AN ADAPTIVE ALGORITHM for PDEs with RANDOM DATA

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Key references:

Alex Bespalov, Catherine Powell and David Silvester
Energy norm a posteriori error estimation for parametric operator equations, SIAM J. Scientific Computing, **36**, A339–A363, 2014.
`https://doi.org/10.1137/130916849`

Alex Bespalov and David Silvester,
Efficient adaptive stochastic Galerkin methods for parametric operator equations, SIAM J. Scientific Computing, **38**, A2118–A2140, 2016.
`https://doi.org/10.1137/15M1027048`

Motivating example (target model problem)

Find $u(\mathbf{x}, \mathbf{y})$ satisfying

$$\begin{aligned} -\nabla \cdot \big(a(\mathbf{x},\mathbf{y})\,\nabla u(\mathbf{x},\mathbf{y})\big) &= f(\mathbf{x}) \qquad \mathbf{x} \in D, \ \ \mathbf{y} \in \Gamma, \\ u(\mathbf{x},\mathbf{y}) &= 0 \qquad\quad\ \ \mathbf{x} \in \partial D, \ \ \mathbf{y} \in \Gamma, \end{aligned} \qquad (D)$$

where $D \subset \mathbb{R}^d$ $(d = 2, 3)$, $\Gamma := \prod_{m=1}^{\infty} [-1, 1]$ and

$$a(\mathbf{x},\mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x})\, y_m, \qquad \mathbf{x} \in D, \ \ \mathbf{y} \in \Gamma.$$

General framework: parametric operator equation

$$A(\mathbf{y})\, u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall\, \mathbf{y} \in \Gamma,$$

where $A : \Gamma \to \mathcal{L}(H, H')$, $f : \Gamma \to H'$, H is a separable Hilbert space (cf. [Schwab, Gittelson '11]).

Parametric operator equation: weak formulation

$$A(\mathbf{y})\, u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall\, \mathbf{y} \in \Gamma.$$

Weak formulation: find $u \in V := L^2_\pi(\Gamma; H)$ such that

$$B(u, v) = F(v) \qquad \forall\, v \in V, \tag{V}$$

where

$$B(u, v) = \int_\Gamma \langle A(\mathbf{y})u, v\rangle \,\mathrm{d}\pi(\mathbf{y}) = \int_\Gamma \langle A_0 u, v\rangle \,\mathrm{d}\pi(\mathbf{y}) + \sum_{m=1}^{\infty} \int_\Gamma \langle A_m u, v\rangle y_m \mathrm{d}\pi(\mathbf{y})$$

$$=: B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v) \quad \forall\, u, v \in V,$$

$$F(v) = \int_\Gamma \langle f(\mathbf{y}), v(\mathbf{y})\rangle \,\mathrm{d}\pi(\mathbf{y}).$$

Parametric operator equation: well posedness

$$A(\mathbf{y})\,u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall\,\mathbf{y} \in \Gamma.$$

Weak formulation: find $u \in V := L^2_\pi(\Gamma; H)$ such that

$$B_0(u, v) + \sum_{m=1}^{\infty} B_m(u, v) = F(v) \qquad \forall\, v \in V. \qquad (V)$$

Assumptions ensuring the well posedness of (V)*:*

- $A_0 \in \mathcal{L}(H, H')$ is symmetric and positive definite;
- $A_m \in \mathcal{L}(H, H')$ for $(m \in \mathbb{N})$ are symmetric and $\exists\, \tau \in [0, 1)$ s.t. $\forall\, \mathbf{y} \in \Gamma$

$$\left| \left\langle \sum_{m=1}^{\infty} y_m A_m v, v \right\rangle \right| \le \tau \, \langle A_0 v, v \rangle \quad \forall\, v \in H$$

(cf. [Schwab, Gittelson '11]).

Parametric operator equation: norm equivalence

$$A(\mathbf{y})\,u(\mathbf{y}) = f(\mathbf{y}), \quad \text{where} \quad A(\mathbf{y}) = A_0 + \sum_{m=1}^{\infty} y_m A_m \quad \forall\, \mathbf{y} \in \Gamma.$$

Weak formulation: find $u \in V := L^2_\pi(\Gamma; H)$ such that

$$\underbrace{B_0(u,v) + \sum_{m=1}^{\infty} B_m(u,v)}_{B(u,v)} = F(v) \qquad \forall\, v \in V. \tag{V}$$

Norm equivalence

Note that $(B(u,u))^{1/2} = \|u\|_B$ is equivalent to the B_0-norm: there exist positive constants $\lambda < 1 < \Lambda$ such that

$$\lambda\, B(v,v) \le B_0(v,v) \le \Lambda\, B(v,v) \qquad \forall\, v \in V. \tag{B_0}$$

Discrete formulation

Solution space: $V := L^2_\pi(\Gamma; H) \sim H \otimes L^2_\pi(\Gamma)$ (isometric isomorphism).

