Let X be a finite dimensional subspace of a Hilbert space H, and let $u \in H$ and $u_h \in X$ satisfy

$$a(u, v) = \ell(v) \qquad \forall v \in H, \tag{V}$$
$$a(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in X, \tag{V_h}$$

respectively, where $a(\cdot,\cdot)$ is a symmetric bilinear form on H and $\ell : H \to \mathbb{R}$ is a linear functional.

1. Show that u satisfying (V) solves the minimization problem: find $u \in H$ satisfying
$$F(u) \le F(v) \qquad \forall v \in H, \tag{M}$$
where $F : H \to \mathbb{R}$ is the "energy functional" given by
$$F(v) = \frac{1}{2}\|v\|_E^2 - \ell(v),$$
and $\|v\|_E^2 = a(v, v)$ is the square of the *energy norm* of the function v.

2. Use the fact that $a(u - u_h, v_h) = 0$, for all $v_h \in X$ to show that the approximation error $e = u - u_h$ satisfies
$$\|u - u_h\|_E^2 = \|u\|_E^2 - \|u_h\|_E^2 .$$
Deduce that $\|u_h\|_E \le \|u\|_E$. Explain why this inequality is consistent with the associated minimisation problem (M).

3. [Theorem 1] Show that the Galerkin solution u_h is the *best approximation* to $u \in H$ when measured in the energy norm $\|u\|_E$, that is
$$\|u - u_h\|_E \le \|u - v_h\|_E \quad \forall v_h \in X.$$

4. [Theorem 2] Suppose that we have an enhanced approximation space W so that $X \subset W \subset H$, with associated solution $u_h^* \in W$ satisfying
$$a(u_h^*, v) = \ell(v) \qquad \forall v \in W.$$
Use the fact that $a(u - u_h^*, v) = 0$, for all $v \in W$ to show that
$$\|u - u_h\|_E^2 = \|u - u_h^*\|_E^2 + \|u_h^* - u_h\|_E^2 .$$
Deduce that $\|u - u_h\|_E \ge \|u - u_h^*\|_E$.

5. Suppose that we can find an alternative symmetric bilinear form $b(\cdot,\cdot)$ that is *equivalent* to $a(\cdot,\cdot)$ in the sense of the eigenvalue bound
$$\lambda \|v\|_E^2 \le b(v, v) \le \Lambda \|v\|_E^2 \qquad \forall v \in H.$$

Consider the representation problem: find $e_h^* \in W$ satisfying

$$b(e_h^*, w_h) = \ell(w_h) - a(u_h, w_h) \qquad \forall w_h \in W. \tag{W_h^*}$$

[Theorem 3] Show that $e_h^* \in W$ is *equivalent* to $\|u_h^* - u_h\|_E \in W$ in the sense that

$$\lambda \, \|e_h^*\|_{E_0}^2 \leq \|u_h^* - u_h\|_E^2 \leq \Lambda \, \|e_h^*\|_{E_0}^2$$

where $\|v\|_{E_0}^2 = b(v, v)$. [Hint: $a(u_h^* - u_h, w_h) = b(e_h^*, w_h)$ for all $w_h \in W$.]

6. Suppose that the enhanced space W can be written as $W = X \oplus Z$ and that a strengthened Cauchy–Schwarz inequality holds

$$|b(v_h, z_h)| \leq \gamma \, \|v_h\|_{E_0} \, \|z_h\|_{E_0} \qquad \forall v_h \in X, \forall z_h \in Z$$

with $0 \leq \gamma < 1$. Consider the simplified error representation problem: find $e_h \in Z$ satisfying

$$b(e_h, z_h) = \ell(z_h) - a(u_h, z_h) \qquad \forall z_h \in Z. \tag{Z_h^*}$$

[Theorem 4] Show that $e_h \in Z$ is *equivalent* to $e_h^* \in W$

$$\|e_h\|_{E_0}^2 \leq \|e_h^*\|_{E_0}^2 \leq \frac{1}{1 - \gamma^2} \, \|e_h\|_{E_0}^2$$

where $0 \leq \gamma < 1$ is the CBS constant. [Hint: to establish the left-hand inequality use that fact that $b(e_h^*, z_h) = b(e_h, z_h)$ for all $z_h \in Z$. To establish the right-hand inequality, write $\|e_h^*\|_{E_0}^2 = \|v_h + z_h\|_{E_0}^2$ and use the inequality $a^2 + b^2 - 2\gamma ab \geq (1 - \gamma^2) b^2$.]

Computational Exercise. The T-IFISS software package offers a choice of two-dimensional domains on which anisotropic diffusion problems can be posed, along with boundary conditions and a choice of finite element approximation on a structured or unstructured triangular mesh.

7. Consider solving Poisson's equation on a square domain with a zero boundary condition and a constant RHS function. By running the driver `diff_testproblem` and choosing problem 1, tabulate the error estimate η that is generated using linear approximation on a sequence of uniform 16×16, 32×32 and 64×64 triangular meshes. From these, estimate the order of convergence of the finite element approximation in the energy norm. Then, repeat the experiment using quadratic approximation. You should find that the experimental order of convergence is increased.

   One way of estimating the exact energy error is to compute a *reference solution* using a fine grid and then to substitute it into the error representation formula $\|\nabla(u - u_h)\|_{L^2(\Omega)}^2 = \|\nabla u\|_{L^2(\Omega)}^2 - \|\nabla u_h\|_{L^2(\Omega)}^2$. Apply this strategy to assess the quality of the error estimate η by repeating the computations made earlier and comparing with a reference quadratic solution computed on a 128×128 grid.