Approximation Theory I
MATH46052|66052 *

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Contents

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1 Piecewise approximation in $\mathbb{R}^1$

Consider a one-dimensional interval $[a, b]$ and let f be a real-valued function defined on this interval: in other words, $f : [a, b] \to \mathbb{R}$. To fix ideas, imagine $f(x)$ to be the depth of the groundwater level over the interval $[a, b]$. It might be the case that we have information about the values of f at a set of points $\{x_0, x_1, \ldots, x_n\}$ only, and we wish to know (approximately) the values of f at intermediate points. Without loss of generality we may take

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < x_{i+1} < \cdots < x_{n-1} < x_n = b,$$
$$h_i \coloneqq x_i - x_{i-1}.$$

(Interpolation problem)
Given a set of $n + 1$ data values $\{(x_k, f_k)\}_{k=0}^{n}$, we want to find a continuous function $S : [x_0, x_n] \to \mathbb{R}$ such that

$$f_k \coloneqq f(x_k) = S(x_k).$$

Observe that there are infinitely many functions that solve the interpolation problem. We will restrict our attention to splines. We denote by $\overline{\Omega}$ the domain of the problem: here it is the closed interval $[a, b]$.

(Splines)
Splines are polynomial interpolating functions that are defined piecewise on each subinterval $[x_{i-1}, x_i]$ with a prescribed degree of smoothness—having continuous derivatives up to a certain order k. We use the notation $S_k \in C^{k-1}(\overline{\Omega})$ to indicate the smoothness of the spline.

1.1 Linear splines

The linear spline $S_1(x) \in C^0[a, b]$ is the piecewise linear interpolant of $f(x)$, that is, $f_k \coloneqq f(x_k) = S_1(x_k), k = 0, \ldots, n$.

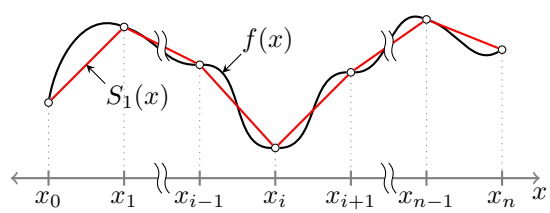


Figure 1.1: The piecewise linear interpolant $S_1(x)$ of $f(x)$.

In Figure 1.1, the symbol $\dashv\!\!\vdash$ means the plot is cut and that portion is not

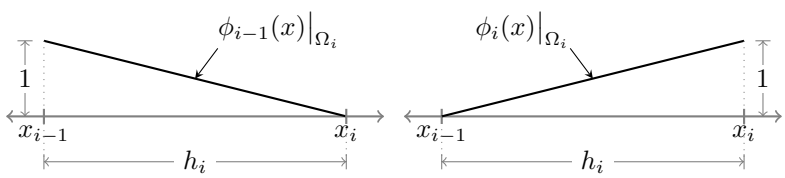
shown. Let us focus on the interval $\Omega_i := [x_{i-1}, x_i]$. In this interval we have

$$\begin{aligned} S_1(x)\big|_{\Omega_i} &= \frac{x_i - x}{h_i} f_{i-1} + \frac{x - x_{i-1}}{h_i} f_i \quad \text{(Lagrange form)} \\ &= f_{i-1} + \frac{f_i - f_{i-1}}{h_i}(x - x_{i-1}) \quad \text{(Newton form).} \end{aligned}$$

Looking at the Lagrange form we can identify two local interpolation basis functions associated to the indices $i-1$ and i:

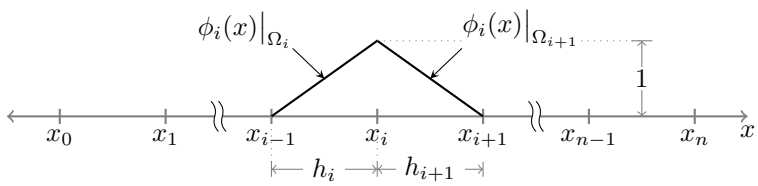
$$\phi_{i-1}(x)\big|_{\Omega_i} = \frac{x_i - x}{h_i}, \quad \phi_i(x)\big|_{\Omega_i} = \frac{x - x_{i-1}}{h_i}.$$

These are shown below.