Finite-dimensional subspace: $X \otimes \mathcal{P}_P =: V_{XP} \subset V$, where

- X is a finite-dimensional subspace of H (corresponding, e.g., to spatial finite element discretization on D);
- $\mathcal{P}_P \subset L^2_\pi(\Gamma)$ is a polynomial space on Γ, more precisely, the set of tensor product polynomials associated with a finite subset P of the index set $\mathfrak{I}$ of finitely supported sequences

$$\mathfrak{I} := \left\{\nu = (\nu_1, \nu_2, \dots) \in \mathbb{N}_0^{\mathbb{N}};\ \#\,\mathrm{supp}\,\nu < \infty\right\},$$

where $\mathrm{supp}\,\nu := \{m \in \mathbb{N};\ \nu_m \neq 0\}$ for any $\nu \in \mathbb{N}_0^{\mathbb{N}}$.

Galerkin projection: find $u_{XP} \in V_{XP} = X \otimes \mathcal{P}_P$ such that

$$B(u_{XP}, v) = F(v) \quad \forall\, v \in V_{XP}.$$

Linear algebra

Galerkin projection: find $u_{XP} \in V_{XP} = X \otimes \mathcal{P}_P$ such that

$$B(u_{XP}, v) = F(v) \quad \forall\, v \in V_{XP}.$$

Discrete system: find $\mathbf{x} \in \mathbb{R}^{n_x \cdot n_\xi}$ such that $\mathcal{A}\mathbf{x} = \mathbf{b}$ with

$$\mathcal{A} = G_0 \otimes K_0 + \sum_{m=1}^{M} G_m \otimes K_m, \qquad \mathbf{b} = \mathbf{g}_0 \otimes \mathbf{f}_0$$

Error estimation I

The estimator $e^*\in V^*_{XP}$ satisfies: $B(e^*,v)=F(v)-B(u_{XP},v)\quad\forall\, v\in V^*_{XP}$.

Recall that $B(u,v)=B_0(u,v)+\sum_{m=1}^{\infty}B_m(u,v)$.

Exploit the structure of the enriched finite-dimensional space $V^*_{XP}\supset V_{XP}$:

$$X^*=X\oplus Z,\quad X\cap Z=\{0\};\quad \mathcal{P}_{P^*}=\mathcal{P}_P\oplus\mathcal{P}_Q,\quad \mathcal{P}_P\cap\mathcal{P}_Q=\{0\};$$

$$V^*_{XP}:=\underbrace{V_{X^*P}}_{X^*\otimes\mathcal{P}_P}\oplus\underbrace{V_{XQ}}_{X\otimes\mathcal{P}_Q}=\underbrace{V_{XP}}_{X\otimes\mathcal{P}_P}\oplus\underbrace{V_{ZP}}_{Z\otimes\mathcal{P}_P}\oplus\underbrace{V_{XQ}}_{X\otimes\mathcal{P}_Q}.$$

This leads to the new error estimate

$$\eta:=\left(\|e_{ZP}\|^2_{B_0}+\|e_{XQ}\|^2_{B_0}\right)^{1/2},$$

where two contributing estimators are defined by

$$\begin{aligned} e_{ZP}\in V_{ZP}: \quad B_0(e_{ZP},v) &= F(v)-B(u_{XP},v)\quad\forall\, v\in V_{ZP},\\ e_{XQ}\in V_{XQ}: \quad B_0(e_{XQ},v) &= F(v)-B(u_{XP},v)\quad\forall\, v\in V_{XQ}.\end{aligned}$$

Error estimation II

The estimator $e^* \in V^*_{XP}$ satisfies: $B(e^*, v) = F(v) - B(u_{XP}, v) \quad \forall\, v \in V^*_{XP}$.

