Analysis of Numerical Methods for Incompressible Fluid Dynamics

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Personnel supported:

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Introduction. Simulation of the motion of an incompressible fluid remains an important but very challenging problem. The resources required for accurate three-dimensional simulation of practical flows test even the most advanced of supercomputer hardware—the necessity for effective and efficient numerical algorithms is uncontrovertible. Computations based on approximations on fixed non-adapted grids are out of the question, and the excellent sparse direct solvers developed over the past 30 years or so (which have enabled many two-dimensional computations) require unreasonable and usually unattainable computational resources.

In this project, we have built and demonstrated a new and complete methodology for the approximation and efficient solution of steady incompressible Navier-Stokes problems. In particular our research evolved along three separate fronts as identified on the original proposal: (i) the approximation by convenient low order mixed finite elements with appropriate pressure-stabilisation (this is described in [7],[8]); (ii) estimation of error a posteriori, using estimators calculated from the solution of local higher order problems together with adaptive mesh (h–)refinement based on these indicators (see [4],[5]); and (iii) preconditioned Krylov subspace iteration employing preconditioners which provide an optimal solver with respect to the mesh size and an almost optimal solver with respect to the Reynolds number associated with the underlying flow (see [3],[10]). In each of these research threads we have made important advances commensurate with those envisaged on the original proposal—these are detailed below. The combination of our ideas is implemented in some research software which is used to compute the example results presented below. This software package is already being used by a number of academic colleagues around the world—making it more widely usable is being undertaken as part of ongoing callaboration with David Kay.

Stabilised mixed finite element methods. The development of stable mixed finite element methods is a fundamental component in the search for efficient numerical methods for solving the Navier-Stokes equations. Using a primitive variable formulation, the importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying a technical 'inf-sup' condition is widely appreciated. In particular, it is well known that conforming low order elements are not stable. This impinges on efficiency since the simple logic and regular data structure associated with low order finite element methods make them particularly attractive on modern parallel processing architectures.

In recent years, computationally convenient stabilised low-order discretisation methods have been developed. Such methods are widely used by practitioners since they permit the use of equal-order velocity and pressure approximations—a combination that is notoriously unstable in the standard finite element framework. The drawback with this methodology is the introduction of stabilisation parameters that must be chosen sensibly if the resulting method is to work well in practice. This choice between stable and stabilised low order finite element methods was addressed by the project student—in the first instance by a series of numerical experiments. These results were published in a special journal issue devoted to stabilisation methods [8], and two conclusions were drawn.

- The three stabilised mixed methods discussed in [8] proved to be superior to their inherently stable counterparts, with respect to the accuracy for a fixed number of degrees of freedom.
- There was no major difference in the performance of the stabilised methods, although the results hinted that the standard equal order P_1–P_1 stabilised method (linear velocity with C^0 linear pressure) may perform poorly compared to a P_1–P_0 (linear velocity and constant pressure) stabilised method if small scale features of the flow really need to be resolved.

Although attention in [8] is restricted to two-dimensional flow problems and triangular meshes, the conclusions are also relevant to the case of three-dimensional flow problems discretised using the lowest order tetrahedral elements. As noted above, the biggest drawback of stabilisation techniques is that (regularisation) parameters are implicitly introduced. To address this issue, a Fourier analysis of periodic Stokes flow problems discretised using a Q_1–Q_1 (C^0 bilinear velocity and pressure) mixed finite element method is given in [9]. In particular, we extended the standard stabilisation technique (originally introduced by Brezzi and Pitkäranta in the case of P_1–P_1 mixed approximation) and deduced the optimal choice of stabilisation parameter which minimises the condition number of the Schur complement matrix that determines stability. Contrary to our expectations, it turns out that the optimal parameter is *not* uniquely determined—there is an interval of parameter values over which optimality is achieved. The upshot of this analysis is that the performance of stabilised methods is less sensitive than might be anticipated in terms of the choice of parameter. Norburn's Ph.D thesis [7] represents a lucid exposition of state-of-the-art *optimally* stabilised mixed methods for incompressible flow equations.

The issue of *inf-sup* stability has increased in importance with the advent of fast iterative solution algorithms, for example, based on multigrid iteration or preconditioning—if the underlying mixed approximation is not stable then stabilisation is essential if methods based on multigrid are to be effective.

