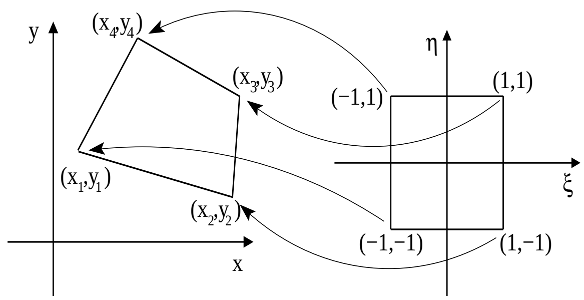
1. Find explicit expressions for the basis functions $\{\chi_i\}_{i=1}^4$ defined on the square $(\xi, \eta) \in [-1, 1] \times [-1, 1]$ illustrated on the right below, given that they satisfy the standard interpolation properties:

$$\chi_i(\xi, \eta) = \begin{cases} 1 & \text{at vertex } i, \\ 0 & \text{at the other three vertices,} \\ & \text{of the form } (a + b\xi)(c + d\eta). \end{cases}$$

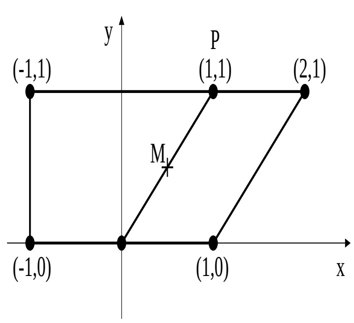


Verify that the bilinear mapping to the general quadrilateral shown on the left of the picture is given by

$$\begin{aligned} x &= x_1\chi_1 + x_2\chi_2 + x_3\chi_3 + x_4\chi_4 \\ y &= y_1\chi_1 + y_2\chi_2 + y_3\chi_3 + y_4\chi_4. \end{aligned}$$

Show that horizontal straight lines $\eta = C$ are mapped onto straight lines in (x, y) coordinates.

2. For the two-dimensional domain illustrated, show that the bilinear interpolant that takes on the value one at vertex P and zero at the other vertices gives different values in the two quadrilaterals at the midpoint M on the common edge. (This means that a global interpolant obtained by piecing together the two local approximations will be discontinuous.) Show that the bilinearly mapped interpolant (as defined in the first question) is however, continuous along the common edge.



3. Given a general triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) the barycentric coordinates (L_1, L_2, L_3) can be defined by the mapping:

$$\begin{aligned} x &= x_1 L_1 + x_2 L_2 + x_3 L_3 \\ y &= y_1 L_1 + y_2 L_2 + y_3 L_3 \\ 1 &= L_1 + L_2 + L_3. \end{aligned}$$

By inverting this mapping, show that $L_i = \frac{1}{2|\triangle|}(a_i + b_i x + c_i y), \quad i = 1, 2, 3,$ where $|\triangle|$ is the area of the triangle and deduce explicit expressions for the constants a_i, b_i and c_i.

4. A superlinear interpolant of the form $u^{①} = aL_1 + bL_2 + cL_3 + dL_1L_2L_3$ where (L_1, L_2, L_3) are barycentric coordinates, is characterised by the local basis set $\{N_i\}_{i=1}^4$ satisfying the standard interpolation property:

$$N_i(x, y) = \begin{cases} 1 & \text{at node } i, \\ 0 & \text{at the other three nodes.} \end{cases}$$

Derive the form of the four basis functions if the nodes correspond to the three vertices and the centroid of the triangle.
By evaluating the interpolant of two triangles sharing a common edge, PQ say, show that the concatenated global interpolant is continuous. Is the normal derivative of the global interpolant also continuous?

5. A linear interpolant of the form $u^{①} = aL_1 + bL_2 + cL_3$ where (L_1, L_2, L_3) are barycentric coordinates, is characterised by the $\boldsymbol{P}_1$ basis set $\{N_i\}_{i=1}^3$ satisfying the standard interpolation property:

$$N_i(x, y) = \begin{cases} 1 & \text{at node } i, \\ 0 & \text{at the other two nodes.} \end{cases}$$

Derive the form of the three basis functions if the nodes correspond to the three mid-edge points of the triangle
By evaluating the interpolant of two triangles sharing a common edge, PQ say, show that the concatenated global interpolant is not continuous.

6. A quadratic interpolant of the form $u^{①} = aL_1 + bL_2 + cL_3 + dL_1L_2 + eL_1L_3 + fL_2L_3$ where (L_1, L_2, L_3) are barycentric coordinates, is characterised by the $\boldsymbol{P}_2$ basis set $\{N_i\}_{i=1}^6$ satisfying the standard interpolation property:

$$N_i(x, y) = \begin{cases} 1 & \text{at node } i, \\ 0 & \text{at the other five nodes.} \end{cases}$$

Derive the form of the six basis functions if the nodes correspond to the three vertices and the three mid-edges of the triangle.
By evaluating the interpolant of two triangles sharing a common edge, PQ say, show that the concatenated global interpolant is continuous. Is the normal derivative of the global interpolant also continuous?