A Review of Preconditioning Techniques for Steady Incompressible Flow

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Outline

- PDEs
- Review : 1984 – 2005
- Update : 2005 – 2009

Outline

- PDEs
  -

$$\left.\begin{aligned}\vec{u}\cdot\nabla\,\vec{u}-\nu\nabla^2\,\vec{u}+\nabla p&=0\\ \nabla\cdot\,\vec{u}&=0\end{aligned}\right\}\quad\text{Navier–Stokes}$$

Outline

- PDEs
  - $$\left.\begin{aligned} \vec{u}\cdot\nabla\,\vec{u} - \nu\nabla^2\,\vec{u} + \nabla p = 0 \\ \nabla\cdot\,\vec{u} = 0 \end{aligned}\right\}\quad \text{Navier–Stokes}$$
  - $$\left.\begin{aligned} -\nabla^2\,\vec{u} + \nabla p = 0 \\ \nabla\cdot\,\vec{u} = 0 \end{aligned}\right\}\quad \text{Stokes}$$

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- PDEs
  - $$\left.\begin{aligned} \vec{u}\cdot\nabla\,\vec{u} - \nu\nabla^2\,\vec{u} + \nabla p &= 0 \\ \nabla\cdot\,\vec{u} &= 0 \end{aligned}\right\}$$ Navier–Stokes
  - $$\left.\begin{aligned} -\nabla^2\,\vec{u} + \nabla p &= 0 \\ \nabla\cdot\,\vec{u} &= 0 \end{aligned}\right\}$$ Stokes
  - $$\left.\begin{aligned} \mathcal{A}^{-1}\,\vec{u} + \nabla p &= 0 \\ \nabla\cdot\,\vec{u} &= 0 \end{aligned}\right\}$$ Darcy

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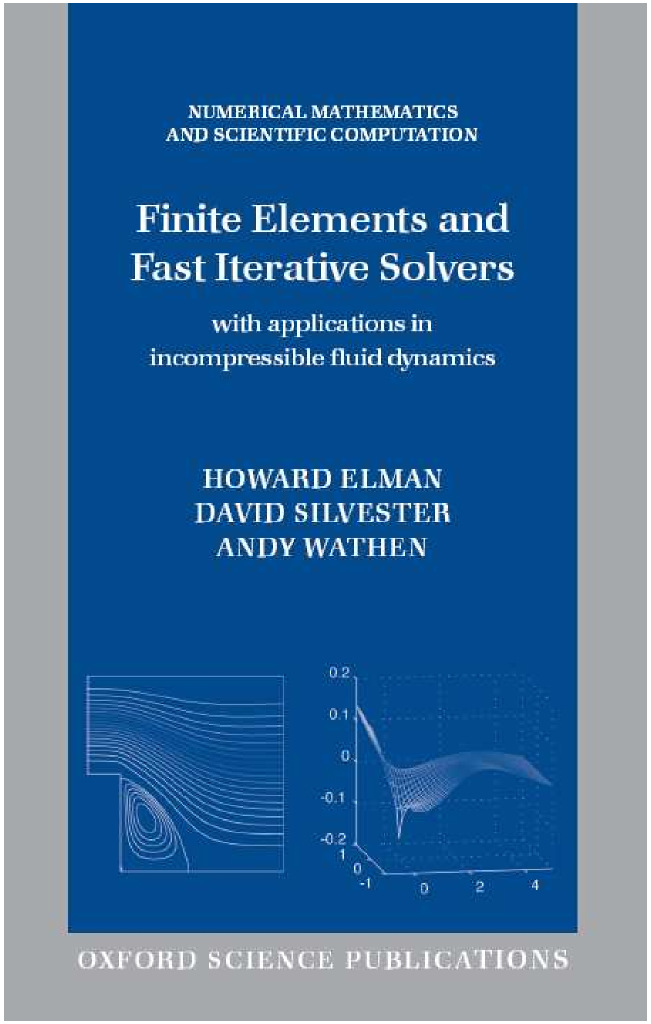