A posteriori error estimation and adaptivity. In the first phase of the project (when David Kay was based at UMIST), the design of an efficient and effective mesh refinement strategy for incompressible flow models was addressed. Our starting point was that of steady incompressible Stokes flow using the stabilised P_1–P_0 approximation methods that is identified in [8] as being the most accurate Stokes mixed finite method based on a linear approximation of velocity. Building on the pioneering work of Ainsworth and Oden, we developed a local error estimator that is computed by solving a local Poisson problem (using a higher order space of edge bubble functions) for each component of velocity (in two-dimensions we need to solve two independent 3×3 systems for every element in the mesh). In [4] we extended the analysis of Verfürth to show that the resulting estimator is equivalent to the discretisation error, where the computable constants in the equivalence relation are independent of the stabilisation parameter that controls the stability of the method. We also devised a novel adaptive refinement strategy that ensures that the stabilisation remains valid in the limit as the mesh size tends to zero.

In a subsequent paper [5] we addressed the reliability of this type of local Poisson estimator in the context of steady convection-diffusion problems. Our analysis illustrates that this type of estimator suffers from a potential source of difficulty in regimes where convection dominates—specifically, in the neighbourhood of unresolved exponential boundary layers, the estimator is likely to grossly over-estimate the discretisation error. Fortunately, in the case of incompressible flow models such exponential boundary layers only arise when outflow boundary conditions are inappropriately specified. Consequently we are confident that a combination of the approaches in [4] and [5] can provide the basis for cheap and effective error indicators for incompressible flow problems, at least for flows associated with Reynolds numbers up to $O(10^4)$ – that is realistic steady flows in most non-trivial flow geometries.

Efficient preconditioning for the linearised Navier-Stokes equations. The second phase of the project on preconditioned iterative solution methods started when David Kay moved from UMIST to Oxford as envisaged in the original proposal.

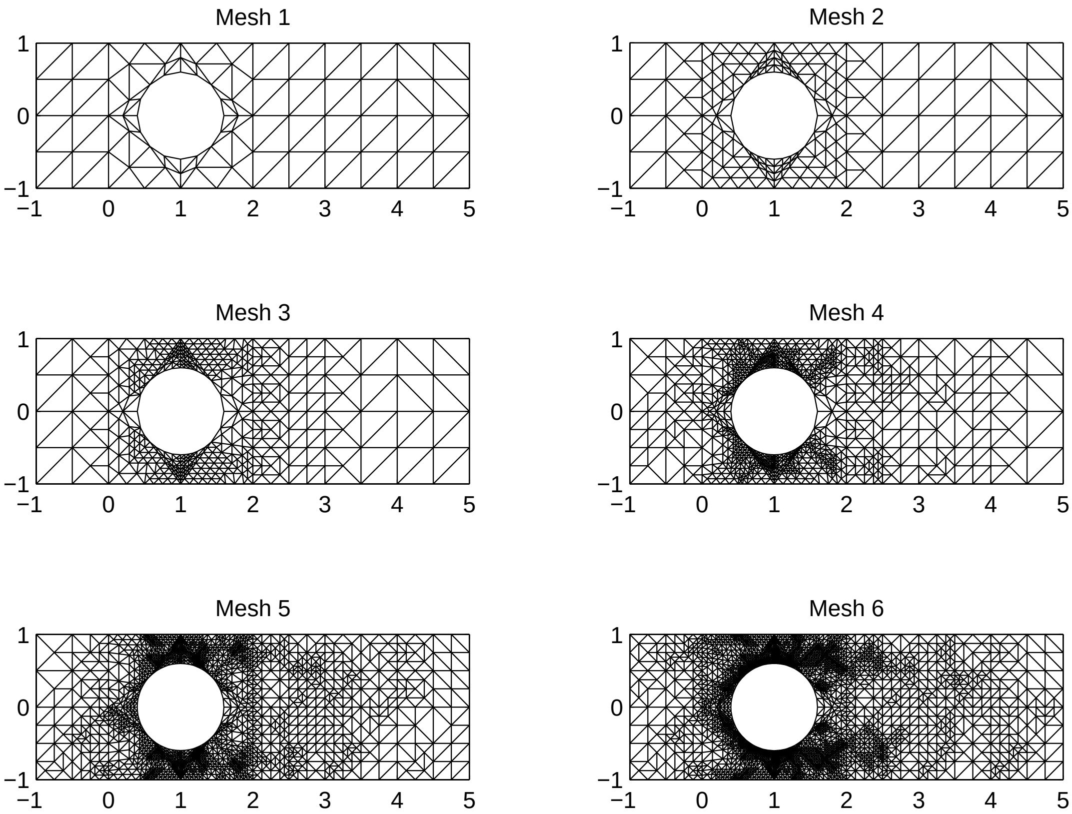
Block preconditioned GMRES iteration had been selected as the most attractive approach for the Navier-Stokes equations based on earlier work of the two principle investigators on the Stokes problem. The requirements were for a multigrid preconditioner for the 'primal' (1,1) operator which could be combined with the pressure scaling matrix in the (2,2) Schur complement operator giving a preconditioner which can be proved to give convergence rates independent of mesh size for the Stokes problem (this had been demonstrated for regular grids by Silvester and Wathen). David Kay quickly implemented a multigrid procedure based on the adaptive levels of refinement and was able to build an efficient Stokes preconditioner which demonstrated the effectiveness of the Silvester–Wathen approach for irregular/adapted grids.

