1. Using the given characterisation of s_3'' on the interval $[x_{j-1}, x_j]$ we have

$$\begin{aligned} s_3''(x) &= \frac{1}{h}(x_j - x)\sigma_{j-1} + \frac{1}{h}(x - x_{j-1})\sigma_j \\ s_3'(x) &= -\frac{1}{2h}(x_j - x)^2\sigma_{j-1} + \frac{1}{2h}(x - x_{j-1})^2\sigma_j + \alpha_j \\ s_3(x) &= \frac{1}{6h}(x_j - x)^3\sigma_{j-1} + \frac{1}{6h}(x - x_{j-1})^3\sigma_j + \alpha_j x + \beta_j. \end{aligned}$$

Applying the interpolation conditions $s_3(x_{j-1}) = f_{j-1}$, $s_3(x_j) = f_j$ gives

$$\begin{aligned} \alpha_j &= \frac{1}{h}(f_j - f_{j-1}) - \frac{h}{6}(\sigma_j - \sigma_{j-1}) \\ \beta_j &= \frac{1}{h}(f_{j-1}x_j - f_j x_{j-1}) + \frac{h}{6}(x_{j-1}\sigma_j - x_j\sigma_{j-1}). \end{aligned}$$

Next, evaluating $s_3'(x)$ defined in the interval $[x_{j-1}, x_j]$ at the right-hand point x_j, we get

$$s_3'(x_j^-) = \frac{1}{h}(f_j - f_{j-1}) + \frac{h}{3}\sigma_j + \frac{h}{6}\sigma_{j-1}.$$

Similarly, evaluating $s_3'(x)$ defined in the interval $[x_j, x_{j+1}]$ at the left-hand point x_j, we get

$$s_3'(x_j^+) = \frac{1}{h}(f_{j+1} - f_j) - \frac{h}{3}\sigma_j - \frac{h}{6}\sigma_{j+1}.$$

Equating these two expressions and rearranging gives

$$\frac{h}{6}\sigma_{j-1} + \frac{2h}{3}\sigma_j + \frac{h}{6}\sigma_{j+1} = \frac{1}{h}(f_{j+1} - 2f_j + f_{j-1}) \qquad (\star)$$

which is the jth equation of the required tridiagonal system $A\boldsymbol{\sigma} = \boldsymbol{b}$.

2. In the specific case of $f(x) = x^3$ and equally spaced knots $h_j = h, j = 1, 2, \ldots, n$, we have that $\sigma_j = f''(x_j) = 6x_j$. Substituting into the left-hand side of $(\star)$ we see that

$$\frac{h}{6}\sigma_{j-1} + \frac{2h}{3}\sigma_j + \frac{h}{6}\sigma_{j+1} = h(x_j - h) + 4hx_j + h(x_j + h) = 6hx_j,$$

which (after some simple algebra) can be shown to be equal to the right-hand side of $(\star)$ with $f_{j-1} = (x_j - h)^3$, $f_j = x_j^3$ and $f_{j+1} = (x_j + h)^3$.

Note that the last equation is not satisfied,

$$\frac{h}{6}\sigma_{n-2} + \frac{2h}{3}\sigma_{n-1} + \frac{h}{6}\sigma_n \neq 6hx_{n-1} = 6h(1 - h)$$

in the case $\sigma_n = 0$, but is satisfied when $\sigma_n = f''(x_n) = f''(1) = 6$. In general s_3 will be identical to f only if the two additional conditions are given by $\sigma_0 = f''(0)$ and $\sigma_n = f''(1)$.

3. Applying the interpolation conditions $s(x_{j-1}) = f_{j-1}$, $s(x_j) = f_j$ gives $a_j = f_{j-1}$ and $b_j = \frac{1}{h_j}(f_j - f_{j-1})$ and hence $s(x)$ is uniquely determined:

$$s(x) = f_{j-1} + \frac{1}{h_j}(f_j - f_{j-1})(x - x_{j-1}) + c_j(x - x_{j-1})(x - x_j), \quad x \in [x_{j-1}, x_j].$$

Evaluating $s'(x)$ defined in the interval $[x_{j-1}, x_j]$ at the right-hand point x_j, we get

$$s'(x_j^-) = \frac{1}{h_j}(f_j - f_{j-1}) + h_j c_j.$$

Similarly, evaluating $s'(x)$ defined in the interval $[x_j, x_{j+1}]$ at the left-hand point x_j, we get

$$s'(x_j^+) = \frac{1}{h_{j+1}}(f_{j+1} - f_j) - h_{j+1} c_{j+1}.$$

Equating these two expressions and rearranging gives the stated result.

4. Consider the case $j = 2$. Evaluating the alternative interpolants in the first interval using the characterisation of $s(x)$ above gives

$$\begin{aligned} s(x) &= f_0 + \frac{1}{h}(f_1 - f_0)(x - x_0) + c_1(x - x_0)(x - x_1), \qquad x \in [x_0, x_1], \\ s^*(x) &= f_0^* + \frac{1}{h}(f_1 - f_0^*)(x - x_0) + c_1(x - x_0)(x - x_1), \qquad x \in [x_0, x_1], \end{aligned}$$

and invoking the update formula for c_2 with $h_j = h_{j+1} = h$ gives

$$\begin{aligned} c_2 &= -c_1 + \frac{1}{h^2}(f_2 - 2f_1 + f_0) \\ c_2^* &= -c_1^* + \frac{1}{h^2}(f_2 - 2f_1 + f_0^*). \end{aligned}$$

Subtracting these equations and noting that $c_1 = c_1^*$ we have that

$$c_2^* = c_2 + \frac{(-1)^2}{h^2}(f_0^* - f_0).$$

This establishes the required result for $j = 2$.