(a) (b)

The local interpolation basis leads to the notion of a global interpolation basis function $\phi_i(x)$ associated to the index i and defined on the whole interval $[a, b]$.



$$\phi_i(x) = \begin{cases} 1 & \text{if } x = x_i, \\ \phi_i(x)\big|_{\Omega_i} = (x - x_{i-1})/h_i & \text{if } x \in (x_{i-1}, x_i), i \neq 0, \\ \phi_i(x)\big|_{\Omega_{i+1}} = (x_{i+1} - x)/h_{i+1} & \text{if } x \in (x_i, x_{i+1}), i \neq n, \\ 0 & \text{otherwise.} \end{cases}$$

We then have the representation

$$S_1(x) = \sum_{i=0}^{n} \phi_i(x) f_i = \bigwedge_{i=1}^{n} S_1(x)\big|_{\Omega_i}, \tag{1.1}$$

where the symbol $\bigwedge$ means assembly over the individual subintervals.

The interpolation error is defined as

$$e(x) := f(x) - S_1(x).$$

The following theorem shows how well S_1 approximates f.

Theorem 1.1 *Suppose that $f \in C^2[a,b]$ and let S_1 be the linear spline interpolant of f defined in (1.1); then the following error bounds hold*

$$\|f - S_1\|_{L^\infty(\Omega)} \leq \frac{h^2}{8}\|f''\|_{L^\infty(\Omega)}, \tag{1.2}$$

$$\|f' - S_1'\|_{L^\infty(\Omega\setminus\{x_k\}_{k=0}^n)} \leq \frac{h}{2}\|f''\|_{L^\infty(\Omega)}, \tag{1.3}$$

where $h = \max\left\{h_k\right\}_{k=1}^n$ and $\|f\|_{L^\infty(\Omega)} = \max_{x\in[a,b]}|f(x)|$.

The error bound in (1.2) means that the linear spline interpolant S_1 converges pointwise at a second-order rate to the underlying function f in the limit $h \to 0$.

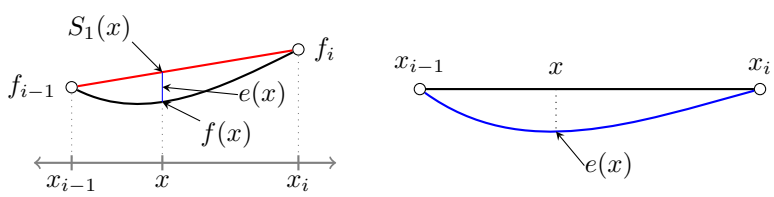
Note that the derivative of the linear spline is not defined at the interior points of interpolation $\{x_k\}_{k=1}^{n-1}$. The error bound in (1.3) means that the derivative S_1' of the linear spline interpolant converges almost everywhere to the derivative f' of the function that generated the data.

We will use Rolle's theorem in the proof of Theorem 1.1.

Theorem 1.2 (Rolle's theorem) *Let $g : [a,b] \to \mathbb{R}$ be a function that is continuous everywhere on $[a,b]$ and has a derivative at each point of the open interval (a,b). Also, assume that $g(a) = g(b)$. Then there is at least one point ξ in the open interval (a,b) such that $g'(\xi) = 0$.*

Proof. See, for example, Apostol's textbook [1, p. 184]. ♡

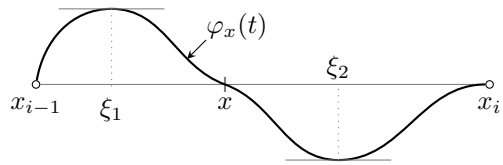
By definition, $e(x_k) = 0$ for all $k \in \{0, 1, \ldots, n\}$. Once again we focus on the interval $[x_{i-1}, x_i]$. In this interval, $f(x)$, $S_1(x)$ and $e(x)$ will take the following form:



Next, consider the function $\varphi_x : [x_{i-1}, x_i] \to \mathbb{R}$:

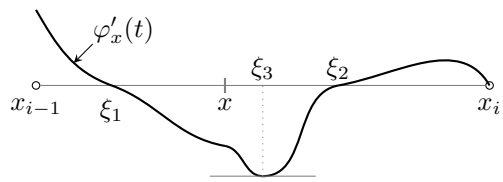
$$\varphi_x(t) := e_i(t) - e_i(x)\frac{(t - x_{i-1})(t - x_i)}{(x - x_{i-1})(x - x_i)},$$

where $x, t \in \Omega_i$ and $e_i := e\big|_{\Omega_i}$. We freeze x and let t be the only active variable. Observe that $\varphi_x(t) = 0$ has three solutions $t = \{x_{i-1}, x_i, x\}$. In the interval $[x_{i-1}, x_i]$, $\varphi_x(t)$ will take the following form:



For $f \in C^2[a,b]$, the fact that $S_1''\big|_{\Omega_i} = 0$ implies that the local error $e_i \in C^2(\Omega_i)$ so that $\varphi_x \in C^2(\Omega_i)$. This means that Rolle's theorem can be applied to both φ_x and φ_x' in Ω_i.

