Math 42142 Analysis, Random Walks and Groups Spring 2020

Week 2 Tutorial Sheet Solutions

1. In the **weak Borel shuffle** one lifts the top card from a deck of 52 cards and inserts it into the deck in a random position.

   a) For $0 \leq j \leq 51$ determine the permutation σ_j corresponding to the outcome of placing the top card in the jth position amongst the remaining cards. (The 0th position is on top of the remaining cards and the 51st position is on the bottom of the remaining cards.)

   Solution. Fix $0 \leq j \leq 51$. The permutation will not alter the cards in positions $j+1, \ldots, 51$. The card from position 0 will go to position j. The cards in positions $1, \ldots, j$ will go to positions $0, \ldots, j-1$. Thus

   $$\sigma_j(n) = \begin{cases} j & n = 0 \\ n-1 & 1 \leq n \leq j \\ n & j+1 \leq n \leq 51 \end{cases}$$

   is correct. □

   b) Supposing that one performs consecutive weak Borel shuffles with $0 \leq j \leq 51$ chosen uniformly at random (e.g. by rolling a 52-sided die each time) what is the probability that the card on the top of the deck before shuffling is on top of the deck after shuffling?

   Solution. If the card initially on top is to be on top after two weak Borel shuffles, it must be in either the top position or the second position after the first weak Borel shuffle. Thus we are only interested in the outcomes where the first weak Borel shuffle is σ_0 or σ_1.

   If the first is σ_0 then the second must be σ_0 as well.

   If the first is σ_1 then the second must not *be σ_0.*

   We are interested therefore in calculating the probability of the event

   $$\{\sigma_0 \circ \sigma_0, \sigma_1 \circ \sigma_1, \sigma_2 \circ \sigma_1, \ldots, \sigma_{51} \circ \sigma_1\}$$

   where the first and second permutations are chosen uniformly at random. This event consists of 52 of the 52^2 possible outcomes, so its probability is $1/52$. □

2. The group $\mathbb{Z}_p$ is the set $\{0, \ldots, p-1\}$ equipped with the binary operation $\oplus$ of addition modulo p.

   a) Calculate $3 \oplus 4$ in $\mathbb{Z}_5$. Calculate $(-2) \oplus 4$ in $\mathbb{Z}_3$. Calculate $34 \oplus 55$ in $\mathbb{Z}_{50}$.

Solution.

$3 \oplus 4 = 7 \bmod 5 = 2$

$(-2) \oplus 4 = 2 \bmod 3 = 2$

$34 \oplus 55 = 34 \oplus 5 = 39$ $\square$

b) Determine all the subgroups of $\mathbb{Z}_5$ and of $\mathbb{Z}_{12}$.

Solution. The subgroups of $\mathbb{Z}_p$ are precisely $\{0\}$, $\mathbb{Z}_p$ and the subgroups generated by the proper divisors of p.

Since 5 is prime it has no proper divisors and its only subgroups are $\{0\}$ and $\mathbb{Z}_5$.

For $\mathbb{Z}_{12}$ the proper divisors of 12 are $2, 3, 4, 6$ giving the subgroups

$$\{0, 2, 4, 6, 8, 10\} \qquad \{0, 3, 6, 9\} \qquad \{0, 4, 8\} \qquad \{0, 6\}$$

respectively, together with $\{0\}$ and $\mathbb{Z}_{12}$. $\square$

c) Which elements of $\mathbb{Z}_{10}$ generate the whole group?

Solution. If x and 10 are coprime then x generates all of $\mathbb{Z}_{10}$. Therefore $1, 3, 7, 9$ each generate all of $\mathbb{Z}_{10}$. $\square$

3. Some counting.

a) How many ways are there of arranging 12 red coins and 8 blue coins in a row?

Solution. Since we cannot distinguish one blue coin from another, each arrangement is determined by the locations $1 \leq j \leq 20$ of the blue coins amongst all the coins. There are $\binom{20}{8}$ ways to choose eight locations for the blue coins from the total number of twenty locations.

By the same argument, there are $\binom{20}{12}$ ways to choose twelve locations for the red coins from the total number of twenty locations. Fortunately $\binom{20}{8} = \binom{20}{12}$.

There are $125,970$ different arrangements. $\square$

b) How many ways are there of arranging 12 red coins, 8 blue coins and 5 green coins in a row?

Solution. The blue and green coins together will take up 13 of the 25 total locations. Of those 13 locations 5 will be taken up by green coins. In total there will be

$$\binom{25}{13}\binom{13}{5} = 5,200,300 \times 1,287 = 6,692,786,100$$

different arrangements. $\square$

c) $(\star)$ How many arrangements of 12 red coins and 8 blue coins in a row are unchanged by reversing the order?

Solution. *If an arrangement is unchanged by reversing the order then there must be* 6 *red coins and* 4 *blue coins in the first* 10 *coins of the arrangement. Moreover, the arrangement of* 6 *red and* 4 *blue coins amongst the first* 10 *coins determines the arrangement if it is to be symmetric. There are therefore totally* $\binom{10}{6}$ *symmetric arrangements.* □

d) ($\star$) How many ways are there of arranging 12 red coins and 8 blue coins in a row if order-reversals of arrangements are not distinguished?

Solution. *If order reversals of arrangements are not distinguished then every arrangement that is not symmetric is not distinguished from its reflection. This gives*

$$\frac{\binom{20}{12} - \binom{10}{6}}{2} + \binom{10}{6} = 63,090$$

arrangements if reflections are not distinguished. □

e) ($\star$) How many ways are there of arranging 13 red and 10 blue coins in a circle if rotations are not distinguished?

Solution. *Since 23 is prime and no circular arrangement is fixed by rotating* $360/23°$ *the answer is* $\binom{23}{10}/23 = 49,742$. □