Three hours

This exam will be worth 80% of the final mark on this course unit.

THE UNIVERSITY OF MANCHESTER

APPROXIMATION THEORY AND FINITE ELEMENT ANALYSIS

May 2023

TBD

Answer ALL the questions (70 marks in total).
All five questions are worth an *equal number* of marks.

The use of electronic calculators is NOT permitted.

General Instructions. You may use the Cauchy–Schwarz inequality or the Poincaré–Friedrichs inequality in answering any question without giving a proof.

Given a general triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and with area $|\triangle|$, the barycentric coordinates (L_1, L_2, L_3) are defined by the mapping

$$L_j = \frac{1}{2|\triangle|}(a_j + b_j x + c_j y) \qquad j = 1, 2, 3,$$

with coefficients

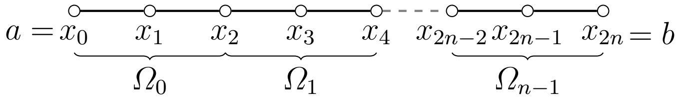
$$\begin{array}{ccc} a_1 = x_2 y_3 - x_3 y_2 & a_2 = x_3 y_1 - x_1 y_3 & a_3 = x_1 y_2 - x_2 y_1, \\ b_1 = y_2 - y_3, & b_2 = y_3 - y_1, & b_3 = y_1 - y_2, \\ c_1 = x_3 - x_2, & c_2 = x_1 - x_3, & c_3 = x_2 - x_1. \end{array}$$

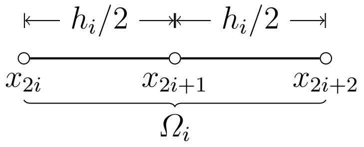
Integrals of polynomial functions can be computed using the *magic* formula

$$\int_\triangle L_1^k L_2^\ell L_3^m = \frac{k!\,\ell!\,m!}{(k+\ell+m+2)!}\,2|\triangle|.$$

1. This question concerns one-dimensional interpolation.

Given the domain $\Omega = [a, b]$, consider a partitioning into n subdomains $\{\Omega_0, \Omega_1, \ldots, \Omega_{n-1}\}$.





Let $I_h f(x)$ denote the piecewise quadratic interpolant of a given function $f(x)$ such that $I_h f(x) \in C^0[a, b]$ is a quadratic polynomial in each Ω_i and $I_h f(x_i) = f_i := f(x_i)$.

(a) For the ith subdomain Ω_i, determine expressions for the basis functions $\phi^e_{2i}(x)$, $\phi^e_{2i+1}(x)$ and $\phi^e_{2i+2}(x)$. (*Hint:* $I_h f(x) = \phi^e_{2i}(x) f_{2i} + \phi^e_{2i+1}(x) f_{2i+1} + \phi^e_{2i+2}(x) f_{2i+2}$.)

[2 marks]

(b) Plot (in a schematic diagram) the basis functions $\phi^e_{2i}(x)$, $\phi^e_{2i+1}(x)$ and $\phi^e_{2i+2}(x)$ for the ith subdomain Ω_i. The diagram should include the function values at the points x_{2i}, x_{2i+1}, x_{2i+2} and the values at the points of local extrema.

[2 marks]

(c) For each x_i construct the global basis functions $\phi_i(x)$ obtained by assembly of local components over a patch and plot them. (*Hint:* all $\phi_{2i}(x)$ where $i = \{1, 2, \ldots, n-1\}$ and all $\phi_{2i+1}(x)$ where $i = \{0, 2, \ldots, n\}$ will have similar plots — you only need to draw one representative from each set.)

[2 marks]

(d) Assume $f \in C^3[a, b]$ and $h := \max\{h_i\}$.

Establish the following error estimate:

$$\|f - I_h f\|_{L^\infty(\Omega)} \leq \frac{h^3}{72\sqrt{3}} \|f'''\|_{L^\infty(\Omega)}.$$

[8 marks]

Hint: Let $e_i = (f - I_h f)|_{\Omega_i}$ and consider the function $\Phi_x : \Omega_i \to \mathbb{R}$ defined by

$$\Phi_x(t) = e_i(t) - e_i(x) \frac{(t - x_{2i})(t - x_{2i+1})(t - x_{2i+2})}{(x - x_{2i})(x - x_{2i+1})(x - x_{2i+2})}.$$

2. This question concerns a convection–diffusion problem defined on the square domain $\Omega := (0,1) \times (0,1)$ with boundary $\partial\Omega$ consisting of two nonoverlapping pieces $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$.

Given a diffusion parameter $\epsilon > 0$, we seek $u : \Omega \to \mathbb{R}$ satisfying

$$\left.\begin{aligned} -\epsilon\nabla^2 u + \frac{\partial u}{\partial x} &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega_D \\ \nabla u \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega_N, \end{aligned}\right\} \quad (D)$$

where $\vec{n}$ is the outward pointing normal.

(a) Given the test space $X = \{v | v \in \mathcal{H}^1(\Omega), v = 0 \text{ on } \partial\Omega_D\}$, show that u satisfying (D) satisfies the variational formulation: find $u \in X$ such that

$$a(u,v) = \int_\Omega v \qquad \forall\, v \in X, \tag{V}$$

where $a : X \times X \to \mathbb{R}$ is the bilinear form: $a(u,v) = \epsilon \int_\Omega \nabla u \cdot \nabla v + \int_\Omega \frac{\partial u}{\partial x} v$.

