Character Analysis for FDTD Based on Frequency Response

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Abstract—A character analysis method for finite difference time domain (FDTD) based on frequency response is presented. By this method, the process of the wave’s traveling from spatial point A to point B can be treated as signal’ passing through a system while the input is the signal on A and the output is the one on B. So the character of FDTD can be obtained by the analysis of the system. In this letter, we deduce the frequency response of the system with respect to one dimensional scalar wave in free space, discuss the distortion of magnitude and phase, build the relation between frequency and attenuation and phase velocity, and reveals the effect of FDTD algorithm on signal and give the criterion to choose appropriate signal for simulation to avoid excessive distortion. At last, the simulation validates the theory analysis.

Index Terms—FDTD, frequency response, character analysis.

I. INTRODUCTION

The finite difference time domain (FDTD) method is an important computational electrodynamics modeling technique. It approximates continuous wave propagation with numerical solution, so the stability and error of it must be studied to ensure the feasibility and efficiency. Dispersion derived from error has also drawn attention [1], [2]. Taflove studied to ensure the feasibility and efficiency. Dispersion in detail [3], and they mainly paid attention to the case of arbitrary value when necessary. From (1), each item of $u_i$ is a polynomial in $K$ of degree $n$ and $u_i$ is obtained by linear algebra operation on signal at adjacent spatial points. If the signal at $x_i$ travels to $x_{i+1}$, then $u_{i+1}$ can be treated as the output of a system while $u_i$ is the input, namely

$$u_{i+1} = u_i * h(n)$$ (2)

where $h(n)$ is impulse response of the system, $*$ denotes convolution. In order to obtain $h(n)$, we set a unit sample signal at point $x_0$, so $u_0 = \delta(n)$. Then we have

$$u_1 = u_0 * h(n) = h(n)$$ (3)

From (3), $h(n)$ can be calculated from $u_1$. The resolving of $u_1^n$ is the same as the problem that how many plane binary trees have $n + 1$ endpoints in combinatorics [4], so

$$u_1^n = -\sum_{m=1}^{n} D_m a_m^n (-K)^m \quad n = 1, 2, 3, \ldots$$ (4)

where $a_m^n = \left(\begin{smallmatrix} n+m-2 \\ 2m-1 \end{smallmatrix}\right)$, $D_m = \frac{1}{m+1}\left(\begin{smallmatrix} 2m \\ m \end{smallmatrix}\right)$, $D_m$ is Catalan number, and $u_0^1 = 0$. Then the impulse response

$$h(n) = u_1^1 = [u_1^1 u_1^2 u_1^3 \ldots]$$ (5)

III. FREQUENCY RESPONSE

It is difficult to obtain the frequency response of the system from (5). We adopt an indirect way. First, we deduce the $z$-transform of $h(n)$, namely the system function

$$H(z) = -\sum_{n=1}^{\infty} \sum_{m=1}^{n} D_m a_m^n (-K)^m z^{-n}$$

$$= -\sum_{m=1}^{\infty} D_m (-K)^m \sum_{n=m}^{\infty} a_m^n z^{-n}$$

$$= -\sum_{m=1}^{\infty} D_m \left[\frac{-zK}{(z-1)^2}\right]^m$$ (6)

According to generating function theory [5],

$$H(z) = 1 + \left(\frac{z-1}{z+1}\right)^2 \sqrt{1 + \frac{4zK}{(z-1)^2}} $$ (7)
Then, substitute $e^{j\omega}$ for $z$. We only consider the the case of $\omega \geq 0$ for convenience, so the frequency response
\[
H(e^{j\omega}) = 1 + \frac{(e^{j\omega} - 1)^2}{2e^{j\omega}K} - \frac{(e^{j\omega} - 1)\sqrt{(e^{j\omega} - 1)^2 + 4e^{j\omega}K}}{2e^{j\omega}K}
\]
\[= 1 + \frac{1}{2K}(e^{j\omega} - 2 + e^{-j\omega}) - \frac{1}{2K}(e^{j\omega} - e^{-j\omega})\sqrt{e^{j\omega} - 2 + e^{-j\omega} + 4K} = 1 - \frac{2}{K}\sin^2\frac{\omega}{2} - j\frac{2}{K}\sin\frac{\omega}{2}\sqrt{K - \sin^2\frac{\omega}{2}}
\]
(8)

According to the stability condition [3], $K \leq 1$. If the sampling period is $\Delta t$, the period of frequency spectrum will be $2\pi/\Delta t$, so we only discuss in one period in frequency domain. If $\Delta t = 1$, the period will be $[0, \pi]$.

