Math 42142 Analysis, Random Walks and Groups Spring 2020

Week 4 Tutorial Sheet

1. Fix $p \geq 7$ and $0 < \alpha < 1$. Define $\mu = \alpha\delta_1 + (1-\alpha)\delta_{-1}$. Compute $\mu \star \mu$ and $\mu \star \mu \star \mu$.

2. Given functions $f, g : \mathbb{Z}_p \to \mathbb{C}$ define

$$(f \star g)(t) = \sum_{s=0}^{p-1} f(t \ominus s) g(s)$$

for all $t \in \mathbb{Z}_p$. Prove the following identities for all $f, g, h : \mathbb{Z}_p \to \mathbb{C}$ and all $\alpha, \beta \in \mathbb{C}$.

- *Commutativity*: $f \star g = g \star f$
- *Associativity*: $(f \star g) \star h = f \star (g \star h)$
- *Linearity*: $f \star (\alpha g + \beta h) = \alpha f \star g + \beta f \star h$

3. Fix distributions μ, ν on $\mathbb{Z}_p$. Prove that $\mu \star \nu$ is a distribution.

4. a) Prove that $\mathsf{max}\{|A|, |B|\} \leq |A \oplus B| \leq |A||B|$ for all sets $A, B \subset \mathbb{Z}_p$.

   b) Given examples of sets $A, B \subset \mathbb{Z}_p$ with $|A \oplus B| = \mathsf{max}\{|A|, |B|\}$.

   c) Given examples of sets $A, B \subset \mathbb{Z}_p$ with $|A \oplus B| = |A||B|$.

5. Prove for all distributions μ, ν that $\mathsf{max}\{\mathsf{H}(\mu), \mathsf{H}(\nu)\} \leq \mathsf{H}(\mu \star \nu) \leq \mathsf{H}(\mu) + \mathsf{H}(\nu)$.

   Hint: For one direction use the fact that $\phi(x) = -x \log x$ is concave. For the other use the inequality

$$\phi\left(\sum_j x_j\right) \leq \sum_j \phi(x_j)$$

   which holds for all $x_i \geq 0$.

6. For which values of $0 < \alpha < 1$ is the probability distribution $\mu = \alpha\delta_0 + (1-\alpha)\delta_1$ on $\mathbb{Z}_5$ ergodic? Compute $\mathsf{d}(\mu_\alpha \star \mu_\alpha, \lambda)$.

7. Fix a probability distribution μ on $\mathbb{Z}_4$ that is not a Dirac distribution. Suppose that the support $\mathsf{spt}(\mu)$ is a coset of a proper non-trivial subgroup of $\mathbb{Z}_4$. Might the limit $\lim_{n\to\infty} \mu^{\star n}$ exist? What could it be?