[2 marks]

(b) Assume that $\partial\Omega_N$ is the right boundary edge of the square where $x = 1$ (so that the condition $u = 0$ holds on some *nonzero piece* of the boundary). Show that there exist positive constants γ and Γ so that

$$a(v,v) \geq \gamma \, \|\nabla v\|^2_{L^2(\Omega)} \qquad \forall\, v \in X,$$
$$a(u,v) \leq |a(u,v)| \leq \Gamma \, \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \quad \forall\, u, v \in X.$$

Hint: try integrating the term $\int_\Omega \frac{\partial u}{\partial x} v$ by parts and then setting $v = u$. [5+3 marks]

(c) Establish Galerkin orthogonality and hence show that a finite element approximation $u_h \in X_h \subset X$ to the solution of (V) is *optimal* in the sense that

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \frac{\Gamma}{\gamma} \|\nabla(u - v_h)\|_{L^2(\Omega)} \qquad \forall v_h \in X_h.$$

[4 marks]

The next two questions are associated with solving an *eigenvalue* problem in a bounded domain $\Omega \subset \mathbb{R}^d$. Thus, we seek $\lambda \in \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ with $u \neq 0$, satisfying

$$\left.\begin{aligned} -\nabla^2 u = \lambda u \quad &\text{in } \Omega \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned}\right\} \quad (E)$$

3. Given the test space $X = \{v | v \in \mathcal{H}^1(\Omega), v = 0 \text{ on } \partial\Omega\}$.

(a) Show that u satisfying (E) satisfies the variational formulation: find the pair $(\lambda \in \mathbb{R}, u \in X)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv \qquad \forall\, v \in X. \qquad (W)$$

Show that any eigenvalue $\lambda \in \mathbb{R}$ satisfying (W) must satisfy $\lambda > 0$.

[2+5 marks]

(b) Given some finite dimensional subspace $X_h \subset X$ with basis set $\{\phi_j\}_{j=1}^k$, show that a Galerkin approximation to the smallest eigenvalue λ^* can be computed by solving a generalized eigenvalue problem:

$$A\,\boldsymbol{x} = \lambda_h\, Q\, \boldsymbol{x}. \qquad (W_h)$$

(You need to identify explicitly the entries $A_{i\,j}$ of the matrix A and $Q_{i\,j}$ of the matrix Q.)

Can Q be a singular matrix? Justify your answer.

[3+4 marks]

4. Consider a square domain $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$. Let X_h be the piecewise linear approximation to the solution of (W) defined on a uniform triangulation of a grid of squares of size h formed by subdividing each square into 4 triangles by constructing the diagonal lines from opposite corners.

(a) Given the usual anticlockwise numbering of the vertices, use the magic formula to show that the 3×3 element matrix that contributes to the matrix Q in (W_h) (see above) is given by

$$Q^{Ⓚ} = \begin{bmatrix} \frac{h^2}{24} & \frac{h^2}{48} & \frac{h^2}{48} \\ \frac{h^2}{48} & \frac{h^2}{24} & \frac{h^2}{48} \\ \frac{h^2}{48} & \frac{h^2}{48} & \frac{h^2}{24} \end{bmatrix}.$$

(*Hint:* you only need to compute one diagonal and one off-diagonal entry.) [4 marks]

(b) By computing the mapping coefficients $a_j, b_j, c_j, j = 1,2,3$ for a generic triangle, construct the 3×3 element matrix $A^{Ⓚ}$ that contributes to the matrix A in (W_h). (*Hint:* you only need to compute $A^{Ⓚ}_{ij}, i \geq j$.)

[6 marks]

(c) Compute the Galerkin approximation to the smallest eigenvalue λ^* in the case $h = 1$ (that is, for a mesh of 4 triangles).

[4 marks]

5. This question is concerned with the solution of a fourth-order partial differential equation.

The deflection of a clamped plate that is subject to a unit load can be modelled as the solution of the following problem: find $u : \overline{\Omega} \to \mathbb{R}$ satisfying

$$\left.\begin{aligned} \nabla^2(\nabla^2 u) = 1 \quad & \text{in } \Omega, \\ u = \partial u/\partial n = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\} \quad (D)$$

where $\partial u/\partial n = \vec{n} \cdot \nabla u$ with $\vec{n}$ representing the outward-pointing normal on the boundary $\partial\Omega$.

(a) Given the function space

$$X = \left\{ v \mid v \in \mathcal{H}^2(\Omega);\ v = 0,\ \vec{n} \cdot \nabla v = 0 \text{ on } \partial\Omega \right\},$$

show that u satisfying (D) also satisfies the following variational formulation:
find $u \in X$ such that

$$\int_\Omega \nabla^2 u \, \nabla^2 v = \int_\Omega v \qquad \forall\, v \in X. \qquad (V)$$

[3 marks]

(b) Starting from the formulation (V), explain why the square of the *bending* energy $\|u\|_E^2 = \int_\Omega (\frac{\partial^2 u}{\partial x^2})^2 + 2(\frac{\partial^2 u}{\partial x \partial y})^2 + (\frac{\partial^2 u}{\partial y^2})^2$ is equal to the average deflection.

[4 marks]

(c) Describe, in two sentences, how you can construct a Galerkin finite element approximation using Hermite spline functions in the case that the plate is rectangular in shape. Draw a picture of the corresponding two-dimensional element showing the component degrees of freedom. Use your picture to justify the assertion that the global approximation is smooth enough to be admissible (you do not need to give a rigorous proof).

[2+5 marks]

END OF EXAMINATION PAPER