1. The basis functions χ_j on the reference element are as follows:

$$\left.\begin{aligned}
\chi_1(\xi,\eta) &= \frac{(\xi-1)(\eta-1)}{4}, && \text{unity on node 1,}\\
\chi_2(\xi,\eta) &= \frac{-(\xi+1)(\eta-1)}{4}, && \text{unity on node 2,}\\
\chi_3(\xi,\eta) &= \frac{(\xi+1)(\eta+1)}{4}, && \text{unity on node 3,}\\
\chi_4(\xi,\eta) &= \frac{-(\xi-1)(\eta+1)}{4}, && \text{unity on node 4.}
\end{aligned}\right\} \quad (\star)$$

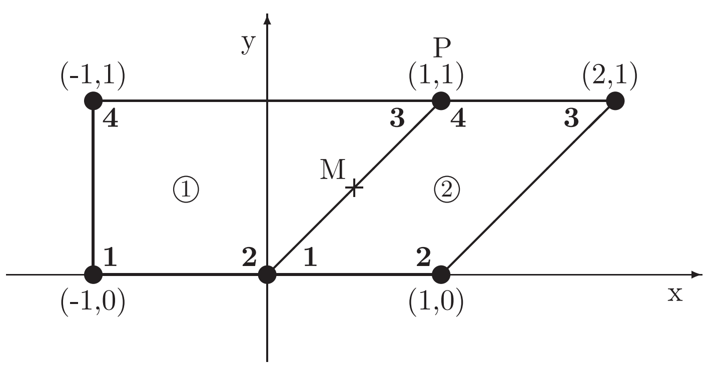
To isoparametrically map from the reference element to a physical element we take

$$x(\xi,\eta) = \sum_{i=1}^{4} x_i\chi_i, \qquad y(\xi,\eta) = \sum_{i=1}^{4} y_i\chi_i,$$

where the (x_i, y_i) are the global coordinates of node i within the element being mapped to. If $\eta = C$ then the basis functions χ_j, $j = 1,2,3,4$ are all linear functions in the ξ coordinate. Moreover the (mapped) function y is a linear function of x. (x and y are both linear functions in ξ.)

2. (**Problem 1.4** in [ESW])

The following picture shows the two elements labelled A and B and defines a local node numbering system.



Similarly to the previous question we want to construct the basis functions. In particular we want the basis function on element ① that is unity at point P. This is defined by

$$\psi_P^{①}(x,y) = \begin{cases} 1 \text{ at } (1,1) \\ 0 \text{ on } x=-1 \\ 0 \text{ on } y=0 \\ \text{bilinear on ①} \end{cases},$$

so $\psi_P^{①}(x,y) = \frac{1}{2}y(x+1)$. Similarly we have

$$\psi_P^{②}(x,y) = \begin{cases} 1 \text{ at } (1,1) \\ 0 \text{ on } x=2 \\ 0 \text{ on } y=0 \\ \text{bilinear on ②} \end{cases},$$

so $\psi_P^{②}(x,y) = y(2-x)$. The point M is at $(\frac{1}{2},\frac{1}{2})$ so we have

$$\psi_P^{①}(M) = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2} = \tfrac{3}{8}, \qquad \psi_P^{②}(M) = \tfrac{1}{2}\cdot\tfrac{3}{2} = \tfrac{3}{4},$$

so we must have a discontinuity at that point.

The isoparametric mapping to element ① is given by

$$x(\xi,\eta) = -\chi_1 + \chi_3 - \chi_4, \qquad y(\xi,\eta) = \chi_3 + \chi_4, \qquad (* \to ①)$$

and the mapping to element ② is

$$x(\xi,\eta) = \chi_2 + 2\chi_3 + \chi_4, \qquad y(\xi,\eta) = \chi_3 + \chi_4. \qquad (* \to ②)$$

