circ, \phi = 45^\circ) \quad (23)
\]

The minimization leads to

\[
c_x = 3.1789, \quad c_y = 3.0222, \quad \text{and} \quad c_z = 0.8118.
\]

The numerical dispersion errors with and without the parameter optimization are shown in Fig. 2. Without the parameter optimization, the phase velocity error is about 0.6%; with the optimization, it is reduced to about 0.01%, 60 times smaller. In other words, the error reduction is quite significant. The dispersion curves of the conventional (2, 4) FDTD with \( CFLN = 1 \) are also plotted in Fig. 2 for comparison. As can be seen, the LOD-FDTD with the parameter optimization has smaller errors that the conventional FDTD method around the intended diagonal directions.

### C. Minimization of Dispersion Errors in Specified Angles

The numerical phase velocity errors can also be minimized in other specified propagation angles. Table I shows the optimized parameters when the objective function (20) is minimized for three sets of specified propagation angles, respectively.

<table>
<thead>
<tr>
<th>Set #</th>
<th>( \theta_1, \phi_1 )</th>
<th>( \theta_2, \phi_2 )</th>
<th>( \theta_3, \phi_3 )</th>
<th>( c_x, c_y, c_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15(^\circ), 30(^\circ)</td>
<td>15(^\circ), 60(^\circ)</td>
<td>15(^\circ), 90(^\circ)</td>
<td>3.5870, 2.6994, 1.4736</td>
</tr>
<tr>
<td>2</td>
<td>30(^\circ), 22.5(^\circ)</td>
<td>30(^\circ), 45(^\circ)</td>
<td>30(^\circ), 67.5(^\circ)</td>
<td>1.4203, 1.4278, 1.5285</td>
</tr>
<tr>
<td>3</td>
<td>45(^\circ), 22.5(^\circ)</td>
<td>45(^\circ), 45(^\circ)</td>
<td>45(^\circ), 67.5(^\circ)</td>
<td>1.4530, 1.4648, 1.6499</td>
</tr>
</tbody>
</table>

Fig. 2. Phase velocity errors of the (2, 4) stencil LOD-FDTD with and without the parameter optimization along the diagonal direction. \( \theta = 45^\circ, CFLN = 2 \) and \( \Delta x = \Delta y = \Delta z = \frac{\lambda}{20} \).

Figs. 3, 4 and 5 show the phase velocity errors with and without the parameter optimizations at the three sets of propagation angles. After the optimization, the phase velocity errors of the (2, 4)-stencil LOD-FDTD are reduced to almost zeros, even smaller those of the conventional FDTD with \( CFLN = 1 \). This is an indication of the effectiveness of the parameter optimization.
D. Minimization of Dispersion Errors in an Array of Angles

To reduce the overall dispersion errors of the method within a specified sector of propagation angles, we can set up the objective function (20) to be minimized over a pre-selected array of angles. To illustrate the process, let’s take the array to be

\[(\theta_i, \phi_j) = (i \times 10^\circ, j \times 10^\circ), \quad i = 0...9, \quad j = 0...9\]  \tag{24}

After the minimization of (20) with the conjugate gradient technique, we obtain the optimized parameters as \(c_x = 1.5171\), \(c_y = 1.5172\), and \(c_z = 1.4883\) under the condition that \(\Delta x = \Delta y = \Delta z = \lambda/20\). The phase dispersion error with and without the optimization is shown in Fig. 6.

To illustrate the effects of different mesh sizes, let \(\Delta x\), \(\Delta y\), and \(\Delta z\) be \(\lambda/10\), \(\lambda/20\), and \(\lambda/30\), respectively, while the time step size retains the default value of \(CFLN = 2\). Again the objective functions (20) to be minimized over a pre-selected array determined by (24). Upon carrying out the optimization, we obtain the optimized parameter \(c_x = 1.2153\), \(c_y = 1.4596\), \(c_z = 1.8149\). Fig. 7 shows the phase dispersion error with and without the optimization.

From Figs. 6 and 7, again, we can see the parameter optimization significantly reduced the dispersion errors.
V. PARAMETER OPTIMIZATION VERSUS TIME STEP

One advantage of using the unconditionally stable LOD-FDTD method is that its time-step size is not constrained by the stability but numerical errors. Therefore, it is useful and meaningful to see whether the optimized parameter values are affected by different time steps, or CFLNs and what the implications are.