Applying Rolle's theorem to φ_x we find that there exist at least two points $\xi_1 \in (x_{i-1}, x)$ and $\xi_2 \in (x, x_i)$ such that $\varphi_x'(\xi_1) = 0$ and $\varphi_x'(\xi_2) = 0$. In the subdomain Ω_i, φ_x' will take the following form:



Applying Rolle's theorem to φ_x' in the interval $[\xi_1, \xi_2]$ we find that there will be at least one point $\xi_3 \in (\xi_1, \xi_2)$ such that $\varphi_x''(\xi_3) = 0$. Now,

$$\varphi_x'(t) = e_i'(t) - e_i(x)\frac{2t - x_{i-1} - x_i}{(x - x_{i-1})(x - x_i)}, \tag{1.4}$$

$$\varphi_x''(t) = e_i''(t) - e_i(x)\frac{2}{(x - x_{i-1})(x - x_i)}. \tag{1.5}$$

Also since $S_1\big|_{\Omega_i}$ is a linear function, we have that $e_i''(t) = f''(t)$ for all points $t \in \Omega_i$. Thus, setting $t = \xi_3$ in (1.5) gives the characterisation

$$e_i(x) = \frac{(x - x_{i-1})(x - x_i)}{2} f''(\xi_3), \quad \text{for some } \xi_3 \in \Omega_i. \tag{1.6}$$

Taking the derivative with respect to x in (1.6) then gives

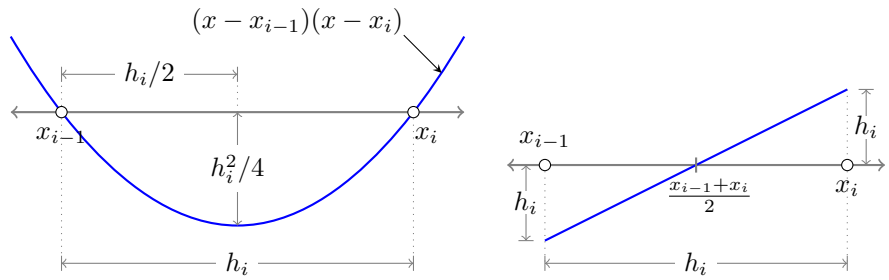
$$e_i'(x) = \frac{2x - x_{i-1} - x_i}{2} f''(\xi_3), \quad \text{for some } \xi_3 \in \Omega_i. \tag{1.7}$$

Next, taking norms of both sides of (1.6) and (1.7) and using the trivial bound $\|f''\|_{L^\infty(\Omega_i)} \geq |f''(\xi_3)|$ gives

$$\|e\|_{L^\infty(\Omega_i)} \leq \tfrac{1}{2}\|(x - x_{i-1})(x - x_i)\|_{L^\infty(\Omega_i)}\, \|f''\|_{L^\infty(\Omega_i)}, \tag{1.8}$$

$$\|e'\|_{L^\infty(\Omega_i)} \leq \tfrac{1}{2}\|2x - x_{i-1} - x_i\|_{L^\infty(\Omega_i)}\, \|f''\|_{L^\infty(\Omega_i)}. \tag{1.9}$$

The maximum absolute value of $(x - x_{i-1})(x - x_i)$ is $h_i^2/4$ and of $2x - x_{i-1} - x_i$ is h_i. One way to see this is to plot the expressions over the interval $[x_{i-1}, x_i]$:



Substituting these bounds into (1.8) and (1.9) gives the local estimates

$$\|f - S_1\|_{L^\infty(\Omega_i)} \le \frac{h_i^2}{8}\|f''\|_{L^\infty(\Omega_i)} \le \frac{h^2}{8}\|f''\|_{L^\infty(\Omega)}, \tag{1.10}$$

$$\|f' - S_1'\|_{L^\infty(\Omega_i)} \le \frac{h_i}{2}\|f''\|_{L^\infty(\Omega_i)} \le \frac{h}{2}\|f''\|_{L^\infty(\Omega)}. \tag{1.11}$$

Equations (1.10) and (1.11) hold for all subintervals. Thus since $f - S_1$ is continuous in the domain $[a, b]$ we deduce that

$$\|f - S_1\|_{L^\infty(\Omega)} \le \frac{h^2}{8}\|f''\|_{L^\infty(\Omega)} \qquad \text{(this is (1.2))}.$$

Moreover, since $f' - S_1'$ is undefined at subinterval end points, we also have

$$\|f' - S_1'\|_{L^\infty(\Omega\setminus\{x_k\}_{k=1}^{n-1})} \le \frac{h}{2}\|f''\|_{L^\infty(\Omega)} \qquad \text{(this is (1.3))}.$$

This concludes the proof of Theorem 1.1. ♡

The characterisation (1.6) is the starting point for establishing least-squares estimates of the interpolation error.

