MATH20401 : Part I
Introduction to Partial Differential Equations

David J. Silvester
School of Mathematics, University of Manchester
d.silvester@manchester.ac.uk.

updated 6 October 2008

Contents

1. Preliminaries

Partial Differential Equations (PDEs) underpin all of applied mathematics and enable us to model practical problems like forecasting the weather, designing efficient aeroplanes and predicting the future price of financial stocks and shares.

There are two very different questions that will consider in these lecture notes. The first of these is obvious: can we find a solution to a given PDE problem—either analytically (that is writing down an explicit formula for the solution in terms of known quantities like position and time) or else numerically? (Using a computer to approximate the continuous solution in a discrete sense.) It turns out that analytic solutions can be obtained in special cases, whereas computers enable the possibility of calculating a numerical solution in any case where the solution make sense from a practical perspective.[1] Our second question is more magical: can we infer generic properties of a solution *without* actually solving the PDE problem? We will show that this is indeed possible: the idea is to characterise PDEs into different types so that the solutions have the same properties. A suitable classification is the topic of this first section.

We will need to establish some notation to begin with. Consider a function, say $u(t, x, y, z, \ldots)$ of several independent variables, representing the temperature of the air at a given point in space and at a given point in time. We define to the *partial derivative*, say

$$\frac{\partial u}{\partial x}, \text{ or } u_x,$$

to be the rate at which u changes when the horizontal position varies, with all the other independent variables held fixed. Other first order derivatives, u_t, u_y, u_z, etc., are defined analogously. Second derivatives are written in the form

$$\frac{\partial^2 u}{\partial x \partial t}, \text{ or } u_{xt},$$

and represent rates of change of first derivatives. Note that in practical applications, u will usually be continuously differentiable so the order of differentiation can be commuted, thus we have that $u_{xt} = u_{tx}$.

We now define a PDE having solution u to be some given relationship of the form

$$F(u, t, x, y, \ldots, u_t, u_x, u_y, \ldots, u_{tt}, u_{tx}, u_{ty}, \ldots) = 0. \tag{1.1}$$

The *order* of the PDE is the highest degree of differentiation that appears in the expression (1.1).

[1] Some PDE problems are taxing in that there may be no solutions. It is also common for there to be infinitely many solutions!

Second order PDEs are very important—they frequently arise in modelling physical phenomena due to Newton's second law of motion. Examples include Laplace's equation, the heat equation, the wave equation, and the Black-Scholes equation. We will focus on these equations later. More complicated physical models involve "systems" of coupled PDEs. Examples include Maxwell's equations governing electromagnetism, the Navier-Stokes equations governing incompressible fluid flow, and Einstein's equations which model the evolution of the universe and the formation of black holes.

1.1. Boundary and initial data

Suppose that $u(t, x)$ satisfies the very simple PDE

$$\frac{\partial u}{\partial t} = 0. \tag{1.2}$$

Integrating once with respect to t gives the general solution

$$u(t, x) = A(x), \tag{1.3}$$

where A is *any* function of x. We will refer to A as a "function of integration" (as opposed to an ODE where we have a "constant" of integration). Now, if in addition to (1.2) we were also given some *initial condition*, say

$$u(0, x) = x^2, \tag{1.4}$$

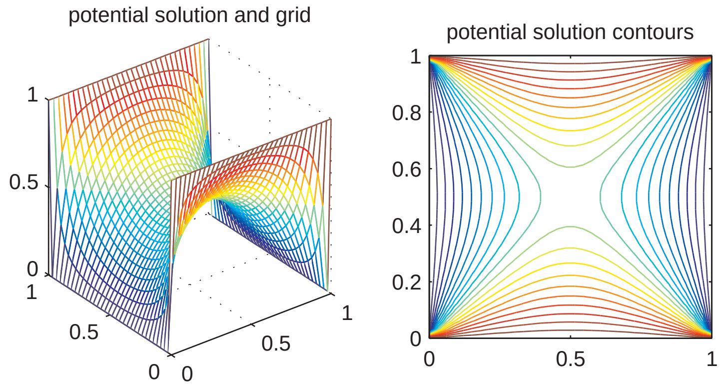
then putting $t = 0$ in (1.3) we see that $A(x) = x^2$ so that the solution is uniquely defined for all time. The combination (1.2)–(1.4) will be collectively referred to as an *initial value problem*.