New error estimate

$$\eta := \left(\|e_{ZP}\|^2_{B_0} + \|e_{XQ}\|^2_{B_0}\right)^{1/2},$$

where two contributing estimators are defined by

$$\begin{aligned} e_{ZP} \in V_{ZP}: \quad B_0(e_{ZP}, v) &= F(v) - B(u_{XP}, v) \quad \forall\, v \in V_{ZP}, \\ e_{XQ} \in V_{XQ}: \quad B_0(e_{XQ}, v) &= F(v) - B(u_{XP}, v) \quad \forall\, v \in V_{XQ}. \end{aligned}$$

The relation between the estimator e^* and the componentwise estimate η:

$$\sqrt{\lambda}\,\eta \le \|e_*\|_B \le \frac{\sqrt{\Lambda}}{\sqrt{1-\gamma^2}}\,\eta,$$

where λ, Λ are the norm-equivalence constants in (B_0), $\gamma \in [0,1)$ is the constant in the strengthened Cauchy–Schwarz inequality for the subspaces $X, Z \subset H$ w.r.t. inner product $\langle A_0 \cdot, \cdot \rangle$.

Error reduction indicators

Recall

$$\eta := \left(\|e_{ZP}\|_{B_0}^2 + \|e_{XQ}\|_{B_0}^2\right)^{1/2},$$

with $e_{ZP} \in V_{ZP}$ and $e_{XQ} \in V_{XQ}$.

Enhanced Galerkin approximations: $u_{X^*P} \in X^* \otimes \mathcal{P}_P$ and $u_{XP^*} \in X \otimes \mathcal{P}_{P^*}$.

Estimates of the error reduction [Bespalov, Powell, Silvester '14]:

$$\sqrt{\lambda}\,\|e_{ZP}\|_{B_0} \le \|u_{X^*P} - u_{XP}\|_B \le \frac{\sqrt{\Lambda}}{\sqrt{1-\gamma^2}}\,\|e_{ZP}\|_{B_0},$$

$$\sqrt{\lambda}\,\|e_{XQ}\|_{B_0} \le \|u_{XP^*} - u_{XP}\|_B \le \sqrt{\Lambda}\,\|e_{XQ}\|_{B_0}.$$

Here, λ, Λ are the constants in (B_0) and $\gamma \in [0,1)$ is the constant in the (deterministic) strengthened Cauchy–Schwarz inequality.

The error estimator e_{XQ}: decomposition

Lemma 1. For any *finite* detail set $Q = \{\mu \in \mathfrak{I}; \mu \notin P\}$ one has

$$e_{XQ} = \sum_{\mu \in Q} e_{XQ}^{(\mu)} \quad \text{with} \quad \|e_{XQ}\|_{B_0}^2 = \sum_{\mu \in Q} \left\|e_{XQ}^{(\mu)}\right\|_{B_0}^2,$$

where $e_{XQ}^{(\mu)} \in X \otimes \mathcal{P}_\mu$ satisfies

$$B_0\left(e_{XQ}^{(\mu)}, v\right) = F(v) - B(u_{XP}, v) \qquad \forall\, v \in X \otimes \mathcal{P}_\mu.$$

Note:

- each $e_{XQ}^{(\mu)}$ can be independently and cheaply computed (the same coefficient matrix for all $\mu \in Q$);
- $\|e_{XQ}^{(\mu)}\|_{B_0}$ provides the error reduction estimate for the corresponding enhanced approximation u_{XP^*} with $P^* = P \cup \mu$.

The error estimator e_{XQ}: choosing Q

Lemma 2. Assume that $f(\mathbf{y})$ has the decomposition

$$f(\mathbf{y}) = f_0 + \sum_{m=1}^{\infty} y_m f_m, \quad \forall\, \mathbf{y} \in \Gamma.$$

For any finite detail index set $Q \subset \mathfrak{I}$ the error estimator $e_{XQ}^{(\mu)} \neq 0$ if

$$\mu \in Q_\infty := \left\{ \mu \in \mathfrak{I} \setminus P;\ \mu = \nu \pm \varepsilon^{(m)},\ \forall \nu \in P,\ \forall m = 1, 2, \ldots \right\}.$$

Here, $\varepsilon^{(m)} = \left(\varepsilon_1^{(m)}, \varepsilon_2^{(m)}, \ldots\right) \in \mathfrak{I}$ represents the Kronecker delta sequence for the coordinate m, i.e., $\varepsilon_j^{(m)} = \delta_{jm}$ for any $j \in \mathbb{N}$.

Proof. Apply the three-term recurrence formula for orthogonal polynomials.