Corollary 1.3 *Suppose that $f \in C^2[a, b]$ and let S_1 be the linear spline interpolant of f defined in (1.1); then there exist constants C_1 and C_2 so that*

$$\|f - S_1\|_{L^2(\Omega)} \le C_1 h^2 \|f''\|_{L^\infty(\Omega)}, \tag{1.12}$$

$$\|f' - S_1'\|_{L^2(\Omega)} \le C_2 h \|f''\|_{L^\infty(\Omega)}, \tag{1.13}$$

where $h = \max\{h_k\}_{k=1}^n$ and $\|f\|_{L^2(\Omega)} = \{\int_a^b f^2 \,\mathrm{d}x\}^{1/2}$.

Proof. Since $e = f - S_1$ is a square integrable function, we can write $\|e\|^2_{L^2(\Omega)} = \sum_{i=1}^n \|e_i\|^2_{L^2(\Omega_i)}$. Squaring both sides of (1.6) and (1.7), integrating over Ω_i and then using the bound $\int_{x_{i-1}}^{x_i} p^2(x)\,\mathrm{d}x \le h_i\|p\|^2_{L^\infty(\Omega_i)}$ that holds for polynomial $p(x)$ gives

$$\|e_i\|^2_{L^\infty(\Omega_i)} \le \frac{h_i}{4}\|(x - x_{i-1})(x - x_i)\|^2_{L^\infty(\Omega_i)}\,\|f''\|^2_{L^\infty(\Omega_i)}, \tag{1.14}$$

$$\|e_i'\|^2_{L^\infty(\Omega_i)} \le \frac{h_i}{4}\|2x - x_{i-1} - x_i\|^2_{L^\infty(\Omega_i)}\,\|f''\|^2_{L^\infty(\Omega_i)}. \tag{1.15}$$

Bounding the terms on the right side as in the proof of Theorem 1.1 and summing over the subintervals then gives

(1.16)
$$\|e\|^2_{L^2(\Omega)} = \sum_{i=1}^{n} \|e_i\|^2_{L^2(\Omega_i)} \le \frac{h^4}{64}\,\|f''\|^2_{L^\infty(\Omega)} \sum_{i=1}^{n} h_i \le \frac{h^4}{64}\,(b-a)\|f''\|^2_{L^\infty(\Omega)},$$

(1.17)
$$\|e'\|^2_{L^2(\Omega)} = \sum_{i=1}^{n} \|e_i'\|^2_{L^2(\Omega_i)} \le \frac{h^2}{4}\,\|f''\|^2_{L^\infty(\Omega_i)} \sum_{i=1}^{n} h_i \le \frac{h^2}{4}\,(b-a)\|f''\|^2_{L^\infty(\Omega_i)}.$$

Taking square roots gives the bounds (1.12) and (1.13). ♡

Observe that the error bounds given in Theorem 1.1 and Corollary 1.3 involve the second derivative of the underlying function. Thus, in cases where the form of the underlying function is not known, the inherent error due to the use of spline interpolation will be difficult to estimate. An alternative way of assessing the quality of the interpolation is to assess the *smoothness* of the approximation. It turns out that the linear spline is the best possible approximation of those taken from a complete[1] infinite-dimensional normed space of functions X.

More precisely, the space of functions X turns out to be the Hilbert space $H^1(a,b)$ containing functions v that are square integrable $v \in L^2(a,b)$ and whose first derivatives are square integrable, that is, $v' \in L^2(a,b)$. One complication here is that the piecewise polynomial spline function s_1 is not differentiable everywhere, so our interpretation of the derivative will need to be extended to ensure that $s_1 \in X$. Consequently, we will refer to v' as the *weak* derivative. In general, a weak derivative v' will take the same finite value as the classical derivative except for a countable set of points; for our spline function this is the set of interior knot points $\{x_k\}_{k=1}^{n-1}$.