There is no need to invert the mapping. The figure suggests that when element ① is mapped from the reference element, the edge containing M is mapped from the right hand edge ($\xi = 1, \eta \in [-1,1]$) of the reference element. This can be rigorously shown by using $(* \to ①)$ and $(\star)$. Thus, for element ① the right hand edge of the reference element maps to

$$\begin{aligned} (x(1,\eta), y(1,\eta)) &= (-\chi_1 + \chi_3 - \chi_4, \chi_3 + \chi_4), \qquad \eta \in [-1,1], \\ &= \left(0 + \tfrac{2(\eta+1)}{4} + 0, \tfrac{2(\eta+1)}{4} + 0\right), \\ &= \left(\tfrac{\eta+1}{2}, \tfrac{\eta+1}{2}\right), \qquad \eta \in [-1,1]. \end{aligned}$$

This is indeed the edge containing M, parameterised linearly by η, starting at (0,0) when $\eta = -1$ and running in a straight line to (1,1) when $\eta = 1$.

In the same way it can be verified using $(* \to ②)$ and $(\star)$ that in element ② the left hand edge of the reference element maps to

$$\begin{aligned} (x(-1,\eta), y(-1,\eta)) &= (\chi_2 + 2\chi_3 + \chi_4, \chi_3 + \chi_4), \qquad \eta \in [-1,1], \\ &= \left(0 + 2\cdot 0 + \tfrac{2(\eta+1)}{4}, 0 + \tfrac{2(\eta+1)}{4}\right), \\ &= \left(\tfrac{\eta+1}{2}, \tfrac{\eta+1}{2}\right), \qquad \eta \in [-1,1]. \end{aligned}$$

Note that this is identical to the mapping of ① of the common edge.

With respect to the basis functions for element ① we need to consider the right-hand edge of the reference element, and for the basis functions of ② we consider the left-hand edge of the reference element. In both cases when we do the mappings, the point P is at the top ($\eta = 1$). So we have the following equivalences between basis functions:

$$\psi_P^{①} \equiv \chi_3 \quad \text{and} \quad \psi_P^{②} \equiv \chi_4.$$

When the reference element is mapped to ①, the function $\psi_P^{①}$ (mapped from χ_3) is 0 at $(0,0)$ and 1 at $(1,1)$ and is linear along the edge. Similarly, when we map to ②, the function $\psi_P^{②}$ (mapped from χ_4) is also 0 at $(0,0)$ and 1 at $(1,1)$ and is also linear along the edge. Finally, $\psi_P^{①}$ and $\psi_P^{②}$ are linear on the common edge and since they both agree at two points we must have continuity along the entire edge.

3. The mapping can be written as a 3×3 linear system where

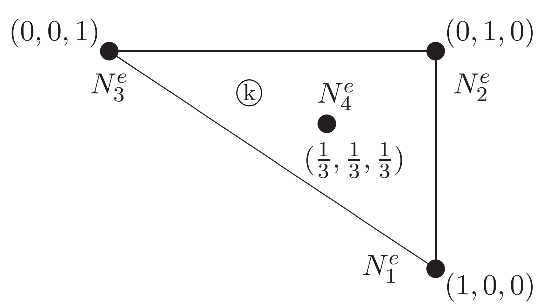
$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}.$$

Computing the inverse of the matrix using Cramer's rule and premultiplying the left-hand side vector by the inverse gives the stated result with coefficients given by

$$\begin{array}{ccc} a_1 = x_2y_3 - x_3y_2 & a_2 = x_3y_1 - x_1y_3 & a_3 = x_1y_2 - x_2y_1, \\ b_1 = y_2 - y_3, & b_2 = y_3 - y_1, & b_3 = y_1 - y_2, \\ c_1 = x_3 - x_2, & c_2 = x_1 - x_3, & c_3 = x_2 - x_1. \end{array}$$

Note that the determinant of the transformation matrix is $2|\Delta| \neq 0$, so the transformation is always well defined.

