MATH46052|66052 **Approximation Theory** Tutorial Sheet IV

Let X be a finite-dimensional subspace of a Hilbert space H, and let $u \in H$ and $u_h \in X$ satisfy

$$a(u,v) = \ell(v) \qquad \forall v \in H, \tag{V}$$
$$a(u_h,v_h) = \ell(v_h) \qquad \forall v_h \in X, \tag{V_h}$$

respectively, where $a(\cdot,\cdot)$ is a symmetric bilinear form on H and $\ell : H \to \mathbb{R}$ is a linear functional.

1. Show that the Galerkin solution u_h is the *best approximation* to $u \in H$ when measured in the energy norm $\|u\|_E$, that is

$$\|u - u_h\|_E \le \|u - v_h\|_E \quad \forall v_h \in X,$$

   where $\|v\|_E^2 = a(v,v)$ for $v \in H$.

2. Suppose that we have an enhanced approximation space W so that $X \subset W \subset H$, with associated solution $u_h^* \in W$ satisfying

$$a(u_h^*, v) = \ell(v) \qquad \forall v \in W.$$

   Use the fact that $a(u - u_h^*, v) = 0$, for all $v \in W$ to show that

$$\|u - u_h\|_E^2 = \|u - u_h^*\|_E^2 + \|u_h^* - u_h\|_E^2 .$$

   Deduce that $\|u - u_h\|_E \ge \|u - u_h^*\|_E$.

3. Suppose that the enhanced space W can be written as $W = X \oplus Z$ and that a strengthened Cauchy–Schwarz inequality holds

$$|a(v_h, z_h)| \le \gamma \|v_h\|_E \|z_h\|_E \qquad \forall v_h \in X, \forall z_h \in Z$$

   with $0 \le \gamma < 1$. Consider the simplified error representation problem: find $e_h \in Z$ satisfying

$$a(e_h, z_h) = \ell(z_h) - a(u_h, z_h) \qquad \forall z_h \in Z. \tag{Z_h^*}$$

   Show that $e_h \in Z$ is *equivalent* to $u_h^* - u_h \in W$

$$\|e_h\|_E^2 \le \|u_h^* - u_h\|_E^2 \le \frac{1}{1-\gamma^2} \|e_h\|_E^2$$

   where $0 \le \gamma < 1$ is the CBS constant. [Hint: to establish the left-hand inequality use that fact that $a(u_h^* - u_h, z_h) = a(e_h, z_h)$ for all $z_h \in Z$. To establish the right-hand inequality, write $u_h^* - u_h = v_h + z_h$ with $v_h \in X$ and $z_h \in Z$ and use the inequality $a^2 + b^2 - 2\gamma ab \ge (1-\gamma^2)b^2$.]