1. Since $u - u_h \in V$, we use the first property and note that $a$ is bilinear:

$$\begin{aligned}\|u-u_h\|^2_{H^1(\Omega)} &\le \tfrac{1}{\gamma} a(u-u_h, u-u_h) \\ &= \tfrac{1}{\gamma} a(u-u_h, u-v_h+v_h-u_h) \qquad \forall v_h \in V_h \\ &= \tfrac{1}{\gamma} a(u-u_h, u-v_h) + a(u-u_h, v_h-u_h).\end{aligned}$$

Next since $a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$ (G–O) and $v_h - u_h \in V_h$ we get

$$\|u-u_h\|^2_{H^1(\Omega)} \le \tfrac{1}{\gamma} a(u-u_h, u-v_h).$$

Then, using the second property

$$\|u-u_h\|^2_{H^1(\Omega)} \le \tfrac{\Gamma}{\gamma} \|u-u_h\|_{H^1(\Omega)} \|u-v_h\|_{H^1(\Omega)}.$$

Assuming $u \ne u_h$, we can divide through by $\|u-u_h\|_{H^1(\Omega)}$ to give

$$\|u-u_h\|_{H^1(\Omega)} \le \tfrac{\Gamma}{\gamma} \|u-v_h\|_{H^1(\Omega)}.$$

The result also holds in the case $\|u-u_h\|_{H^1(\Omega)} = 0 \iff u = u_h$. ♡

2. (a) Let $u \in X$ be the solution of $(V)$. Suppose $v \in X$ and define $w = v - u \in X$. Using the symmetry of the bilinear form gives

$$\begin{aligned}F(v) &= F(u+w) \\ &= \frac{1}{2} a(u+w, u+w) - \ell(u+w) \\ &= \frac{1}{2} a(u,u) - \ell(u) + \frac{1}{2} a(u,w) + \frac{1}{2} \underbrace{a(w,u)}_{\frac{1}{2}a(u,w)} - \ell(w) + \frac{1}{2} a(w,w) \\ &= \frac{1}{2} a(u,u) - \ell(u) + \frac{1}{2} a(w,w) + \underbrace{a(u,w) - \ell(w)}_{=0} \\ &= F(u) + \frac{1}{2} a(w,w).\end{aligned}$$

Finally, $a(w,w) = \|w\|^2_E \ge 0$, thus $F(v) \ge F(u)$ as required. ♡

(b) Noting that $u_h \in X_h \subset X$ and using Galerkin orthogonality gives

$$\begin{aligned}\|u-u_h\|^2_E &= a(u-u_h, u-u_h) \\ &= a(u-u_h, u) - \underbrace{a(u-u_h, u_h)}_{=0} \\ &= a(u-u_h, u) + a(u-u_h, u_h) \\ &= a(u-u_h, u+u_h) \\ &= a(u,u) - a(u_h,u_h) = \|u\|^2_E - \|u_h\|^2_E. \qquad ♡\end{aligned}$$

Thus, since $\|u - u_h\|_E^2 \geq 0$, we have shown that $\|u\|_E \geq \|u_h\|_E$. $(*)$
To see the connection with $(M)$ we note that $F(u) \leq F(u_h)$. Thus, $\frac{1}{2}\|u\|_E^2 - \ell(u) \leq \frac{1}{2}\|u_h\|_E^2 - \ell(u_h)$. Setting $v = u$ in $(V)$ and $v_h = u_h$ in the Galerkin formulation, we find that $-\frac{1}{2}\|u\|_E^2 \leq -\frac{1}{2}\|u_h\|_E^2$ which can be arranged to give the result $(*)$ above.

3. (a) First, if $u$ satisfies $(E)$ then $u \in C^2(\Omega)$, which means that $u \in \mathcal{H}^1(\Omega)$. Since $u = 0$ on $\partial\Omega$ we deduce that $u \in X$.