More generally, if the PDE of interest is defined in time and space so that $(x, y) \in \Omega$ with Ω a bounded region of $\mathbb{R}^2$ (or perhaps $(x, y, z) \in \Omega$ with $\Omega \subset \mathbb{R}^3$) then appropriate conditions also need to be specified on the spatial boundary (denoted here by $\partial\Omega$) if we are to find a unique solution. Typical boundary conditions would involve specifying the PDE solution u on the boundary (this is called Dirichlet data), or else specifying the normal derivative $\frac{\partial u}{\partial n}$ on the boundary (called Neumann data). Such a combination of a PDE and initial condition together with a suitable boundary condition will be referred to here as a *boundary value problem*. (It is also called an *initial-boundary value problem* in some textbooks.)

As an example, a solution $u(x, y)$ to Laplace's equation

$$u_{xx} + u_{yy} = 0, \tag{1.5}$$

defined over the unit square $(x, y) \in (0, 1) \times (0, 1)$ together with specified Dirichlet data: $u = 1$ on the top and bottom edges and $u = 0$ on the left



and right edges is shown in the above figure.[2] Note that whenever the PDE problem is defined in $\mathbb{R}^2$ it turns out that $\partial\Omega$ is essentially in $\mathbb{R}^1$ (the union of straight lines in this example). Also, when solving PDE problems in $\mathbb{R}^3$ the boundary is essentially in $\mathbb{R}^2$ and we have one normal derivative and two tangential derivatives at every boundary point.

We conclude this section with a really important definition.

Definition x.1 (Well Posed Problem)
A boundary value problem which has a *unique* solution that varies *continuously* with the initial and boundary data is said to be *well posed*.

Not all boundary value problems are well posed. A good example is the "backward" heat equation for $u(t, x)$

$$u_t + u_{xx} = 0, \tag{1.6}$$

for all $x \in \mathbb{R}$, together with the initial data $u(0, x) = 0$. This problem has the unique solution, $u(t, x) = 0$.

However, if we make *tiny* changes in the initial data to

$$u(0, x) = 10^{-99}\cos(10^{99}x)$$

say, then the unique solution changes to

$$u(t, x) = 10^{-99}e^{10^{198}t}\cos(10^{99}x).$$

[2]The solution u is computed using the MATLAB function `laplacefd`. The precise calling sequence is `[u,A] = laplacefd(32,1,1,0,0);`.

In this case the solution becomes enormous even for very small values of t. For example, if $t = 10^{-195}$ then $\max|u| = 10^{-99}e^{1000} > 10^{200}$. This happens in spite of the fact that the size of the change in initial data would probably be subatomic in any practical example—it reflects the fact that "anti-diffusive" behaviour violates the second law of thermodynamics and is something like having time running backwards!

1.2. Linear boundary value problems

A very important class of boundary value problems are those that are linear. To test linearity we need to express the PDE (1.1) and any associated boundary conditions in the form

$$\mathcal{L}(u) = f \tag{1.7}$$

where $\mathcal{L}$ is a differential operator, $u(t, \vec{x})$ is the solution with $\vec{x} \in \mathbb{R}^d$ (typically the spatial dimension $d = 1, 2,$ or 3) and $f(t, \vec{x})$, the "right hand side", does not depend on u or on partial derivatives of u.

Definition x.2 (Linearity)
The operator $\mathcal{L}$ is *linear* if for any two functions u and v and any $\alpha \in \mathbb{R}$ we have the following two properties satisfied:

❶ $\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v)$;

❷ $\mathcal{L}(\alpha u) = \alpha\mathcal{L}(u)$.

Note that conventional Dirichlet/Neumann boundary conditions are always linear. Thus linearity of the PDE component of the operator is sufficient for a linear problem. Conversely, if the differential operator does not satisfy ❶ or ❷ then the boundary value problem is said to be *nonlinear*.

We give some examples below.

