ONE AND A HALF HOURS

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL METHODS 3

xx January 2018

[Time of Examination]

Answer ALL parts of the question in **Section A** (worth 20 marks). Answer any **TWO** of the three questions in **Section B** (each worth 10 marks). If more than two questions from Section B are attempted, then credit will be given for the best two answers.

Calculators may be used in accordance with the university regulations.
Tables of mathematical formulae are available and may be used without restriction.

SECTION A

Question 1. [20 marks]

Given a load function $f(x)$, the deflection $u(x)$ in an elastic string of unit length can be shown to satisfy the boundary value problem,

$$-u''(x) = f(x), \quad x \in (0,1); \qquad u(0) = 0,\ u(1) = 0. \tag{D}$$

(a) Show that $u(x)$ satisfying equations (D) can be written as a Fourier series:

$$u(x) = \sum_{j=1}^{\infty} \frac{c_j}{\lambda_j} \sin(j\pi x),$$

where the c_j's are the Fourier coefficients of $f(x)$ and $\lambda_j = j^2\pi^2$ are the eigenvalues satisfying the problem

$$-u''(x) = \lambda\, u(x), \quad x \in (0,1); \qquad u(0) = 0,\ u(1) = 0. \tag{E}$$

[5 marks]

(b) We would like to construct approximations to the deflection $u(x)$ in the case that the load function $f(x)$ in (D) satisfies

$$f(x) = \begin{cases} x & \text{when } x \le 1/2, \\ 1 - x & \text{when } x > 1/2. \end{cases}$$

(i) Compute the first three Fourier coefficients $\{c_1, c_2, c_3\}$ of the function $f(x)$. Hint: use

$$\int_a^b x \sin(j\pi x)\, dx = \frac{1}{j^2\pi^2} \Big[\sin(j\pi x) - j\pi x \cos(j\pi x) \Big]_a^b .$$

[5 marks]

(ii) Construct the *truncated* Fourier series approximation $u^j(x)$ with $j = 1$ and $j = 3$ terms. Compare the approximate solutions at the point $x = 0.5$ with the exact value $u(0.5) = 0.0417$ (working to 4 decimal places).

[2 marks]

(c) Describe the construction of a *centered difference* approximation to the function $u(x)$ solving (D) when the elastic string is subdivided into n=4 subintervals of equal length h.

[4 marks]

(d) With $f(x)$ given in part (b), compute the centered difference approximation to $u(x)$ in the case n=4. Compare the exact and approximate solutions at the point $x = 0.5$.

Hint: you can simplify the linear algebra using the following result

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

[4 marks]

End of Section A.

SECTION B

Answer **TWO** of the three questions

Question 2. [10 marks]
A sphere floating in water (in the absence of friction and with its centre y metres above the water level) experiences a downward force due to gravity and an upward force due to buoyancy. Its motion can be determined by solving the second-order ODE

$$y''(t) = -10 + 10\,\frac{\rho_{\text{water}}}{4\rho}\left[2 + \frac{y^3}{r^3} - 3\frac{y}{r}\right],$$

where ρ is the density of the sphere and r is the radius.
You are given that a 0.5 metre radius sphere with density 0.75 that of water is initially at rest with its centre 0.25 metres below the level of the water.

(a) Express the equation of motion as a system of two first-order equations and state appropriate initial conditions.

[3 marks]

(b) Take a step size of $h = 0.1$ and carry out one step of Runge–Kutta to estimate the position of the centre of the sphere when $t = 0.1$. Work to an accuracy of 4 decimal places. [7 marks]

Question 3. [10 marks]
The amplitude of waves in a tank of water can be modelled by the solutions $u(x, t)$ of the wave equation

$$u_{tt} - u_{xx} = 0 \quad \text{in } (-1, 1) \times (0, \infty),$$

subject to the end conditions

$$u(-1, t) = 0, \quad u(1, t) = 0, \quad \text{for } t > 0.$$

(a) Using separation of variables, construct a set of linearly independent solutions $u(x, t)$ to this problem. (You will need to include all details to be awarded full marks.) [6 marks]

(b) Write down an infinite series representation of the solution to the above problem. Then, determine a precise expression for $u(x, t)$ in the case that the wave amplitude in the tank satisfies the initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin(2\pi x), \quad \text{for } x \in (-1, 1).$$

[4 marks]

Question 4. [10 marks]
An iron bar of uniform cross-section and length 2 units is differentially heated (one side only) so that the initial temperature is the unit step function $u(z)$, which is zero when $z < 0$ and unity when $z \geq 0$. Both ends of the bar ($z = \pm 1$) are then insulated and the bar is allowed to cool. The resulting temperature $T(z, t)$ inside the bar satisfies the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} \qquad (H)$$

together with the end conditions $\frac{\partial T}{\partial z} = 0$ at $z = \pm 1$.

(a) The boundary value problem is to be solved using the FTCS (forward time centred space) approximation method with a step size in z of $h = 0.4$.

   (i) For what values of z will the temperature T be computed? [1 mark]

   (ii) What is the maximum allowable time step that can be used? [1 mark]

   (iii) For a time step $k = 0.01$, carry out one step of the FTCS approximation method. Record all your answers to an accuracy of four decimal places. [5 marks]

(b) Describe an implicit finite difference scheme that could be used instead of FTCS. Explain how this method differs from the explicit (FTCS) scheme and state one advantage of implicit approximation methods.

[3 marks]

End of Section B.

END OF EXAMINATION PAPER