NUMERICAL MATHEMATICS
AND SCIENTIFIC COMPUTATION

Finite Elements and
Fast Iterative Solvers

with applications in
incompressible fluid dynamics

HOWARD ELMAN
DAVID SILVESTER
ANDY WATHEN

OXFORD SCIENCE PUBLICATIONS

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- PDEs
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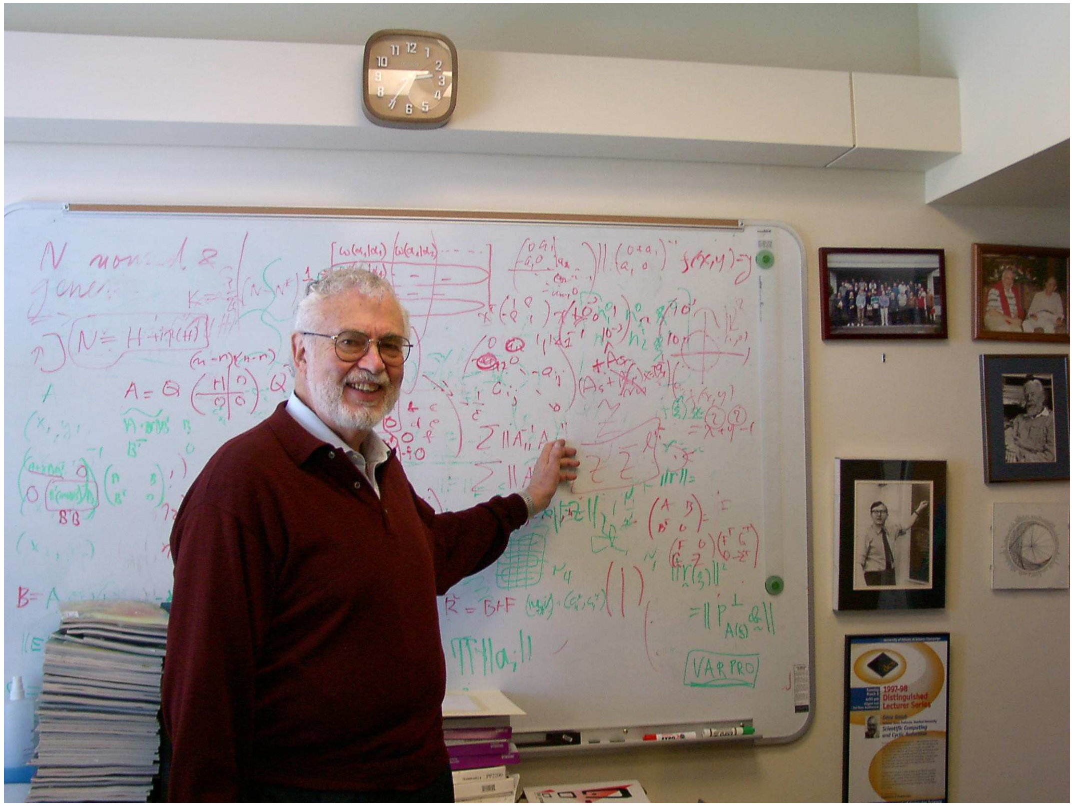