Example x.2.1 The heat equation with conductivity $\kappa > 0$ is linear

$$u_t - \kappa u_{xx} = f. \qquad \heartsuit \tag{1.8}$$

Example x.2.2 The Poisson equation is linear (so Laplace's equation is too)

$$-(u_{xx} + u_{yy}) = f. \qquad \heartsuit \tag{1.9}$$

Example x.2.3 The wave equation with wave speed c is linear

$$u_{tt} - c^2 u_{xx} = f. \qquad \heartsuit \tag{1.10}$$

Example x.2.4 The steady-state convection-diffusion equation with viscosity $\nu > 0$ and horizontal "wind" w is linear

$$-\nu(u_{xx} + u_{yy}) + wu_x = f. \qquad \heartsuit \tag{1.11}$$

Example x.2.5 The Black-Scholes equation with stock price x, interest rate r and volatility σ is linear

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0. \qquad \heartsuit \tag{1.12}$$

Example x.2.6 The inviscid Burgers' equation is nonlinear

$$u_t + uu_x = 0. \qquad \clubsuit \tag{1.13}$$

Example x.2.7 The Korteweg-de Vries (KdV) equation is nonlinear

$$u_t + 6uu_x + u_{xxx} = 0. \qquad \clubsuit \tag{1.14}$$

If $\mathcal{L}$ is a linear operator, then the boundary value problem $\mathcal{L}(u) = 0$, that is, with the right hand side set to zero, is referred to as a *homogeneous* problem This is important because of the "principle of superposition", which is summarized by the following two theorems.

Theorem 1.1 *If u_1 and u_2 are any two solutions of a homogeneous boundary value problem, then any linear combination $v = \alpha u_1 + \beta u_2$ with $\alpha, \beta \in \mathbb{R}$ is also a solution.*

Proof.

$$\mathcal{L}(v) = \mathcal{L}(\alpha u_1 + \beta u_2) = \alpha \underbrace{\mathcal{L}(u_1)}_{0} + \beta \underbrace{\mathcal{L}(u_2)}_{0} = 0.$$

□

Theorem 1.2 *Suppose that u_* is a "particular" solution of the linear boundary value problem $\mathcal{L}u = f$, and that v is a solution of the associated homogeneous problem, then $w = u_* + v$ is also a solution of the boundary value problem $\mathcal{L}u = f$.*

Proof.

$$\mathcal{L}(w) = \mathcal{L}(u_* + v) = \underbrace{\mathcal{L}(u_*)}_{f} + \underbrace{\mathcal{L}(v)}_{0} = f.$$

□

This principle will prove to be invaluable when we construct analytic solutions to linear boundary value problems in Section 2.

1.3. Classifying second order PDEs

For simplicity, we now restrict attention to second order PDEs in just two variables t and x. Thus (1.1) can be expressed in the form:

$$au_{tt} + bu_{tx} + cu_{xx} + du_t + eu_x + gu = f \tag{1.15}$$

where a, b and c can be functions of t, x, u and also of any of the first and second partial derivatives of u; d and e can be functions of t, x, u and either u_t and u_x; g can be a function of t, x and u, and f is a function of t and x.

Note that (1.15) is *linear* if and only if all the coefficients a, b, c, d, e and f do not depend on u or on any of the partial derivatives of u. In other cases, the leading coefficients a, b, c determine the degree of nonlinearity. If these coefficients are independent of u_{tt}, u_{tx} and u_{xx} then (1.15) is said to be *quasi-linear*. If the leading coefficients are linear, that is, independent of u, u_t and u_x, then (1.15) is said to be *semi-linear*.

Example x.2.8 The viscous Burgers' equation with viscosity $\nu > 0$ is semi-linear

$$u_t + uu_x - \nu u_{xx} = f. \qquad \heartsuit \tag{1.16}$$

The sign of the discriminant associated with the leading coefficients determines the classical "type" of the PDE (1.15).