Numerical experiments: model problem

[Eigel, Gittelson, Schwab, Zander '14]

We solve the steady-state diffusion problem

$$\begin{aligned} -\nabla \cdot \big(a(\mathbf{x},\mathbf{y})\,\nabla u(\mathbf{x},\mathbf{y})\big) &= f(\mathbf{x}) \qquad \mathbf{x} \in D,\ \mathbf{y} \in \Gamma, \\ u(\mathbf{x},\mathbf{y}) &= 0 \qquad \mathbf{x} \in \partial D,\ \mathbf{y} \in \Gamma \end{aligned} \tag{D}$$

with $\mathbf{x} = (x_1, x_2) \in D := [0,1]^2$, $\mathbf{y} \in \Gamma = \prod_{m=1}^{\infty}[-1,1]$, $f(\mathbf{x}) = 1$ and

$$a(\mathbf{x},\mathbf{y}) = \underbrace{1}_{a_0 = \mathbb{E}[a]} + \sum_{m=1}^{\infty} \bar{\alpha}\, m^{-\tilde{\sigma}} \cos(2\pi\beta_1(m)\,x_1)\cos(2\pi\beta_2(m)\,x_2)\,y_m.$$

Here $\tilde{\sigma} > 1$ and $0 < \bar{\alpha} < 1/\zeta(\tilde{\sigma})$ (here, ζ denotes the Riemann zeta function); y_m are the images of uniformly distributed independent mean-zero r.v.

The weak formulation (V) of this problem is well posed because $\tau = \bar{\alpha}\zeta(\tilde{\sigma}) < 1$.

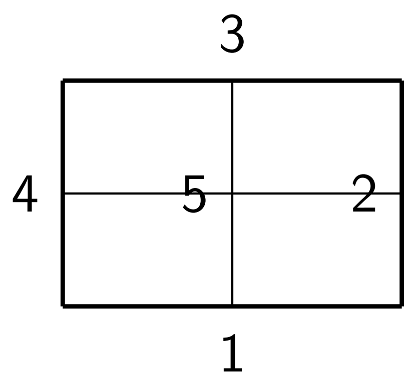
Two cases are considered: $\tilde{\sigma} = 4$ (fast decay) and $\tilde{\sigma} = 2$ (slow decay).

model problem | spatial detail space

The detail spaces $Z(h_k)$ for bilinear (Q_1) spatial approximation satisfy

$$X_{h/2} = X_h \oplus Z_h.$$

Thus, Z_h spans the set of bilinear bubble functions corresponding to the edge midpoints and element centroids of the original mesh $\square_h$ (defined locally)



This gives a decomposition of the error estimate η with

$$V_{XP}^{*} := V_{XP} \oplus (Z_h \otimes \mathcal{P}_P) \oplus (X_h \otimes \mathcal{P}_Q).$$

The smart choice of the detail space Z_h gives rise to local problems defined on all elements $K \in \square_h$.

model problem | spatial detail space

This gives a decomposition of the error estimate η with

$$V_{XP}^{*} := V_{XP} \oplus (Z_h \otimes \mathcal{P}_P) \oplus (X_h \otimes \mathcal{P}_Q).$$

For example, we compute $\bar{e}_{ZP}|_K \in Z_h\big|_K \otimes \mathcal{P}_P$ satisfying

$$\begin{aligned} B_{0,K}(\bar{e}_{ZP}|_K, v) &= F_K(v) \\ &+ \int_\Gamma \int_K \nabla \cdot \big(a(\mathbf{x},\mathbf{y})\, \nabla u_{XP}(\mathbf{x},\mathbf{y})\big)\, v(\mathbf{x},\mathbf{y})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\pi(\mathbf{y}) \\ &- \tfrac{1}{2} \int_\Gamma \int_{\partial K \setminus \partial D} a(s,\mathbf{y}) \left[\!\left[\frac{\partial u_{XP}}{\partial n} \right]\!\right] v(s,\mathbf{y})\, \mathrm{d}s\, \mathrm{d}\pi(\mathbf{y}), \end{aligned}$$

for all $v \in Z_h|_K \otimes \mathcal{P}_P$.

The coefficient matrix associated with this local problem is the Kronecker product of a 5×5 (stiffness) matrix and an identity matrix of dimension $|P| = \dim(\mathcal{P}_P)$.