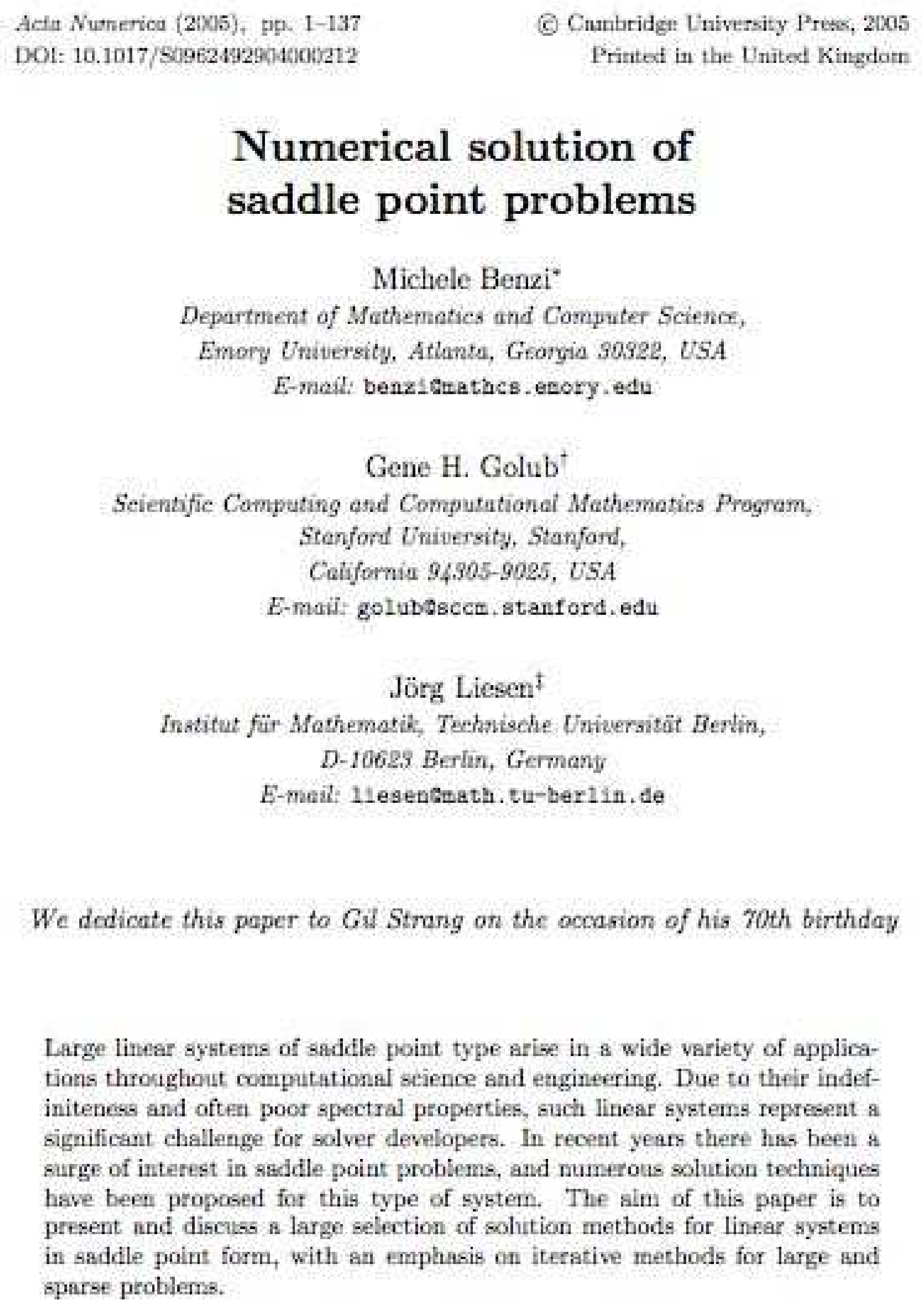

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Numerical solution of saddle point problems

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We dedicate this paper to Gil Strang on the occasion of his 70th birthday

Large linear systems of saddle point type arise in a wide variety of applications throughout computational science and engineering. Due to their indefiniteness and often poor spectral properties, such linear systems represent a significant challenge for solver developers. In recent years there has been a surge of interest in saddle point problems, and numerous solution techniques have been