A. $K = 1$

This is the case of magic time-step [3], and we have
\[
H(e^{j\omega}) = e^{-j\omega}
\]
(9)
From (9), it’s just a unit time-delay. If the sampling period is $\Delta t$, then the time-delay will be $\Delta t$. So the signal has no distortion.

B. $K < 1$

From (8), $(K - \sin^2\frac{\omega}{2})$ can be negative or positive, so we call $\omega_B = 2\arcsin\sqrt{K}$ boundary radian frequency.

1) $\omega \leq \omega_B$:

We have the square of magnitude response
\[
|H(e^{j\omega})|^2 = (1 - \frac{2}{K}\sin^2\frac{\omega}{2})^2 + (\frac{2}{K}\sin\frac{\omega}{2}\sqrt{K - \sin^2\frac{\omega}{2}})^2 = 1
\]
and phase response
\[
\angle H(e^{j\omega}) = -\arccos(1 - \frac{2}{K}\sin^2\frac{\omega}{2})
\]
(11)
\[
2) \omega_B < \omega \leq \pi:

\[
H(e^{j\omega}) = 1 - \frac{2}{K}\sin^2\frac{\omega}{2} - j\frac{2}{K}\sin\frac{\omega}{2}(\sqrt{\sin^2\frac{\omega}{2} - K})
\]
\[= 1 - \frac{2}{K}\sin^2\frac{\omega}{2} + \frac{2}{K}\sin\frac{\omega}{2}\sqrt{\sin^2\frac{\omega}{2} - K}
\]
(12)
When $\omega > \omega_B$,
\[
1 - \frac{2}{K}\sin^2\frac{\omega}{2} < 0
\]
(13)
At the same time,
\[
(1 - \frac{2}{K}\sin^2\frac{\omega}{2} - (\frac{2}{K}\sin\frac{\omega}{2}\sqrt{\sin^2\frac{\omega}{2} - K})^2 = 1 > 0
\]
(14)
From (13) and (14), $H(e^{j\omega})$ is negative, so
\[
|H(e^{j\omega})| = -1 + \frac{2}{K}\sin^2\frac{\omega}{2} - \frac{2}{K}\sin\frac{\omega}{2}\sqrt{\sin^2\frac{\omega}{2} - K}
\]
(15)
and phase response
\[
\angle H(e^{j\omega}) = -\pi
\]
(16)
From (13), $1 - \frac{2}{K}\sin^2\frac{\omega}{2} < 0$ and
\[
\min(1 - \frac{2}{K}\sin^2\frac{\omega}{2}) > -1
\]
(17)
At the same time,
\[
\frac{2}{K}\sin\frac{\omega}{2}\sqrt{\sin^2\frac{\omega}{2} - K} \geq 0
\]
(18)
so
\[
|H(e^{j\omega})| < 1
\]
(19)

IV. DISTORTION ANALYSIS

The distortion we discuss here refers to the one caused by FDTD algorithm. Distortion consists of amplitude distortion and phase distortion. And these two kinds of distortion could introduce attenuation and variation of phase velocity.

A. Amplitude Distortion

Through the analysis above, the system is actually low-pass filter. For the signal with $\omega < \omega_B$, the system has unit gain and nonlinear phase. But for the one with $\omega > \omega_B$, the system is the same as an attenuator. From (12), the attenuation will be greater with the increase of radian frequency. When traveling from one spatial point to another, the component of the signal with $\omega > \omega_B$ will attenuate quickly for multi-multiplication with $H(e^{j\omega})$ in frequency domain. So for the signals whose bandwidth could exceed boundary radian frequency, such as impulse signal and step signal, the distortion of waveform will be serious after travel a distance. When the sampling period is $\Delta t$, the boundary radian frequency
\[
\omega_B = \frac{2\arcsin\sqrt{K}}{\Delta t}
\]
(20)
Hence, the boundary radian frequency will give the reference for choosing simulation signal.