The work proceded on to the Oseen problem where the result of [6] indicated that a multigrid preconditioning procedure for the discrete scalar advection-diffusion operator was required as well as an approximate Schur complement.

The multigrid procedure was achieved by constructing stable finite element approximations on each of the different grid levels using the Streamline Upwind Petrov Galerkin (SUPG) procedure of Hughes and Brookes with an appropriate stabilisation parameter as identified by Fischer, Ramage, Silvester and Wathen (Comput. Meths. Appl. Mech. Engrg. **179**, pp. 179-196 (1999)). By this approach, the unwanted oscillations on coarser grids associated with Galerkin approximation were avoided enabling standard grid transfer operations to be employed. Note that even if a Galerkin approximation could be used on the finest level of adapted mesh, it was always (and will generally) be the case that mesh-Peclet conditions will be violated on the coarser meshes and so appropriate advection-stabilisation is required. We avoided complicated 'trajectory-following' variable orderings for the smoother as these seem unlikely to be applicable for realistic flows but employed a small number of Gauss-Seidel sweeps through the variables in the forward and reverse order. The ordering of the discrete variables was simply as computed in the adaptive algorithm, employing a simple 'stack' for new degrees of freedom in a finer level of refined mesh. This multigrid component of the overall block preconditioner worked very well in tests and attention switched to the Schur complement part of the preconditioner.

Initial work on the so-called BFBT idea of Howard Elman (SIAM J. Sci. Comput. **20**, pp. 1299-1316 (1999)) encountered some difficulties, particularly for the stabilised low order elements which we desired to use. Results for higher order stable mixed spaces (such as P_2–P_1) were a little better, but not as good as we had hoped. Coincidently, a graduate student, Daniel Loghin, was working on the use of Green's functions and Green's tensors for preconditioning partial differential equation problems and when he heard about this problem, he was able to come up with an idea for a Schur complement preconditioner motivated by the Green tensor for the Oseen operator. The preconditioner is described in [3] where also an excellent set of computational results which demonstrate the effectiveness of this so-called **Fp** preconditioner are given. Analysis of this preconditioner is ongoing and has involved also Howard Elman of the University of Maryland. The numerical results indicate that the combination of one multigrid V-cycle employing all of the levels of adapted mesh, and the Fp approximation of the Schur complement as a preconditioner for the GMRES iterative method is an extremely efficient solver - the number of iterations remains constant with respect to the number of discrete variables and also seems to depend only mildly on the Reynolds number.

A set of sample results is given here: the problem is the full non-linear incompressible Navier-Stokes equations for flow past a cylinder at Reynolds number 50. The figure below shows six levels of refined mesh computed for the final non-linear iteration and the table gives the number of preconditioned GMRES iterations required over *all* of the computation. $\mathcal{P}_{\mathcal{G},2}$ is the fully efficient preconditioner employing multigrid and the Fp approximation. Note that use of multigrid rather than a direct inversion of the (1,1) block results in at most two more iterations and more profoundly that the number of iterations *goes down* as the number of adapted degrees of freedon (in brackets for each level) increases. This is a feature of the Green's function approach to preconditioning since the underlying preconditioning operator is better approximated on finer grids. Further results can be found in [10]. David Kay has computed simple three-dimensional flows using these techniques and they appear equally effective. Analysis of the Fp preconditioner and the computation of more realistic three dimensional flows are ongoing.