proposed for this type of system. The aim of this paper is to present and discuss a large selection of solution methods for linear systems in saddle point form, with an emphasis on iterative methods for large and sparse problems.

Steady-state Navier-Stokes equations

$$\begin{aligned} \vec{u}\cdot\nabla\,\vec{u} - \nu\nabla^2\,\vec{u} + \nabla p &= 0 \qquad \text{in } \Omega \\ \nabla\cdot\,\vec{u} &= 0 \qquad \text{in } \Omega. \end{aligned}$$

Boundary conditions:

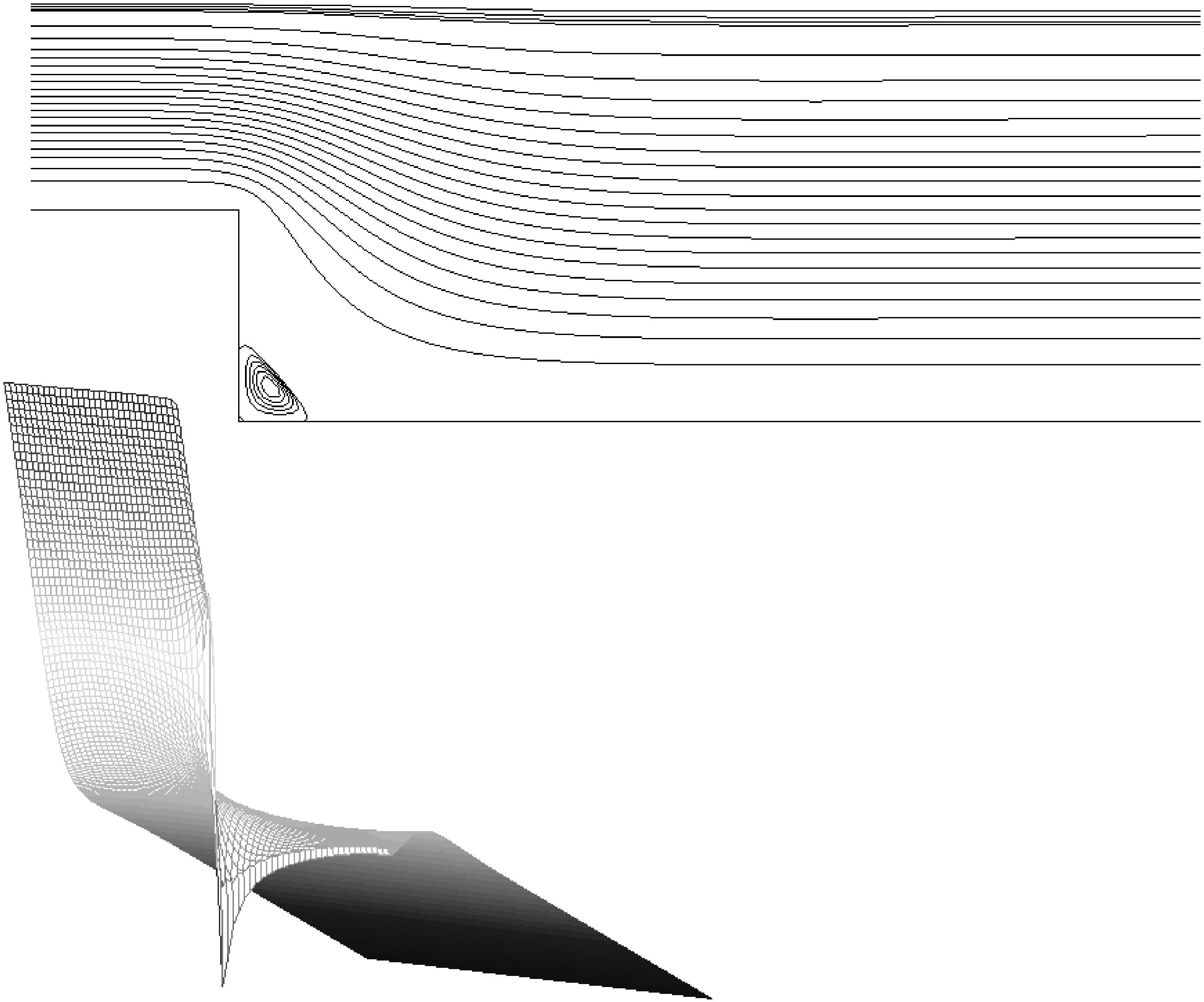
$$\vec{u} = \vec{w} \text{ on } \partial\Omega_D, \qquad \nu\frac{\partial\,\vec{u}}{\partial n} - \vec{n}p = \vec{0} \text{ on } \partial\Omega_N.$$

