<u>**Three hours**</u>

This exam will be worth 80% of the final mark on this course unit.

THE UNIVERSITY OF MANCHESTER

APPROXIMATION THEORY AND FINITE ELEMENT ANALYSIS

May 2025

TBD

Answer all FIVE questions (70 marks in total).
All five questions are worth an *equal number* of marks.

The use of electronic calculators is NOT permitted.

General Instructions. You may use the Cauchy–Schwarz inequality or the Poincaré–Friedrichs inequality in answering any question without giving a proof.

Given a general triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and with area $|\triangle|$, the barycentric coordinates (L_1, L_2, L_3) are defined by the mapping

$$L_j = \frac{1}{2|\triangle|}(a_j + b_j x + c_j y) \qquad j = 1, 2, 3,$$

with coefficients

$$\begin{array}{ccc} a_1 = x_2y_3 - x_3y_2 & a_2 = x_3y_1 - x_1y_3 & a_3 = x_1y_2 - x_2y_1, \\ b_1 = y_2 - y_3, & b_2 = y_3 - y_1, & b_3 = y_1 - y_2, \\ c_1 = x_3 - x_2, & c_2 = x_1 - x_3, & c_3 = x_2 - x_1. \end{array}$$

Integrals of polynomial functions can be computed using the *magic* formula

$$\int_\triangle L_1^k L_2^\ell L_3^m = \frac{k!\,\ell!\,m!}{(k + \ell + m + 2)!}\,2|\triangle|.$$

1. This question concerns a fourth-order differential equation.

The deflection of a beam subject to a unit load that is clamped at one end can be modelled as the solution of the following problem: find $u\colon [0,\ell] \to \mathbb{R}$ satisfying

$$\left.\begin{aligned} u'''' &= 1 \quad \text{in } (0,\ell), \\ u''(0) = u'''(0) &= 0; \quad u(\ell) = u'(\ell) = 0. \end{aligned}\right\} \quad (D)$$

(a) Given the test space $X := \{v | v \in H^2(0,\ell); v(\ell) = 0, v'(\ell) = 0\}$, where $H^2(0,\ell)$ is the standard Sobolev space, show that u solving (D) also satisfies the variational formulation: find $u \in X$ such that

$$\int_0^\ell u''v''\,dx = \int_0^\ell v\,dx \quad \forall v \in X. \qquad (V)$$

Distinguish between the *essential* and *natural* boundary conditions in the formulation (V) and show that the integral on the left-hand side of (V) is finite.

[3+3 marks]

(b) Define the Galerkin approximation $u_h \in X_h = \operatorname{span}\{\phi_j\}_{j=1}^n \subset X$ and show that the approximation method leads to a $n \times n$ matrix system

$$A\boldsymbol{x} = \boldsymbol{f}.$$

Identify explicitly the entries $A_{i,j}$ of the matrix A and f_i of the vector $\boldsymbol{f}$.

Can A be a singular matrix? Justify your answer.

[4+4 marks]

2. This question is concerned with smooth interpolation on the domain $\Omega = (0,\ell)$.

(a) Explain the difference between *natural* cubic splines and *Hermite* cubic splines. Give two reasons why one might prefer to use Hermite spline interpolation rather than natural splines when interpolating a continuous function $f\colon [0,\ell] \to \mathbb{R}$ at a uniform set of knots $K = \{x_j\}_{j=0}^n$.

[3 marks]

(b) Suppose that we interpolate a continuous function $f\colon [0,\ell] \to \mathbb{R}$, using the natural spline function $s_3(x) \in C^2(\overline{\Omega})$ defined on a set of uniformly spaced knots $K = \{x_j\}_{j=0}^n$. Prove that

$$\|s_3''\|_{L^2(\Omega)} \le \|v''\|_{L^2(\Omega)}$$

for any other function $v \in H^2(\Omega)$ interpolating f on the set K.

(Hint: use the fact that $\int_0^\ell (v'' - s_3'')s_3''\,dx = -\sum_{j=1}^n \int_{x_{j-1}}^{x_j} (v' - s_3')s_3'''\,dx$.)

[8 marks]

(c) Hermite splines are the starting point for the design of a numerical method for solving the clamped beam problem (D) above. In no more than four sentences, explain how Hermite splines can be used to construct a *conforming* finite element approximation to the variational problem (V).

[3 marks]

3. This question is concerned with a reaction–diffusion problem in two or three dimensions. Given a constant reaction coefficient $r \geq 1$, we seek $u \colon \Omega \to \mathbb{R}$ satisfying

$$\left.\begin{aligned} -\nabla^2 u + ru &= 0 \quad \text{in } \Omega, \\ \nabla u \cdot \vec{n} &= 1 \quad \text{on } \partial\Omega, \end{aligned}\right\} \quad (R)$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is the spatial domain, $\partial\Omega$ is the boundary and $\vec{n}$ is the outward pointing normal.