Again, minimization of the dispersion errors along the three axial directions was carried out. Table II shows the optimized parameters with different CFLNs, while Figs. 8 and 9 show the dispersion errors with the optimized parameters; for the comparison purpose, the dispersion errors of the conventional (2, 4) explicit FDTD is also plotted.

Table II Optimized parameters with different CFLNs

<table>
<thead>
<tr>
<th>CFLN</th>
<th>$c_x$, $c_y$, $c_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2095, 1.2095, 1.2095</td>
</tr>
<tr>
<td>2</td>
<td>1.4628, 1.4628, 1.4628</td>
</tr>
<tr>
<td>4</td>
<td>2.4984, 2.4984, 2.4984</td>
</tr>
</tbody>
</table>

As can be seen from Figs. 8 and 9, it is found that when the time step increases, the phase velocity error increase correspondingly. However, with the optimized parameters, the dispersion errors of the (2, 4) LOD-FDTD are reduced significantly to a level comparable to that of the conventional explicit FDTD but the LOD-FDTD is with CFLN=1. For instance, with CFLN=4, the maximum dispersion errors of the LOD-FDTD are reduced from 2.0% without the optimization to 0.01% with the optimized parameters of Table II; 0.01% is very close to the dispersion errors of the conventional (2,4) FDTD with CFLN=1.

To further illustrate the case with different time steps in an array of angles, let CFLN be 1, 2, and 4 respectively. In other words, (20) was minimized one more with different CFLNs respectively but the incident angles $(\theta, \phi)$ were selected to be an array of angles specified by (24). Table III presents the
optimized parameters with different CFLNs and array of specified angles.

Table III Optimized parameters with different CFLNs

<table>
<thead>
<tr>
<th>CFLN</th>
<th>(c_x, c_y, c_z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2217, 1.2217, 1.2129</td>
</tr>
<tr>
<td>2</td>
<td>1.5122, 1.5123, 1.4727</td>
</tr>
<tr>
<td>4</td>
<td>2.5977, 2.6685, 2.5194</td>
</tr>
</tbody>
</table>

Figs. 10 and 11 show the numerical phase velocity errors with and without the optimized parameters. For comparison purpose, the dispersion errors of the conventional (2, 4) FDTD are also plotted. Again, as can be seen, the dispersion errors are reduced significantly to the level comparable to those of the conventional FDTD. Without the optimization, the phase velocity error is about 1.2% and 1.7% for \(\theta = 45^\circ\) and 90° without CFLN=4. After the optimization, the phase velocity is about 0.12% and 0.14%, almost 10 times smaller.

Based on Fig. 8-11, we can conclude that by applying the parameter optimization, numerical dispersion errors of the LOD-FDTD can be reduced to the level of that of the conventional FDTD method but with larger time step (CFLN>1). In other words, the parameter-optimized LOD-FDTD may require less simulation time without sacrificing the accuracy.

VI. PARAMETER OPTIMIZATION VERSUS FREQUENCY

Numerical dispersion or phase velocity errors also change with the frequency. Normally, the errors increase with the increase of frequency (or decrease of wavelength \(\lambda\)). To study this effect, let’s consider \(\omega = \omega_0 = 10\pi \times 10^9\) rad/s with \(\Delta x = \Delta y = \Delta z = \delta = \lambda / 60\) and CFLN = 3. Minimization of (20) is done with different values of \(\omega\), and the objective function is selected as the same as (21) for minimization of the dispersion errors along the three axial directions. Table IV gives the optimized parameters obtained with the application of the conjugate gradient technique.