Picard linearization:
Given $\vec{u}^0$, compute $\vec{u}^1$, $\vec{u}^2$, …, $\vec{u}^k$ via

$$\begin{aligned} \vec{u}^k\cdot\nabla\,\vec{u}^{k+1} - \nu\nabla^2\,\vec{u}^{k+1} + \nabla p^{k+1} &= 0, \\ \nabla\cdot\,\vec{u}^{k+1} &= 0 \qquad \text{in } \Omega \end{aligned}$$

together with appropriate boundary conditions.

Example: Flow over a Step



Finite element matrix formulation

Introducing the basis sets

$$\mathbf{X}_h = \text{span}\{\vec{\phi}_i\}_{i=1}^{n_u}, \quad \text{Velocity basis functions;}$$
$$M_h = \text{span}\{\psi_j\}_{j=1}^{n_p}, \quad \text{Pressure basis functions.}$$

gives the discretized system:

$$\begin{pmatrix} N + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix},$$

with associated matrices

$$N_{ij} = (\mathbf{w}_h \cdot \nabla \vec{\phi}_i, \vec{\phi}_j), \quad \text{convection}$$
$$A_{ij} = (\nabla \vec{\phi}_i, \nabla \vec{\phi}_j), \quad \text{diffusion}$$
$$B_{ij} = -(\nabla \cdot \vec{\phi}_j, \psi_i), \quad \text{divergence .}$$

In the Stokes limit $\nu \to \infty$, we obtain the saddle-point problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

with $A \in \mathbb{R}^{n_u \times n_u}$, $B \in \mathbb{R}^{n_p \times n_u}$, and for a stable approximation

- $A = A^T$, $\mathbf{u}^T A \mathbf{u} > \mathbf{0}$, $\forall \mathbf{u} \neq \mathbf{0}$,
- rank$(B) = n_p$ if $\partial\Omega_N \neq \emptyset$, otherwise null$(B^T) = \{\mathbf{1}\}$.

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- rank$(B) = n_p$ if $\partial\Omega_N \neq \emptyset$, otherwise null$(B^T) = \{\mathbf{1}\}$.

Note that the Stokes system matrix is “highly indefinite”:

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{pmatrix} \begin{pmatrix} I & A^{-1}B^T \\ 0 & I \end{pmatrix}$$

where $S = BA^{-1}B^T$ is the Schur complement of A in $\mathcal{A}$.

A Simple Iterative Solver ...

Given scalar parameter α

Algorithm: Arrow, Hurwicz, Uzawa, 1958

```
for $k = 0, 1, \ldots$
    solve      $A\mathbf{u}_{k+1} = \mathbf{f} - B^T\mathbf{p}_k$
    compute    $\mathbf{p}_{k+1} = \mathbf{p}_k + \alpha B\mathbf{u}_{k+1}$
end
```

A Simple Iterative Solver ...