B. Phase Distortion

From (11) and (16), the phase of the system is nonlinear. Now we consider the variation of phase velocity introduced by this nonlinearity. Let $\Delta t_\alpha = \frac{\angle H(e^{j\omega})}{\omega}$ be the time shift of the signal with sampling period $\Delta t$, $\omega$, then
\[
\Delta t_\alpha = \begin{cases} 
-\arccos(1 - \frac{2}{K}\sin^2\frac{\Delta t}{2}) : \omega \in (0, \frac{\omega_B}{\Delta t}] \\
-\frac{\pi}{\omega} : \omega \in (\frac{\omega_B}{\Delta t}, \frac{\pi}{\Delta t})
\end{cases}
\]
(21)
And
\[
\frac{d\Delta t_\alpha}{d\omega} = \begin{cases} 
-\frac{\Delta t \cos \frac{\Delta t}{2}}{\sqrt{K - \sin^2 \frac{\Delta t}{2}}} : \omega \in (0, \frac{\omega_B}{\Delta t}] \\
-\frac{\arccos(1 - \frac{2}{K}\sin^2\frac{\Delta t}{2}) + \frac{\pi}{\omega^2} : \omega \in (\frac{\omega_B}{\Delta t}, \frac{\pi}{\Delta t})}
\end{cases}
\]
(22)
From (22), when $\omega \in (0, \frac{\omega_B}{\Delta t}]$, $\frac{d\Delta t_\omega}{d\omega} < 0$. So $\Delta t_\omega$ is a decreasing function. When $\omega = \frac{\omega_B}{\Delta t}$

$$\Delta t_\omega = -\frac{\pi \Delta t}{2 \arcsin \sqrt{K}} \quad (23)$$

When $\omega \in \left(\frac{\omega_B}{\Delta t} \cdot \frac{\pi}{\Delta t}\right)$, $\frac{d\Delta t_\omega}{d\omega} > 0$. So $\Delta t_\omega$ is a increasing function.

The phase velocity

$$v_p = \frac{\Delta x}{\Delta t_\omega} = \begin{cases} \frac{\omega \Delta t}{\arccos(1 - \frac{K}{2} \sin^2 \frac{\omega \Delta t}{2}) \sqrt{K}} & \omega \in (0, \frac{\omega_B}{\Delta t}] \\ \frac{\omega \Delta t}{\pi \sqrt{K}} & \omega \in \left(\frac{\omega_B}{\Delta t} \cdot \frac{\pi}{\Delta t}\right) \end{cases} \quad (24)$$

From (24), the phase velocity $v_p$ equals to the product of a coefficient and speed of light $c$ and this coefficient is function of $\omega$. Because

$$\lim_{\omega \to 0} v_p = \sqrt{K - \sin^2 \frac{\omega \Delta t}{2}} \frac{\Delta t}{\Delta t \cos \frac{\pi}{\Delta t}} c = c \quad (25)$$

and $v_p$ is decreasing function when $\omega \in (0, \frac{\omega_B}{\Delta t}]$, so $v_p$ will less than $c$ in this interval. And when $\omega = \frac{\pi}{\Delta t}$, $v_p$ reaches the minimum, namely

$$\min v_p = \frac{2 \arcsin \sqrt{K}}{\pi \sqrt{K}} c \quad (26)$$

While $\omega \in \left(\frac{\omega_B}{\Delta t} \cdot \frac{\pi}{\Delta t}\right)$, $v_p$ is increasing function. And when $\omega = \frac{\pi}{\Delta t}$, $v_p$ reaches the maximum, namely

$$\max v_p = \frac{1}{\sqrt{K}} c = \frac{\Delta x}{\Delta t} \quad (27)$$

From (27), if $K < 1$, $v_p$ will exceed $c$. This is introduced by FDTD algorithm. From above analysis, the $\omega$ that make $v_p = c$ must locate in $\omega \in \left(\frac{\omega_B}{\Delta t} \cdot \frac{\pi}{\Delta t}\right)$, so we have

$$\omega|_{v_p = c} = \frac{\pi \sqrt{K}}{\Delta t} = \frac{\pi c}{\Delta x} \quad (28)$$

V. Simulation

Let $u_0 = \delta(n)$, and calculate $u_1$ by (1). Because $h(n) = u_1$, so the Fourier transform of $u_1$ gives the impulse response. Fig. 1 shows the Fourier transform of $u_1$ with $K = 0.4$ and $K = 0.8$ respectively. We can see that the simulation validates the theory analysis above. With the decrease of $K$, the boundary radian frequency $\omega_B$ is much less and attenuation is much greater.

VI. Conclusion

In this letter, we present a character analysis method for FDTD based on frequency response. Other than traditional methods, we map the FDTD algorithm into a system function. So the character of FDTD algorithm can be obtained by studying the system. This will be significant when considering the frequency of the simulation signal. The frequency response of the system is deduced, and the attenuation and variation of phase velocity are analyzed from the perspective of amplitude and phase distortion. This method can help to choose appropriate signal for simulation and evaluate the performance of some FDTD algorithm. The simulation validates the theory analysis at last. Although we discuss the case of one-dimensional scalar wave in free-space, the method can be generalized to higher dimension wave propagating in dielectric medium.

REFERENCES