David Silvester, 24 August 2020

Notes for the video tutorial on
PRESSURE-ROBUST APPROXIMATION

18.1 The case for using pressure-robust approximation in incompressible flow modelling is cogently made by John et al. in the recent review [john17]. The potential flow test problem was introduced by Linke & Merdon in [linke16a] and [linke16b].

18.2 Two-dimensional channel flow "over" a backward-facing step has been used as a test flow problem since the early days of computational fluid dynamics; see, for example, Biswas et al. [biswas04]. One reason for this is that comparison with laboratory experiments is relatively straightforward, as discussed by Armaly et al. [armaly83]. The solution to the flow problem at moderate Reynolds number was the subject of minor controversy in the early 1990s, see Gresho et al. [gresho93]. The problem is one of two two-dimensional test cases discussed by Farrell et al. [farrell20], where a detailed comparison of TH and SV approximation strategies is presented.

18.3 The numerical results shown in the video were computed using a refined version of the T-IFISS* toolbox [tifiss]. Sample output for the two test problems can be found at
http://www.manchester.ac.uk/ifiss/referenceresults1.18.txt.

18.4 A standard way of representing a pointwise divergence-free velocity field is to combine a $H(div)$-conforming velocity field, in which the normal component of the velocity is continuous across edges/faces, together with a discontinuous pressure field from a lower-order approximation space. An efficient strategy for constructing such approximations spaces has recently been developed by Rhebergen & Wells [rhebergen18] and involves hybridised discontinuous Galerkin (HDG) discretisation. The resulting bespoke $\mathbb{P}_2$–$\mathbb{P}_{-1}$ approximation strategy is stable and pressure-robust in two and three dimensions. The basic HDG components are included within the NGSolve finite element library developed by Schöberl and others at https://ngsolve.org.

18.5 The Scott–Vogelius approximation is included as an option in a number of contemporary research codes, see for example https://github.com/florianwechsung/alfi/.

References

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* A beta release (version 2.0) of the software package is available on request for anyone wanting to reproduce the results.

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