Given scalar parameter α

```
Algorithm:    Arrow, Hurwicz, Uzawa, 1958
for $k = 0, 1, \ldots$
    solve     $A\mathbf{u}_{k+1} = \mathbf{f} - B^T\mathbf{p}_k$
    compute   $\mathbf{p}_{k+1} = \mathbf{p}_k + \alpha B\mathbf{u}_{k+1}$
end
```

Eliminating $\mathbf{u}_{k+1}$ gives

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \alpha(BA^{-1}\mathbf{f} - BA^{-1}B^T\mathbf{p}_k).$$

This is simple (first order) Richardson iteration for the Schur Complement problem

$$BA^{-1}B^T\mathbf{p} = BA^{-1}\mathbf{f}.$$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

- 1984

Idea I : Introduce the pressure mass matrix $Q \sim I$ and use an iterated penalty approach:

```
for $k = 0, 1, \ldots$
    $A\mathbf{u}_{k+1} + B^T\mathbf{p}_{k+1} = \mathbf{f}$
    $B\mathbf{u}_{k+1} - \epsilon\, Q\mathbf{p}_{k+1} = -\epsilon\, Q\mathbf{p}_k$
end
```

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```
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    $B\mathbf{u}_{k+1} - \epsilon\, Q\mathbf{p}_{k+1} = -\epsilon\, Q\mathbf{p}_k$
end
```

Eliminating p_{k+1} gives

$$(A + \tfrac{1}{\epsilon}B^T Q^{-1} B)\mathbf{u}_{k+1} = \mathbf{f} - B^T\mathbf{p}_k$$

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \tfrac{1}{\epsilon}Q^{-1}B\mathbf{u}_{k+1}$$

- Michel Fortin & Roland Glowinski
  Augmented Lagrangian Methods
  North-Holland, 1984.

Idea II : Apply CG to the Schur Complement system, but replace inversion of A with n_A multigrid V-cycles A_*^{-1}:

$$BA_*^{-1}B^T\mathbf{p} = BA^{-1}\mathbf{f}.$$

- Rudiger Verfürth
  A Combined Conjugate Gradient-Multigrid algorithm for the Numerical Solution of the Stokes Problem
  IMA J. Numer. Anal., 4, 1984.

where it is established that this approach leads to “textbook” (grid independent) convergence if n_A is sufficiently big.

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

- 1984
- 1991

Silvester & Wathen : A Fledgling-Fast Solver

We introduce the pressure mass matrix Q and solve the (symmetric–) system $\mathcal{A}x = f$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

with the Conjugate Residual method and the diagonal preconditioning

$$\mathcal{P}^{-1} = \begin{pmatrix} D_A^{-1} & 0 \\ 0 & D_Q^{-1} \end{pmatrix}.$$

Analysis is elegant ...

Given

- ▷ “inf–sup” stability:

$$\gamma^2 \leq \frac{\mathbf{p}^T B A^{-1} B^T \mathbf{p}}{\mathbf{p}^T Q \mathbf{p}} \leq \Gamma^2 \qquad \forall \mathbf{p} \in \mathbb{R}^{n_p}$$

- ▷ the mass matrix preconditioner satisfies

$$\theta^2 \leq \frac{\mathbf{p}^T Q \mathbf{p}}{\mathbf{p}^T D_Q \mathbf{p}} \leq \Theta^2, \qquad \forall \mathbf{p} \in \mathbb{R}^{n_p}$$

- ▷ the vector Laplacian satisfies:

$$g(h) \leq \frac{\mathbf{u}^T A \mathbf{u}}{\mathbf{u}^T D_A \mathbf{u}} \leq 1 \qquad \forall \mathbf{u} \in \mathbb{R}^{n_u}$$

with $g(h) \to 0$ as $h \to 0$.

if $g(h) \to 0$ as $h \to 0$, then the eigenvalues of the preconditioned Stokes operator lie in the union of two real intervals

$$\left[-\Gamma^2\Theta^2\,,\,-\gamma\theta\sqrt{g(h)}\,\right] \cup \left[\,g(h)\,,\,1+\Gamma^2\Theta^2\right]$$

In particular, using simple diagonal scaling $g(h) = O(h^2)$, and the eigenvalues lie in

$$\left[\,-a\,,\,-bh\,\right] \cup \left[\,ch^2\,,\,d\right].$$

- Andrew Wathen & David Silvester
  Fast iterative solution of stabilised Stokes systems
  Part I: Using simple diagonal preconditioners,
  Report NA–91–04, Stanford University, 1991.
  SIAM J. Numer. Anal., 30, 1993.

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

- 1984
- 1991
- 1992

Silvester & Wathen : A Fast Solver

We solve the (symmetric–) system $\mathcal{A}x = f$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

using MINRES (due to Bernd Fischer) with the spectrally equivalent preconditioning

$$\mathcal{P}^{-1} = \begin{pmatrix} A_*^{-1} & 0 \\ 0 & D_Q^{-1} \end{pmatrix},$$

so that the vector Laplacian satisfies

$$\lambda^2 \leq \frac{\mathbf{u}^T A \mathbf{u}}{\mathbf{u}^T A_* \mathbf{u}} \leq 1 \qquad \forall \mathbf{u} \in \mathbb{R}^{n_u}$$

with $\lambda \geq \lambda_* > 0$ as $h \to 0$.

In this case the eigenvalues of the preconditioned Stokes operator lie in the union of two real intervals

$$\left[-\Gamma^2\Theta^2\,,\frac{1}{2}(\lambda-\sqrt{\lambda^2+4\gamma^2\theta^2\lambda}\,)\right]\cup\left[\lambda,1+\Gamma^2\Theta^2\,\right]$$

giving a convergence rate which is bounded away from one independently of the grid.

- Torgeir Rusten & Ragner Winther
  A Preconditioned Iterative Method for Saddle-Point Problems, SIAM J. Matrix. Anal. Appl., 13, 1992.
- Andrew Wathen & David Silvester
  Fast iterative solution of stabilised Stokes systems Part II: Using general block preconditioners,
  Report NA–92–xx, Stanford University.
  SIAM J. Numer. Anal., 31, 1994.

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

- 1984
- 1991
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- 1993

Gene's Idea: Use efficient inexact solve (e.g. multigrid V-cycle)