Now suppose that the result is true for $j = n$, that is

$$c_j^* = c_j + \frac{(-1)^j}{h^2}(f_0^* - f_0). \qquad (\ddagger)$$

Invoking the update formula for c_{j+1} and $c_{j+1}^*$with $h_j = h_{j+1} = h$ gives

$$\begin{aligned} c_{j+1} &= -c_j + \frac{1}{h^2}(f_{j+1} - 2f_j + f_{j-1}) \\ c_{j+1}^* &= -c_j^* + \frac{1}{h^2}(f_{j+1} - 2f_j + f_{j-1}). \end{aligned}$$

Subtracting these equations gives

$$c_{j+1}^* = c_{j+1} + (-1)(c_j^* - c_j),$$

and using ($\ddagger$) we see that the desired result holds for $j = n+1$. Hence by induction the result is true for all $j = 2, 3, 4, \ldots$.

5. By definition

$$\begin{aligned} ||f - s_1||^2_{L^2(\Omega)} &= ||f - \sum_j \alpha_j \phi_j||^2_{L^2(\Omega)} \\ &= \int_0^1 \{f(x) - \sum_j \alpha_j \phi_j(x)\}^2 \, dx = F(\boldsymbol{\alpha}). \end{aligned}$$

To minimise this we require that

$$\frac{\partial F}{\partial \alpha_i} = 0, \quad i = 0, \ldots, n,$$

which gives

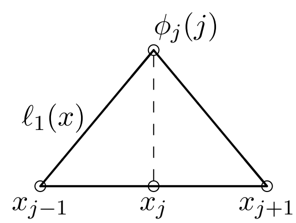
$$-2 \left[\int_0^1 f(x) - \sum_j \alpha_j \phi_j(x) \right] \phi_i(x) \, dx = 0, \quad i = 0, \ldots, n.$$

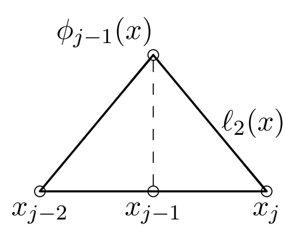
Rearranging this expression gives the system $(\star)$.

6. The matrix Q having entries given by

$$Q_{i\,j} = \int_0^1 \phi_j(x) \phi_i(x) \, dx,$$

is called the *mass* matrix. The basis functions are only nonzero in the two intervals adjoining the jth knot as illustrated below.





The general diagonal entry $Q_{j,j}$ is thus given using Simpson's rule by

$$\begin{aligned} Q_{j,j} &= \int_{x_{j-1}}^{x_j} \phi_j(x)^2 \, dx + \int_{x_j}^{x_{j+1}} \phi_j(x)^2 \, dx \\ &= \frac{h}{6}(1 \cdot 0^2 + 4 \cdot \frac{1}{2^2} + 1 \cdot 1^2) + \frac{h}{6}(1 \cdot 1^2 + 4 \cdot \frac{1}{2^2} + 1 \cdot 0^2) = \frac{2h}{3}. \end{aligned}$$

The first and last basis functions are nonzero in a single subinterval so we only get one contribution to the integral above and thus $Q_{0,0} = \frac{h}{3} = Q_{n,n}$.

The off-diagonal entries are nonzero only when the two basis functions are both nonzero in the same interval. Thus

$$\begin{aligned} Q_{j,j-1} &= \int_{x_{j-1}}^{x_j} \phi_j(x)\phi_{j-1}(x)\,dx = \int_{x_{j-1}}^{x_j} \ell_1(x)\,\ell_2(x)\,dx \\ &= \frac{h}{6}(1\cdot 0\cdot 1 + 4\cdot\frac{1}{2}\cdot\frac{1}{2} + 1\cdot 1\cdot 0) = \frac{h}{6}. \end{aligned}$$

The matrix is symmetric: $\frac{h}{6} = Q_{j,j-1} = Q_{j-1,j} = Q_{j,j+1}$.

The elements of $\boldsymbol{b}$ are

$$b_i = \int_0^1 f(x)\,\phi_i(x)\,dx.$$

Consider the case $f(x) = x$. We want to show that the jth equation

$$\frac{h}{6}\alpha_{j-1} + \frac{2h}{3}\alpha_j + \frac{h}{6}\alpha_{j+1} = \int_0^1 x\phi_j(x)\,dx,$$

is satisfied by setting $\alpha_j = x_j = jh$. The left-hand side is given by

$$\frac{h}{6}[(j-1)h + 4jh + (j+1)h] = jh^2.$$

The right-hand side can be computed exactly by Simpson's rule (the integrand is cubic) thus:

$$\begin{aligned} b_j &= \int_{x_{j-1}}^{x_j} x\phi_j(x)\,dx + \int_{x_j}^{x_{j+1}} x\phi_j(x)\,dx \\ &= \frac{h}{6}((x_j - h)\cdot 0 + 4\cdot(x_j - h/2)\cdot\frac{1}{2} + x_j\cdot 1) \\ &\qquad + \frac{h}{6}(x_j\cdot 1 + 4\cdot(x_j + h/2)\cdot\frac{1}{2} + (x_j + h)\cdot 0) \\ &= \frac{h}{6}(6x_j) = jh^2. \quad \square \end{aligned}$$

The first and last equations can be shown to be satisfied using the same technique. The fact that $\alpha_j = x_j$ is the only solution of the linear system equations follows from the fact that $\boldsymbol{\alpha}^T Q\boldsymbol{\alpha} > 0$ for all nonzero vectors $\boldsymbol{\alpha}$. (Positive-definiteness implies that the mass matrix Q is *nonsingular*.)