Flow past a cylinder: levels of adaptive mesh refinement used also in the multigrid preconditioner

Level	1 (264)	2 (540)	3 (1242)	4 (2709)	5 (5484)	6 (9003)
$\mathcal{P}_{\mathcal{G},2}$	49	51	44	44	41	40

GMRES iteration counts for the P_1–P_1 adaptive mesh solution to the flow past a cylinder problem

Research Seminars given by David Silvester: *Fast iterations imply optimally stabilised discretisation methods*, University of Bradford, December 1996; *Fast and accurate solution of the incompressible Navier-Stokes equations*, University of Greenwich, March 1997; *Error estimation and aadaptivity for incompressible flow problems*, University of Bath, June 1997; *Multigrid preconditioning in computational fluid dynamics*, University of Manchester, September 1997; *Multigrid preconditioning for elliptic PDEs*, Manchester Metropolitan University, January 1998; *A posteriori error estimation for elliptic PDEs*, University of Strathclyde, February 1998, Imperial College, December 1998, University of Maryland, USA, January 1999, University of Reading, February 1999, University of Minnesota, USA, June 1999.

Lecture Series given by David Silvester: University of Utrecht, 3 lectures on *Fast solution techniques for incompressible Navier-Stokes equations*, *Stable & unstable finite element methods for incompressible flow problems* and *Fast iterations imply optimally stabilised discretisation methods* in a workshop on Theoretical and Practical Aspects of Incompressible CFD, January 8–10, 1997; Technical University of Denmark, Lyngby, 4 lectures on *Finite element multigrid methods* in an International Graduate Research Course on Iterative and Finite Element Multigrid Methods, June 25—July 3, 1998.

Plenary Lectures given by David Silvester: *Mathematical Aspects of CFD*, Oberwolfach, January 1997; Meeting in honour of Professor Godunov, Manchester Metropolitan University, October, 1999.

Research Seminars and Conference Contributions given by Andy Wathen: Imperial College, November 1996, University of Leeds, November 1996, University of Sussex, January 1997, University of Greenwich, April 1997, University of Minnestota, January 1998, University of Cambridge, February 1998, University of Manchester, February 1998, Copper Mountain Conference on Iterative Methods, April 1998, University of Reading, May 1998, University of Liverpool, February 1999, University of Maryland, February 1999, Lawrence Livermore National Laboratory, California, March 1999, Stanford University, March 1999, University of Leicester, May 1999, Preconditioning techniques for large sparse matrix problems in industrial applications, University of Minnesota, June 1999, Biennial conference on Numerical Analysis, University of Dundee, July 1999.

Lecture Series given by Andy Wathen: University of Utrecht, 3 lectures on *Introduction to Stokes problems/inf-sup stability*, *Eigenvalue bounds/estimates for Stokes* and *Preconditioning discrete Oseen systems* in a workshop on Theoretical and Practical Aspects of Incompressible CFD, January 8–10, 1997

Plenary Lectures given by Andy Wathen: Meeting in honour of Professor K.W. Morton, Oxford, April 1997, Biennial Conference on Numerical Analysis, Dundee, Scotland, June 1997, Householder Meeting on Numerical Algebra, Vancouver, Canada, June 1999, International Conference on Numerical Algebra and Optimization, Nanjing, China, Sept 1999.

Research Seminars and Conference Contributions given by David Kay: University of Durham, March 1997, University of Dundee, 17th Biennial Conference on Numerical Analysis, 1997, University of Strathclyde, February 1998, Stockholm, Royal Institute of Technology, Workshop on Adaptive Methods for Differential Equations, March 1998, University of Leicester, April 1998, Ninth Copper Mountain conference on multigrid methods, 1999, Stanford University, May 1999, University of Dundee, 18th Biennial Conference on Numerical Analysis, 1999. Oxford University Computing Laboratory, November 1999.

Publications: GR/K91262 and GR/L05617

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3. Kay, D. and Loghin, D. A Green's function preconditioner for the steady state Navier-Stokes equations, Oxford University Computing Laboratory Report # 99–06, 1999, submitted to SIAM J. Scientific Computing.
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10. Silvester, D., Elman, H., Kay, D. and Wathen, A. Efficient preconditioning of the linearised Navier-Stokes equations, Manchester Centre for Computational Mathematics Report # 352, submitted to the Journal of Computational and Applied Mathematics millennium issue.