```
for $k = 0, 1, \ldots$
    $A\mathbf{u}_{k+1} = \mathbf{f} - B^T\mathbf{p}_k + \delta_k$
    $\mathbf{p}_{k+1} = \mathbf{p}_k + \alpha B\mathbf{u}_{k+1}$
end
```

with stopping criterion $\|\delta_k\| \leq \tau\|B\mathbf{u}_k\|$.

- Howard Elman & Gene Golub
  Inexact and Preconditioned Uzawa Algorithms for Saddle Point Problems
  SIAM J. Numer. Anal., 31, 1994.

where it is established that this approach leads to "textbook" (grid independent) convergence.

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

- 1984
- 1991
- 1992
- 1993
- 1994

Elman & Silvester : A Fledgling-Fast Solver

Now for the Navier-Stokes Equations. We solve the nonsymmetric system $\mathcal{F}x = f$

$$\begin{pmatrix} N + \nu A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

using GMRES with the spectrally equivalent preconditioning

$$\mathcal{P} = \begin{pmatrix} F_* & 0 \\ 0 & D_Q \end{pmatrix},$$

so that the vector convection-diffusion operator F_* satisfies

$$\lambda^2 \le \frac{\mathbf{u}^T(N + \nu A)\mathbf{u}}{\mathbf{u}^T F_* \mathbf{u}} \le \Lambda^2 \qquad \forall \mathbf{u} \in \mathbb{R}^{n_u}.$$

In this case the eigenvalues of the preconditioned Oseen operator lie in two boxes on either side of the imaginary axis in the complex plane.

- Gives a convergence rate which is bounded away from one independently of the grid.

In this case the eigenvalues of the preconditioned Oseen operator lie in two boxes on either side of the imaginary axis in the complex plane.

- Gives a convergence rate which is bounded away from one independently of the grid.

- Convergence deteriorates as $\nu \to 0$ because of low-quality Schur complement approximation

$$B(N + \nu A)^{-1} B^T \sim D_Q.$$

- Howard Elman & David Silvester
  Fast Nonsymmetric Iterations and Preconditioning for Navier-Stokes Equations,
  Report UMIACS–TR–94–66, University of Maryland.
  SIAM J. Sci. Comput, 17, 1996.

Block triangular preconditioning

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

A perfect preconditioner is given by

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}$$

Here $F = N + \nu A$ and $S = BF^{-1}B^T$.

Block triangular preconditioning

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

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$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}$$

Here $F = N + \nu A$ and $S = BF^{-1}B^T$. Note that

$$\underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} \underbrace{\begin{pmatrix} F & B^T \\ 0 & -S \end{pmatrix}}_{\mathcal{P}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

- 1984
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- 1999

A Fast Solver

Given

$$\begin{pmatrix} N+\nu A & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}$$

With discrete matrices

$$\begin{aligned} N_{ij} &= (\,\vec{w}_h \cdot \nabla \vec{\phi}_i,\, \vec{\phi}_j), && \text{convection} \\ A_{ij} &= (\nabla \vec{\phi}_i, \nabla \vec{\phi}_j), && \text{diffusion} \\ B_{ij} &= -(\nabla \cdot \vec{\phi}_j, \psi_i), && \text{divergence} \end{aligned}$$

For an efficient block diagonal (or triangular) preconditioner $\mathcal{P}$ we require a sparse approximation to the “exact” Schur complement

$$S^{-1} = (B(N+\nu A)^{-1}B^T)^{-1} =: (BF^{-1}B^T)^{-1}$$

Schur complement approximation – I

Introducing associated pressure matrices

$$\begin{aligned} A_p &\sim (\nabla\psi_i, \nabla\psi_j), && \text{diffusion} \\ N_p &\sim (\vec{w}_h \cdot \nabla\psi_i, \psi_j), && \text{convection} \\ F_p &= \nu A_p + N_p, && \text{convection-diffusion} \end{aligned}$$

gives the “pressure convection-diffusion preconditioner”:

$$(BF^{-1}B^T)^{-1} \approx Q^{-1} F_p \underbrace{A_p^{-1}}_{\text{AMG}}$$

- David Kay & Daniel Loghin (& Andy Wathen)
  A Green’s function preconditioner for the steady-state Navier-Stokes equations
  Report NA–99/06, Oxford University Computing Lab.
  SIAM J. Sci. Comput, 24, 2002.

Schur complement approximation – II

Introducing the diagonal of the velocity mass matrix

$$M_* \sim M_{ij} = (\vec{\phi}_i, \vec{\phi}_j),$$

gives the “least-squares commutator preconditioner”:

$$(BF^{-1}B^T)^{-1} \approx (\underbrace{BM_*^{-1}B^T}_{\text{AMG}})^{-1}(BM_*^{-1}FM_*^{-1}B^T)(\underbrace{BM_*^{-1}B^T}_{\text{AMG}})^{-1}$$

- Howard Elman (& Ray Tuminaro et al.)
  Preconditioning for the steady-state Navier-Stokes equations with low viscosity,
  SIAM J. Sci. Comput, 20, 1999.
  Block preconditioners based on approximate commutators,
  SIAM J. Sci. Comput, 27, 2006.

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

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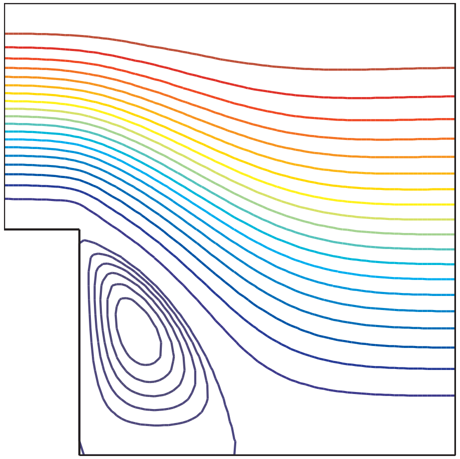
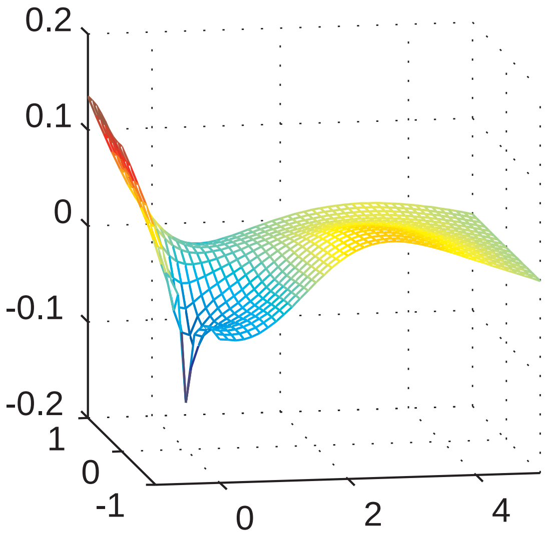
IFISS2.2 computational results





Final step of Oseen iteration : $R = 2/\nu$

GMRES iterations using Q_2–Q_1 (Q_1–P_0) — tol $= 10^{-6}$

Exact Least Squares Commutator

$1/h$	$R = 10$	$R = 100$	$R = 200$
5	15 (15)	17 (16)	
6	19 (21)	21 (22)	29 (32)
7	23 (31)	29 (32)	29 (30)

- Update : 2005 – 2009

- Howard Elman & Alison Ramage & David Silvester,
Algorithm 866 : IFISS, a Matlab toolbox for modelling incompressible flow,
ACM Trans. Math. Soft. 33, 2007.
- Howard Elman (& Silvester et al.)
Least squares preconditioners for stabilized discretizations of the Navier-Stokes equations,
SIAM J. Sci. Comput, 30, 2007.
- Howard Elman & Ray Tuminaro
Boundary conditions in approximate commutator preconditioners for the Navier-Stokes equations,
Report UMIACS–TR–2009–02, University of Maryland.

Darcy Flow Equations

Given boundary “pressure head” data g, and an isotropic permeability matrix $\mathcal{A} = \mu I_2$, such that

$$0 < \mu_* \leq \mu(\vec{x}) \leq \mu^* < \infty \quad \forall \vec{x} \in D \subset \mathbb{R}^2 :$$

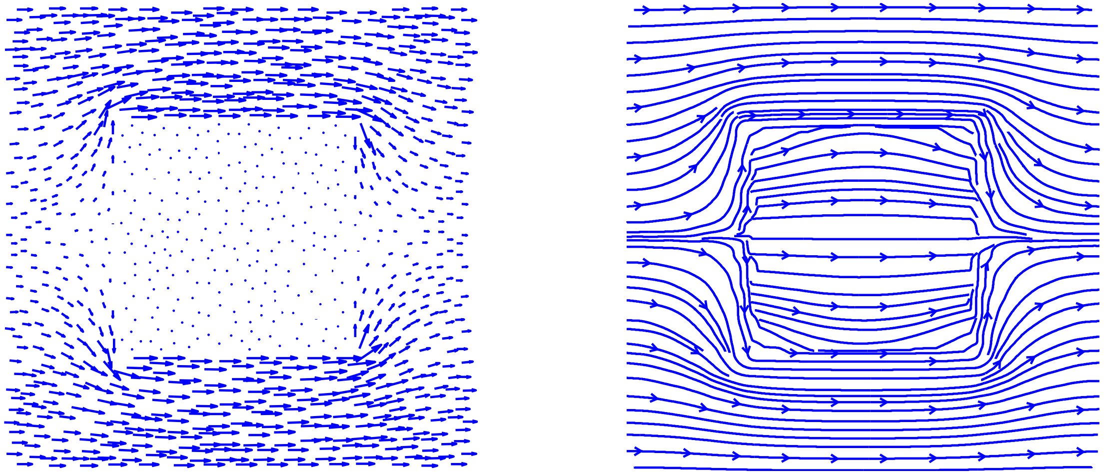
we want to compute the pair $(\vec{u}, p)$ such that

$$\mathcal{A}^{-1}\,\vec{u} + \nabla p = 0 \quad \text{in } D,$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } D,$$

$$p = g \quad \text{on } \Gamma_D; \qquad \vec{u} \cdot \vec{n} = 0 \quad \text{on } \Gamma_N.$$

Example: Groundwater Flow



Finite element matrix formulation

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$$M_h = \text{span}\{\psi_j\}_{j=1}^{n_p}; \quad \text{Pressure basis functions.}$$

gives the discretized system

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix},$$

with associated matrices:

$$M_{ij} = (\mu^{-1}\,\vec{\phi}_i,\, \vec{\phi}_j), \quad \text{local mass}$$

$$B_{ij} = -(\nabla \cdot \vec{\phi}_j, \psi_i), \quad \text{divergence}$$

Fast Saddle-Point Solver

We solve the (symmetric–) system $\mathcal{L}x = f$

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix}$$

with $M \in \mathbb{R}^{n_u \times n_u}$, and $B \in \mathbb{R}^{n_p \times n_u}$, using MINRES with the block diagonal preconditioning

$$\mathcal{P}^{-1} = \begin{pmatrix} M_*^{-1} & 0 \\ 0 & Q_*^{-1} \end{pmatrix}.$$

Given that the blocks M_* (mass matrix diagonal) and Q_* (via AMG) satisfy

$$\gamma^2 \le \frac{\mathbf{u}^T M \mathbf{u}}{\mathbf{u}^T M_* \mathbf{u}} \le \Gamma^2 \qquad \forall \mathbf{u} \in \mathbb{R}^{n_u},$$

$$\theta^2 \le \frac{\mathbf{p}^T B M_*^{-1} B^T \mathbf{p}}{\mathbf{p}^T Q_* \mathbf{p}} \le \Theta^2 \qquad \forall \mathbf{p} \in \mathbb{R}^{n_p},$$

then the eigenvalues of the preconditioned problem,

$$\begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \lambda \begin{pmatrix} M_* & 0 \\ 0 & Q_* \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix},$$

lie in the union of intervals that are bounded away from zero and $\pm\infty$, independently of h and μ.

- Catherine Powell & David Silvester
  Optimal Preconditioning for Raviart-Thomas Mixed Formulation of Second-Order Elliptic Problems
  SIAM J. Matrix Anal. Appl., 25, 2004.
- David Silvester & Catherine Powell
  PIFISS Potential (Incompressible) Flow & Iterative Solution Software guide,
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  MIMS Eprint 2007.14.

For an extension to the more practical case of a random permeability field $\mathcal{A}^{-1} := T^{-1}(\vec{x}, \omega)$:

- Oliver Ernst, Catherine Powell, David Silvester & Elisabeth Ullmann, Efficient Solvers for a Linear Stochastic Galerkin Mixed Formulation of Diffusion Problems with Random Data,
  SIAM J. Sci. Comput, 31, 2009.