4. The nodes are labelled 1, 2, 3 and 4 and have the area coordinates illustrated in the following figure.



We can find the element basis functions using the interpolation property. For example,

$$N_1^e = \alpha_1 L_1 + \beta_1 L_2 + \gamma_1 L_3 + \delta_1 L_1 L_2 L_3.$$

Thus,

$$1 = \alpha_1, \quad 0 = \beta_1, \quad 0 = \gamma_1,$$
$$0 = \tfrac{1}{3}\alpha_1 + \tfrac{1}{3}\beta_1 + \tfrac{1}{3}\gamma_1 + \tfrac{1}{27}\delta_1 \implies \delta_1 = -9.$$

That is,

$$N_1^e = L_1 - 9L_1L_2L_3.$$

Similarly the other two vertex basis functions are

$$N_2^e = L_2 - 9L_1L_2L_3, \quad N_3^e = L_3 - 9L_1L_2L_3,$$

and the centroid basis function is

$$N_4^e = 27L_1L_2L_3.$$

Note that the basis functions (necessarily) sum to unity. Note also that the centroid basis function is zero on the boundary of the triangle. The element trial function is

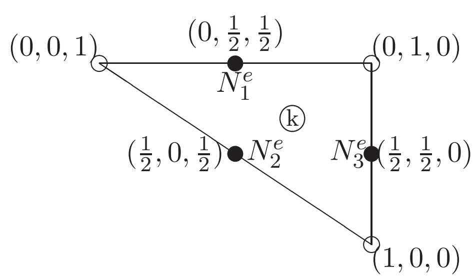
$$u_h^{(k)} = u_{k,1}N_1^e + u_{k,2}N_2^e + u_{k,3}N_3^e + u_{k,4}N_4^e.$$

On an element edge, say the edge e_j opposite vertex j, we have that $L_j = 0$ so that the approximation is linear; e.g.

$$u_h^{(k)}|_{e_1} = u_{k,2}L_2 + u_{k,3}L_3.$$

This function is uniquely defined by the two end values $u_{k,2}$ and $u_{k,3}$ and so we see that the approximation must be continuous across the edge. The normal derivative is quadratic and is so it must be discontinuous across the edge.

5. The nodes are labelled 1^*, 2^* and 3^* and have the area coordinates illustrated in the following figure.



We can find the element basis functions using the interpolation property. For example

$$N_1^e = \alpha_1 L_1 + \beta_1 L_2 + \gamma_1 L_3$$

So

$$1 = \frac{\beta_1}{2} + \frac{\gamma_1}{2}, \quad 0 = \frac{\alpha_1}{2} + \frac{\gamma_1}{2}, \quad 0 = \frac{\alpha_1}{2} + \frac{\beta_1}{2}.$$

Solving this system gives

$$\alpha_1 = -1, \beta_1 = 1, \gamma_1 = 1 \implies N_1^e = L_2 + L_3 - L_1.$$

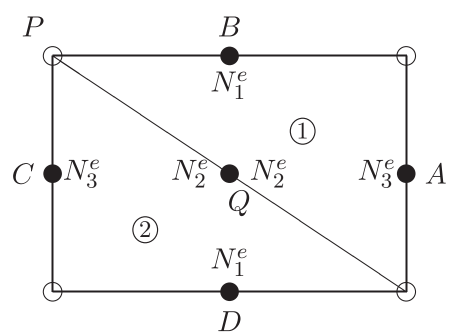
Similarly the other two basis functions are

$$N_2^e = L_3 + L_1 - L_2,$$
$$N_3^e = L_1 + L_2 - L_3.$$

Hence the element trial function is

$$u_h^{ⓚ} = u_{k,1^*}(L_2 + L_3 - L_1) + u_{k,2^*}(L_3 + L_1 - L_2) + u_{k,3^*}(L_1 + L_2 - L_3).$$

To investigate continuity consider two adjoining triangles as shown below.