Definition x.3 (PDE type)
There are three generic types of PDE associated the discriminant $d = b^2 - 4ac$. These are associated with conic sections:

$$\begin{aligned} \text{“hyperbolic”} &\iff d > 0; \\ \text{“parabolic”} &\iff d = 0; \\ \text{“elliptic”} &\iff d < 0. \end{aligned}$$

Note that if the coefficients are independent of t and x, that is, if the PDE in (1.15) has "constant coefficients" then the type of PDE is easily determined. If the PDE is linear but has variable coefficients, then the PDE can "change type" as t and x varies. If the PDE is nonlinear then the type could well be tricky to determine since it will depend on the unknown solution u.

Example x.3.1 Since $d = 0$, the heat equation is *parabolic*

$$u_t - \kappa u_{xx} = f. \qquad \heartsuit$$

Example x.3.2 Since $d = -4 < 0$, the Poisson equation and Laplace's equation are both elliptic

$$-(u_{tt} + u_{xx}) = f. \qquad \heartsuit$$

Example x.3.3 Since $d = 4c^2 > 0$, the wave equation is hyperbolic

$$u_{tt} - c^2 u_{xx} = f. \qquad \heartsuit$$

We will see in later sections that knowing the type of the PDE is absolutely crucial if numerical methods for solving the associated boundary value problems are to be effective.

1.4. Constructing PDEs from the solution

Finally we note that it easy to show that a given function will satisfy any number of PDEs—just keep differentiating!

Example 1.1

$$\begin{aligned} u &= x^2t^2 + A(t) + B(x) \\ u_t &= 2x^2t + A'(t) \\ u_{tx} &= 4xt \\ u_{txt} &= 4x \\ u_{txtx} &= 4. \end{aligned}$$

Note that two specific differentiations kill the two functions of integration.

Example 1.2

$$\begin{aligned} u &= t^{-1/2}e^{-x^2t^{-1}} \\ u_t &= -\frac{1}{2}t^{-3/2}e^{-x^2t^{-1}} + x^2t^{-5/2}e^{-x^2t^{-1}} \\ u_x &= -2xt^{-3/2}e^{-x^2t^{-1}} \\ u_{xx} &= -2t^{-3/2}e^{-x^2t^{-1}} + 4x^2t^{-5/2}e^{-x^2t^{-1}} \end{aligned}$$

$$\text{thus} \quad u_t - \frac{1}{4}u_{xx} = 0.$$

The function $u = \frac{1}{\sqrt{t}}e^{-x^2/t}$ is pretty special—it is called a *fundamental solution* of the heat equation. This function is also extremely important in probability theory, but that is another story...

Example 1.3

$$\begin{aligned} u &= (x^2+y^2+z^2)^{-1/2} \\ u_x &= -x(x^2+y^2+z^2)^{-3/2} \\ u_{xx} &= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} \\ u_{yy} &= -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} \\ u_{zz} &= -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} \\ \text{thus} \quad u_{xx}+u_{yy}+u_{zz} &= 0. \end{aligned}$$

Thus we see that the function $u = 1/r$ with $r^2 = x^2+y^2+z^2$ is also special—it satisfies Laplace's equation in three dimensions. This means that it a *harmonic* function. Interestingly, $u = 1/r$ does not satisfy Laplace's equation in two dimensions.

Example 1.4

$$\begin{aligned} u &= \frac{1}{2}\ln(x^2+y^2) \\ u_x &= x(x^2+y^2)^{-1} \\ u_{xx} &= (x^2+y^2)^{-1} - 2x^2(x^2+y^2)^{-2} \\ u_{yy} &= (x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2} \\ \text{thus} \quad u_{xx}+u_{yy} &= 0. \end{aligned}$$

Thus we see that $u = \frac{1}{2}\ln r^2 = \ln r$ is the sought-after harmonic function in two dimensions.

Example 1.5

$$\begin{aligned} u &= A(x-ct) \quad \text{with } c > 0 \text{ constant} \\ u_t &= -cA'(x-ct) \\ u_x &= A'(x-ct) \\ \text{thus} \quad u_t + cu_x &= 0. \end{aligned}$$

This PDE is known as the one-way wave equation. The solution u is also special—the "initial profile" defined by the function $A(x)$ simply translates to the left with speed c.

In practical applications we would like to reverse the above process. That is, given a well posed boundary value problem: can we explicitly construct the solution? We focus on this task in the rest of the notes.