Adaptive algorithm

```
Adaptive_sGFEM [tol, A, f] → u_n
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input $h_0,\ P_0$

for $k = 0, 1, 2, \ldots$ **do**

  $u_k \leftarrow \text{Solve}\,[A, f, X(h_k), P_k]$

  $\delta_X \leftarrow \text{Error\_Estimate\_1}\,[A, f, u_k, Z(h_k)]$

  $Q_k \leftarrow \text{Detail\_Index\_Set}\,[P_k]$ % Lemma 2

  for $i = 1, 2, \ldots, \#(Q_k)$ **do**

    $\delta_{P,i} \leftarrow \text{Error\_Estimate\_2}\,[A, f, u_k, \mu_i]$

  end

  $\eta_k := \left(\delta_X^2 + \sum_{i=1}^{\#(Q_k)} \delta_{P,i}^2\right)^{1/2}$ % Lemma 1

  if $\eta_k < tol$ **then** $n := k$, **break**

  if $\delta_X \geq \max\left\{\delta_{P,i};\ i = 1, 2, \ldots, \#(Q_k)\right\}$ **then**

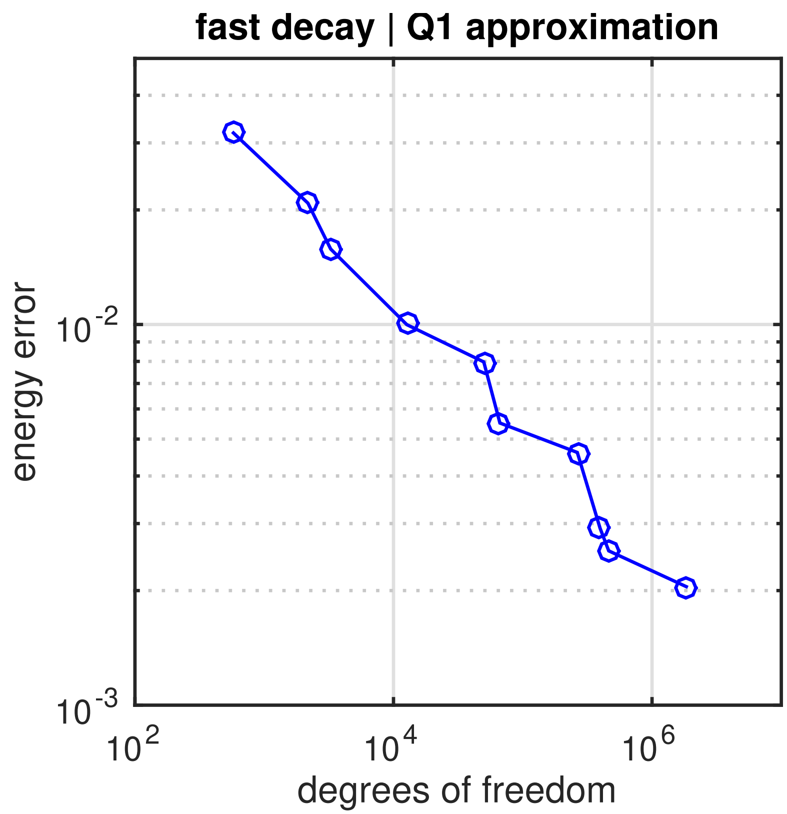
   $X(h_{k+1}) := X(h_k) \oplus Z(h_{k+1}),\ P_{k+1} := P_k$

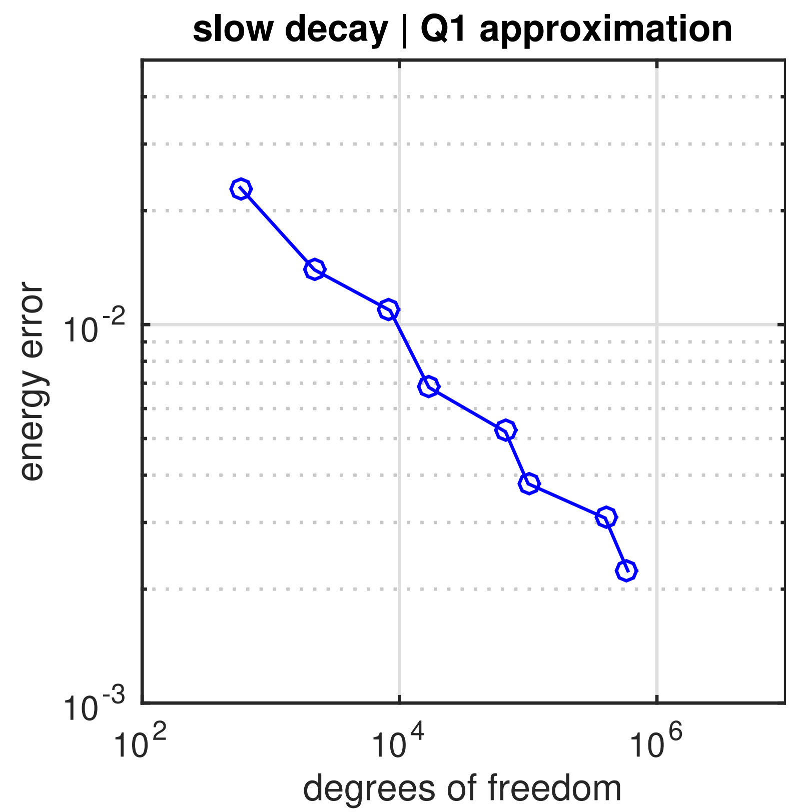
  else $X(h_{k+1}) := X(h_k),\ P_{k+1} := P_k \cup \left\{\mu_i \in Q_k;\ \delta_{P,i} \geq \delta_X\right\}$