Table IV Optimized parameters with different values of \(\omega\)

<table>
<thead>
<tr>
<th>(\omega)</th>
<th>(c_x, c_y, c_z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1(\omega_0)</td>
<td>1.8749, 1.8756, 1.8756</td>
</tr>
<tr>
<td>1.0(\omega_0)</td>
<td>1.8787, 1.8787, 1.8787</td>
</tr>
<tr>
<td>1.5(\omega_0)</td>
<td>1.8833, 1.8833, 1.8833</td>
</tr>
<tr>
<td>3.0(\omega_0)</td>
<td>1.9089, 1.9089, 1.9089</td>
</tr>
</tbody>
</table>

Fig. 12 shows that the numerical phase velocity errors with and without the optimized parameters of Table IV and with different \(\omega\), respectively. As can be seen from Fig. 12, when the frequency increases, the phase velocity error increase correspondingly. However, with the parameter optimization, the impact of the frequency increase is substantially minimized.
\[ \frac{\gamma}{\Delta t} = \frac{1}{R} \] for comparison, numerical phase velocity errors are also plotted in Figs. 15 and 16.

\[ \lambda_{\text{LOD-FDTD}}(2,4) = 1.9563, 1.9671, 1.9206 \]

\[ \lambda_{\text{LOD-FDTD}}(2,4) = 1.9910, 1.9907, 1.8904 \]

To further illustrate the case with different frequency, the optimization is applied for an array of propagation angles with fixed parameters. Minimization of (20) was carried out over the pre-selected array of incident angles defined by (24). Table V presents the optimized parameters with different values of \( \omega \).

Table V Optimized Parameters with Different Values of \( \omega \)

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( c_x, c_y, c_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1( \omega_0 )</td>
<td>1.9879, 1.9933, 1.8975</td>
</tr>
<tr>
<td>1.0( \omega_0 )</td>
<td>1.9910, 1.9907, 1.8904</td>
</tr>
<tr>
<td>1.5( \omega_0 )</td>
<td>1.9952, 1.9952, 1.9118</td>
</tr>
<tr>
<td>3.0( \omega_0 )</td>
<td>1.9563, 1.9671, 1.9206</td>
</tr>
</tbody>
</table>

As can be seen from Figs. 13 and 14, when the frequency increases, the phase velocity error increase correspondingly. However, the parameter optimization minimizes the impact of the frequency increase. For instance, without the optimization, the phase velocity error is about 1.2% and 2.1% for \( \theta = 45^\circ \) and \( 90^\circ \) at 3.0\( \omega_0 \). After the optimization, the average phase velocity is about 0.2% and 0.13%, which are the error level of the conventional FDTD. Therefore, numerical dispersion error is reduced significantly.

To assess the sensitivity of the optimized parameters to frequency, numerical dispersions were computed with varying frequencies but fixed parameters. Figs. 15 and 16 show the numerical phase velocity errors with different frequencies with \( c_x = c_y = c_z = 1.8787 \) which were obtained after minimization of numerical phase velocity errors along the three axial three directions at \( \omega_0 \). For comparison, numerical phase velocity errors of the conventional FDTD with \( CFLN=1 \) and \( \omega = \omega_0 \) are also plotted in Figs. 15 and 16.

As can be seen from Figs. 15 and 16, the parameter optimization is not sensitive to frequency variations. This is reflected by Table IV where the optimized parameters at different frequencies are little different from each other. Therefore, the parameters obtained at a frequency are also good at other frequencies. In particular, the parameters obtained at a higher frequency even present lower phase velocity errors at lower frequencies. In other words, the parameter optimization is preferred to be carried out at a higher frequency than at a lower frequency. In addition, like what was discussed previously, the parameter optimization does bring the numerical phase velocity errors down to the level of the conventional FDTD.
VII. Conclusion

This paper has presented a parameter optimized LOD-FDTD method based on the (2, 4) stencil to achieve better dispersion performance. The method has been proved to be unconditionally stable for any real parameters. By setting different optimization objectives, the method proposed here can satisfy different accuracy requirements, such as minimum dispersion error in the axial directions, minimum dispersion error in the diagonal direction, and minimum dispersion error for a few arbitrary angles. The variation of the performance of the (2, 4) parameter-optimized LOD-FDTD method with different frequencies and time steps has also been studied. Our results show that the parameter optimization proposed in this paper can dramatically bring the dispersion errors down to the level of the conventional (2, 4) stencil FDTD but with larger time steps and without introducing additional computational cost. In addition, the optimized parameters are not very sensitive to the frequencies. The optimization is then recommended to be carried out at a high frequency point.

REFERENCES