(a) Briefly explain (two sentences) why a refined triangular/tetrahedral subdivision is needed near the boundary of the domain if we are to get accurate finite element approximation in cases where the reaction term dominates; that is, in cases where $r \gg 1$.

[2 marks]

(b) Given the test space $X = H^1(\Omega)$ (where $H^1(\Omega)$ is the usual Sobolev space), show that u satisfying (R) satisfies the following variational formulation: find $u \in X$ such that

$$a(u, v) = \int_{\partial\Omega} v \, ds \qquad \forall\, v \in X, \qquad (W)$$

where $a : X \times X \to \mathbb{R}$ is a bilinear form that will need to be identified.

Show further that there exist positive constants γ and Γ so that

$$a(v, v) \geq \gamma \, \|v\|^2_{H^1(\Omega)} \qquad \forall\, v \in X,$$
$$a(u, v) \leq |a(u, v)| \leq \Gamma \, \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall\, u, v \in X.$$

Hint: the space $H^1(\Omega)$ has a natural inner product. [4+3 marks]

(c) Establish Galerkin orthogonality and hence show that a finite element approximation $u_h \in X_h \subset X$ to the solution of (W) is *optimal* in the sense that

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{\Gamma}{\gamma} \|u - v_h\|_{H^1(\Omega)} \qquad \forall v_h \in X_h.$$

[5 marks]

4. Consider solving the problem (R) above with fixed $r = 1$ on the square domain $\Omega = (0, 1) \times (0, 1)$, using piecewise linear approximation on a triangular mesh.

(a) Denoting the approximation $u_h \in X_h = \text{span}\{\phi_j\}_{j=1}^n$, the Galerkin solution is determined by solving an $n \times n$ matrix system

$$A\,\boldsymbol{u} = \boldsymbol{f}. \qquad (W_h)$$

Identify the entries $A_{i,j}$ of the matrix A and f_i of the vector $\boldsymbol{f}$.

[2 marks]

(b) By computing the mapping coefficients $a_j, b_j, c_j, j = 1, 2, 3$ for a generic triangle and using the magic formula, construct the 3×3 element matrix $A^{ⓚ}$ that contributes to the matrix A in (W_h). (*Hint:* you only need to compute $A^{ⓚ}_{ij}, i \geq j$.)

[4 marks]

(c) Assemble the 4×4 Galerkin matrix corresponding to a grid of two triangles that is formed by constructing a diagonal from the bottom left to the top right corner of the domain.

[8 marks]

5. The final question concerns finite element error estimation. Given some bounded two-dimensional domain Ω, suppose that $X_h \subset X := \{v \mid v \in H^1(\Omega);\ v = 0 \text{ on } \partial\Omega\}$ corresponds to a piecewise *quadratic* finite element approximation space defined on a triangular subdivision of Ω. Let $u \in X$ and $u_h \in X_h$ satisfy

$$(\nabla u, \nabla v) = (f, v) \qquad \forall\, v \in X,$$
$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall\, v_h \in X_h,$$

respectively, where $(\cdot, \cdot)$ is the standard $L^2(\Omega)$ inner product and $f \in L^2(\Omega)$ is a given function.

(a) Prove that the Galerkin solution u_h is the *best approximation* to $u \in X$ when measured in the energy norm $\|u\|_E$, that is

$$\|u - u_h\|_E \leq \|u - v_h\|_E \quad \forall v_h \in X,$$

where $\|v\|_E^2 = (\nabla v, \nabla v)$ for $v \in X$.

[3 marks]

(b) Using Green's theorem, show that the error $u - u_h$ satisfies

$$(\nabla u - \nabla u_h, \nabla\, v) = \sum_{ⓚ} (f + \nabla^2 u_h, v)_{\triangle_k} - \sum_{ⓚ} \left\{ \frac{1}{2} \sum_{E \in \partial\triangle_k} \int_E [\![\frac{\partial u_h}{\partial n}]\!]\, v\, ds \right\}, \quad \forall\, v \in X, \qquad (\star)$$

where $(\cdot, \cdot)_{\triangle_k}$ is the $L^2(\triangle_k)$ inner product, $\partial\triangle_k$ is the set of edges of triangle $\triangle_k$ (excluding edges on $\partial\Omega$) and

$$[\![\frac{\partial v}{\partial n}]\!] = (\nabla v|_T - \nabla v|_S) \cdot \vec{n}_{E,T}$$

denotes the flux jump across the edge E between elements T and S with $\vec{n}_{E,T}$ the outward-pointing normal from T on E.

[7 marks]

(c) By localising the error representation in $(\star)$, suggest an appropriate *local problem* that might be solved in order to estimate the error on triangle $\triangle_k$ and identify a suitable local approximation space that one might use in order to approximate the local error on $\triangle_k$.

[4 marks]

END OF EXAMINATION PAPER