end

Numerical experiments I: performance of the adaptive algorithm

Bilinear (Q_1) spatial approximation on uniform grids $\square_h$





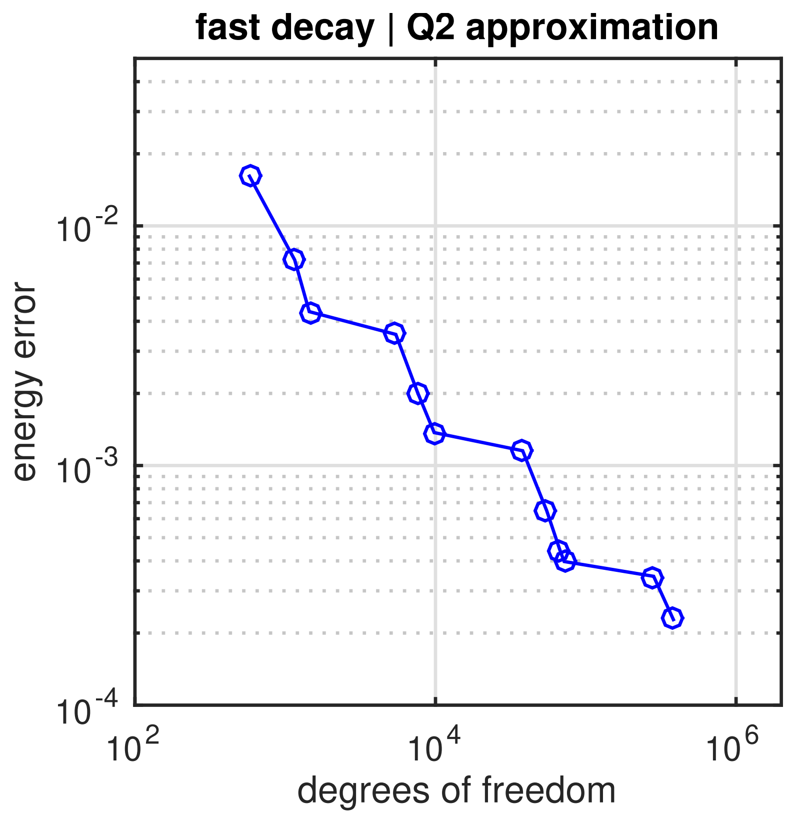
Numerical experiments II: evolution of the index set

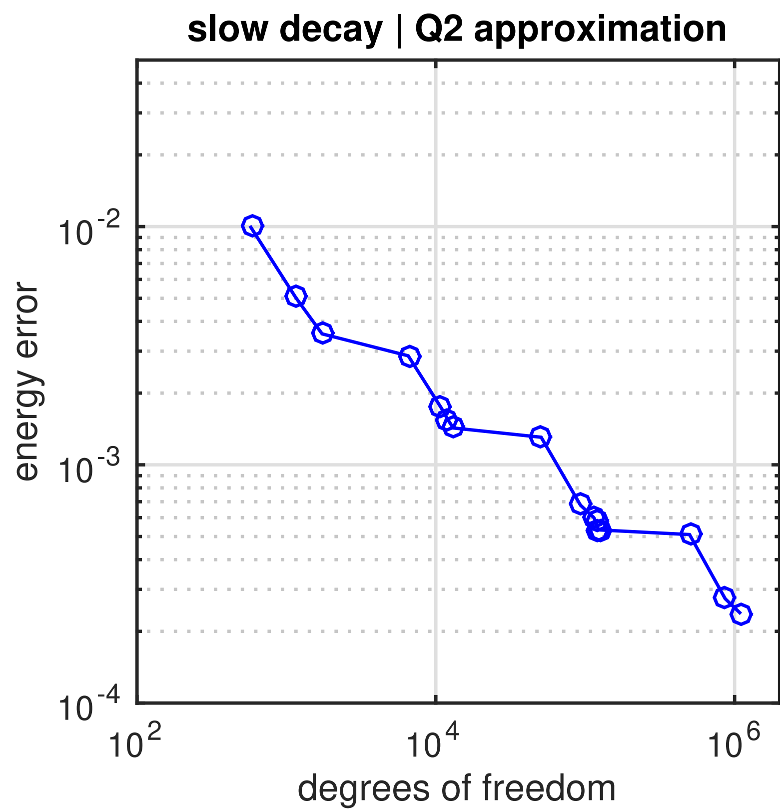
Bilinear (Q_1) spatial approximation on uniform grids $\square_h$

k	fast decay	slow decay
0	(0 0) (1 0)	(0 0 0 0) (1 0 0 0)
1	—	—
2	(2 0)	—
3	—	(0 1 0 0) (2 0 0 0)
4	—	—
5	(3 0)	(0 0 1 0) (1 1 0 0)
6	—	—
7	(0 1) (4 0)	(0 0 0 1) (3 0 0 0) (1 0 1 0)
8	(1 1)	
9	—	

Numerical experiments III: performance of the adaptive algorithm

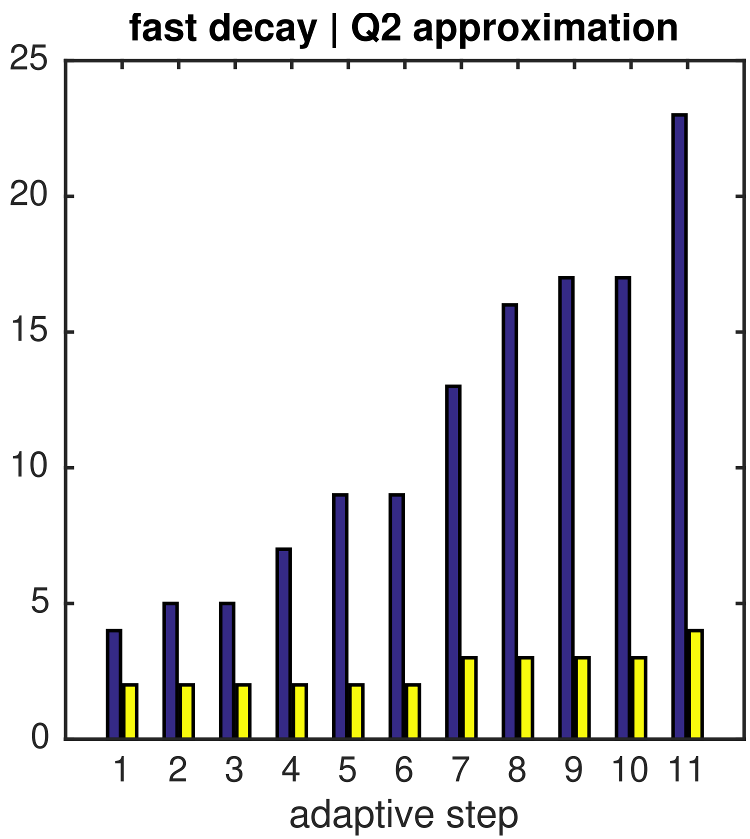
Biquadratic (Q_2) *spatial approximation on uniform grids* $\square_h$

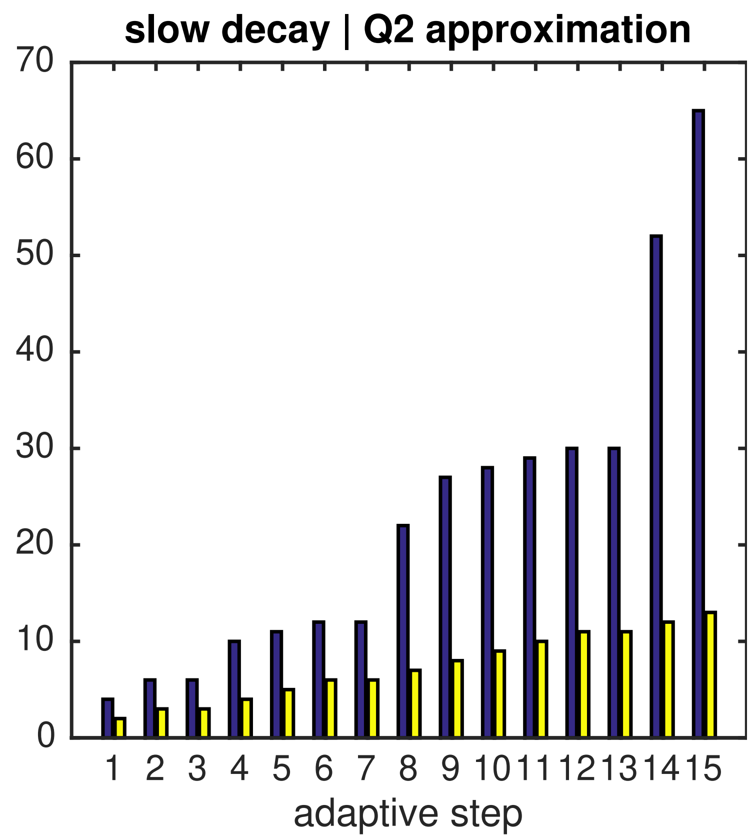




Numerical experiments IV: evolution of the index set

Biquadratic (Q_2) spatial approximation on uniform grids $\square_h$





Numerical experiments V: effectivity