Multiplying both sides of $(E)$ by $v \in X$ and integrating over $\Omega$ gives the *strong* formulation

$$-\int_\Omega v\nabla^2 u = \lambda \int_\Omega uv \quad \forall\, v \in X.$$

Next, integrating by parts gives the *weak* formulation

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv + \int_{\partial\Omega} v\nabla u \cdot \vec{n} \quad \forall\, v \in X,$$

where $\vec{n}$ is the outward pointing normal to the boundary: this integration by parts is permitted because $v \in \mathcal{H}^1(\Omega)$. The fact that $v = 0$ on $\partial\Omega$ establishes that $u$ satisfies $(V)$.

To establish that $\lambda > 0$ we we make the specific choice $v = u$ in $(V)$. This gives

$$\int_\Omega \nabla u \cdot \nabla u = \lambda \int_\Omega u^2.$$

Applying the (P–F) inequality then leads to the estimate

$$0 \leq \|u\|^2 \leq L^2 \|\nabla u\|^2 = L^2 \lambda \|u\|^2. \qquad (\star)$$

This establishes that $\lambda \geq 0$. To show that $\lambda > 0$ we suppose that $\lambda = 0$ in $(\star)$ so that

$$0 \leq \|u\|^2 \leq L^2 \|\nabla u\|^2 = 0.$$

This means that $\|u\| = 0 \implies u = 0$ almost everywhere in $\Omega$. This is a contradiction because an eigenfunction satisfying $(E)$ cannot be the zero function. Thus $\lambda > 0$ as required. ♡

(b) Introducing a bilinear form $a(\cdot,\cdot) : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$, the weak formulation can be written as follows:

$$\text{find } (\lambda \in \mathbb{R}, u \in X) \text{ such that } a(u, v) = \lambda\,(u, v) \quad \forall v \in X.$$

The corresponding Galerkin formulation is the following:

$$\text{find } (\lambda_h \in \mathbb{R}, u_h \in X_h) \text{ such that } a(u_h, v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in X_h \subset X.$$

Writing $u_h = \sum_{j=1}^k u_j\phi_j$ and setting $v_h = \phi_i$ gives

$$a\left(\textstyle\sum_{j=1}^k u_j\phi_j, \phi_i\right) = \lambda_h\left(\textstyle\sum_{j=1}^k u_j\phi_j, \phi_i\right).$$

That is,

$$\sum_{j=1}^{k} u_j \, a(\phi_j, \phi_i) = \lambda_h \sum_{j=1}^{k} u_j \, (\phi_j, \phi_i).$$

This represents a $k \times k$ system $A\mathbf{x} = \lambda_h \, Q\mathbf{x}$ with matrix coefficients $A_{ij} = a(\phi_j, \phi_i)$ and $Q_{ij} = (\phi_j, \phi_i)$. $\heartsuit$

(c) case (i)

When $h = 1/2$ there are 8 triangles and one interior degree of freedom at the central point $(\frac{1}{2}, \frac{1}{2})$. There are six elements that meet at this vertex. The $1 \times 1$ Galerkin system is thus given by

$$4u_1 = \lambda_h \cdot 6 \cdot \frac{h^2}{12} u_1$$

and the eigenvalue estimate is given by $\lambda_h = 8/h^2 = 32$.

case (ii)

When $h = 1/3$ there are 18 triangles and four interior degrees of freedom at the points $(\frac{1}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{2}{3})$. Assembling the $3 \times 3$ Galerkin system gives the generalised eigenvalue problem

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \lambda_h \cdot \tfrac{h^2}{12} \begin{bmatrix} 6 & 1 & 1 & 1 \\ 1 & 6 & 0 & 1 \\ 1 & 0 & 6 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Solving this problem the *smallest* eigenvalue is given by $\lambda_h^1 = 25.3763$. Note that the analytic solution to this eigenproblem can be found using Fourier analysis. This gives

$$\begin{aligned} \lambda &= (i^2 + j^2)\,\pi^2, \qquad i = 1, 2, 3 \ldots \quad j = 1, 2, 3, \ldots \\ u_{i,j} &= \sin(i\pi x)\sin(j\pi y). \end{aligned}$$

Thus the smallest eigenvalue is given by $\lambda^1 = 2\pi^2 = 19.7392\ldots$.