At the point P, in element ①

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 1.$$

$$\begin{aligned} u_h^{①}(P) &= u_A N_3^e + u_B N_1^e + u_Q N_2^e \\ &= u_A(L_1 + L_2 - L_3) + u_B(L_2 + L_3 - L_1) + u_Q(L_1 + L_3 - L_2) \\ &= -u_A + u_B + u_Q. \end{aligned}$$

At the point P, in element ②

$$L_1 = 1, \quad L_2 = 0, \quad L_3 = 0.$$

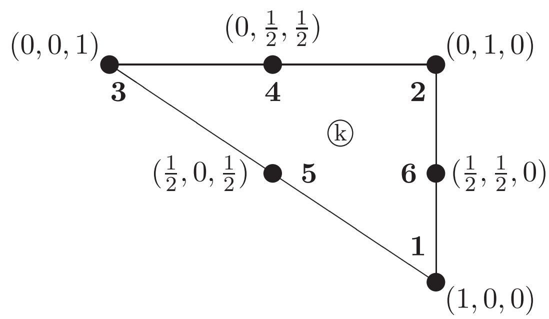
$$\begin{aligned} u_h^{②}(P) &= u_C N_3^e + u_D N_1^e + u_Q N_2^e \\ &= u_C(L_1 + L_2 - L_3) + u_D(L_2 + L_3 - L_1) + u_Q(L_1 + L_3 - L_2) \\ &= u_C - u_D + u_Q. \end{aligned}$$

Thus, in general $u_h^{①}(P) \neq u_h^{②}(P)$ and the global trial function is not continuous.

6. The $\boldsymbol{P}_2$ element basis functions are defined by

$$\psi_j^{ⓚ} = \begin{cases} 1 & \text{at node } j \\ 0 & \text{at node } i \neq j \\ \text{quadratic on the element ⓚ} \end{cases}.$$

Suppose that nodes are numbered as in the following figure.



Let us consider node **1** (a vertex). We want $\psi_1^{Ⓚ}(0, L_2, L_3) = 0$ (top edge of the triangle) and also $\psi_1^{Ⓚ}(\frac{1}{2}, L_2, L_3) = 0$ on the line between nodes **5** and **6**. This gives two linear factors, since $L_1 = 0$ and $L_1 = \frac{1}{2}$, thus $\psi_1^{Ⓚ}(L_1, L_2, L_3) = \alpha L_1(L_1 - \frac{1}{2})$. Finally we need to set α so that the function is unity at node **1**. Since $L_1 = 1$ at node **1**, we deduce that $\alpha = 2$. By a symmetry argument we can then write down the basis functions for the vertex nodes:

$$\begin{aligned} \psi_1^{Ⓚ}(L_1, L_2, L_3) &= 2L_1(L_1 - \tfrac{1}{2}) \\ \psi_2^{Ⓚ}(L_1, L_2, L_3) &= 2L_2(L_2 - \tfrac{1}{2}) \\ \psi_3^{Ⓚ}(L_1, L_2, L_3) &= 2L_3(L_3 - \tfrac{1}{2}). \end{aligned}$$

Similarly let us consider node **6** at the mid-point of the edge $L_3 = 0$. Since $\psi_6^{Ⓚ}$ is zero on the other two edges, we have two factors $L_1 = 0$ and $L_2 = 0$ and so $\psi_6^{Ⓚ} = \beta L_1 L_2$. Then since the function is unity at node **6** we deduce that $\beta = 4$. A symmetry argument gives the other basis functions for the mid-edge nodes:

$$\begin{aligned} \psi_4^{Ⓚ}(L_1, L_2, L_3) &= 4L_2L_3 \\ \psi_5^{Ⓚ}(L_1, L_2, L_3) &= 4L_1L_3 \\ \psi_6^{Ⓚ}(L_1, L_2, L_3) &= 4L_1L_2. \end{aligned}$$

To explore the continuity let us look at the edge $L_2 = 0$ between vertices **1** and **3** in the figure. The trial function on this edge takes the form

$$u_h^{Ⓚ}(L_1, 0, L_3) = 2L_1(L_1 - \tfrac{1}{2})u_{k,1} + 4L_1L_3u_{k,5} + 2L_3(L_3 - \tfrac{1}{2})u_{k,3},$$

This is a quadratic function in the variable $s = L_3 = (1 - L_1)$ that runs along the edge from **1** to **3**.

For an adjoining element the local trial function must also be quadratic in the variable s. Since the two trial functions agree at three points then the quadratic is uniquely defined and so the overall approximation must be continuous along the edge.

The normal derivative of the global finite element solution will not be continuous in general.