$u \in L^2_\pi(\Gamma, H^1_0(D))$ – the (unknown) exact solution;

$u_k \in X_k \otimes \mathcal{P}_{P_k}$ – the Galerkin solution at the step $k = 0, 1, 2, \ldots$;

$e_k := u - u_k$ – the true error ($\|e_k\|_B \approx \eta_k$).

The error representation:

$$\begin{aligned} \|e_k\|_B &= \|u - u_k\|_B = \left(\|u\|_B^2 - \|u_k\|_B^2\right)^{1/2} \\ &\approx \left(\|u_{\mathrm{ref}}\|_B^2 - \|u_k\|_B^2\right)^{1/2} := \|e_k^{\mathrm{ref}}\|_B, \end{aligned}$$

where $u_{\mathrm{ref}} \in X_{\mathrm{ref}} \otimes \mathcal{P}_{P_{\mathrm{ref}}}$ is an accurate (reference) solution.

The effectivity index:

$$\theta_k = \frac{\eta_k}{\|e_k^{\mathrm{ref}}\|_B}.$$

In the case of Q_1 approximations:

u_{ref} is generated by running the adaptive algorithm with Q_2 approximations on a fine spatial grid until a small tolerance ($tol = 2.5\text{e-}4$) is reached.

Numerical experiments V: effectivity

Model problem with $\tilde{\sigma} = 2$ (slow decay).

The Galerkin solution u_k using Q_1 approximation ($k = 0, 1, 2, \ldots$).

The reference solution u_{ref} using Q_2 approximation

$$\|u_{\text{ref}}\|_B = 1.90117\text{e-}01,\ N_{\text{ref}} = 1'081'665.$$
$$\text{effectivity indices: } \theta_k = \eta_k / \|e_k^{\text{ref}}\|_B.$$

k	N_k	$\|u_k\|_B$	η_k	$\|e_k^{\text{ref}}\|_B$	θ_k
0	578	1.89179e-01	2.29925e-02	1.88606e-02	1.22
1	2178	1.89633e-01	1.39596e-02	1.35540e-02	1.03
2	8450	1.89746e-01	1.08927e-02	1.18594e-02	0.92
3	16900	1.89996e-01	6.82414e-03	6.76065e-03	1.01
4	66564	1.90025e-01	5.21155e-03	5.89584e-03	0.88
5	99846	1.90074e-01	3.79683e-03	4.03873e-03	0.94
6	396294	1.90081e-01	3.08163e-03	3.68263e-03	0.84
7	594441	1.90100e-01	2.23643e-03	2.55596e-03	0.87

Conclusions

What have we achieved?

⋆ Novel error estimation strategy: this exploits the tensor product structure of the approximation spaces.

⋆ Effective adaptive algorithm: adaptive refinement is driven by dominant error reduction estimates.