Let v be an interpolant of the underlying function having a weak derivative v'. We will measure the smoothness ("flatness") of a function from X by computing $\int_a^b \big(v'(x)\big)^2\,\mathrm{d}x$. The optimality of the linear spline approximation is expressed in the following theorem.

Theorem 1.4 *Let us denote by V the space of functions that are in $H^1(a,b)$ and that interpolates a given but otherwise arbitrary set of data points in $[a,b]\times\mathbb{R}$. The linear spline interpolant S_1 is the best approximation from V in the sense that*

$$\int_a^b \big(S_1'(x)\big)^2\,\mathrm{d}x \le \int_a^b \big(v'(x)\big)^2\,\mathrm{d}x \quad \forall v \in V. \tag{1.18}$$

Proof. The function S_1 is continuous over $[a,b]$. This ensures that S_1 is square integrable over the domain of interest, $S_1 \in L^2(a,b)$. The fact that S_1' is piecewise constant ensures that $S_1 \in H^1(a,b)$, so that $S_1 \in V$. For any $v \in V$, we

[1] A *complete* space contains the limits of all Cauchy sequences that are defined with respect to the norm in the function space (intuitively, this means that there are no "holes").

let $\eta := v - S_1 \in H^1(a,b)$. As v and S_1 are both interpolants, we have $\eta = 0$ at all the abscissae of the given set of data points. Squaring both sides of the equality $v' = S_1' + \eta'$ and integrating over the domain gives

$$\int_b^a \big(v('x)\big)^2 \,\mathrm{d}x = \int_b^a \big(S_1'(x)\big)^2 \,\mathrm{d}x + \int_b^a \big(\eta'(x)\big)^2 \,\mathrm{d}x + 2\int_b^a S_1'(x)\,\eta'(x)\,\mathrm{d}x. \tag{1.19}$$

We will show that the third term on the right side of (1.18) simplifies to zero. First, we break the integral into the pieces

$$\int_a^b S_1'(x)\,\eta'(x)\,\mathrm{d}x = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} S_1'(x)\,\eta'(x)\,\mathrm{d}x.$$

Second, since S_1 is smooth (a polynomial) in the open interval (x_{i-1}, x_i) we can integrate each piece by parts

$$\int_{x_{i-1}}^{x_i} S_1'(x)\,\eta'(x)\,\mathrm{d}x = \underbrace{\eta(x_i)}_{=0} S_1'(x_i^-) - \underbrace{\eta(x_{i-1})}_{=0} S_1'(x_{i-1}^+) - \int_{x_{i-1}}^{x_i} \underbrace{S_1''(x)}_{=0}\,\eta(x)\,\mathrm{d}x.$$

Thus (1.19) can be rewritten as

$$\int_b^a \big(S_1'(x)\big)^2 \,\mathrm{d}x = \int_b^a \big(v('x)\big)^2 \,\mathrm{d}x - \int_b^a \big(\eta'(x)\big)^2 \,\mathrm{d}x,$$

which leads to (1.18) since $\int_b^a \big(\eta'(x)\big)^2 \,\mathrm{d}x \geq 0$. ♡

Note that we have been careful to write $S_1'(x_i^-)$ and $S_1'(x_{i-1}^+)$ in the penultimate step of the proof to indicate the direction in which the limiting value is taken. Measuring the degree of discontinuity in the derivative of the spline provides a simple mechanism for assessing the accuracy of the approximation in scenarios where the unknown function f is known to have a continuous derivative. A simple strategy for *adaptively* improving the approximation in such cases is to compute the weighted *gradient jump* (residual) associated with knot x_j[2]

$$R_j = [\![S']\!]_j / \{\{h_j\}\} = \big|S'(x_j^+) - S'(x_j^-)\big| \cdot 2/(h_j + h_{j+1}) \tag{1.20}$$

and to consider the size of each contribution to the total residual $R = \sum_{j=1}^{n-1} R_j$. Note that R_j provides a simple (centered) difference approximation of the magnitude of the second derivative function f'' at the point x_j. The heuristic for local adaptivity is that making the interval size h_i small in regions where $|f''|$ is large will give a more equally balanced error distribution. Specifically, the local error bounds (1.10) and (1.11) suggest that adding new interpolation points at the center of the two subintervals on either side of any knot x_j where R_j contributes more than $\frac{1}{n}R$ to the residual estimate will significantly improve the accuracy of the approximation overall.

[2] The reason that this is a residual is that if $f \in C^1[a,b]$ then there are no derivative jumps at the knots.

Remark 1.1 Strategies based on equidistributing gradient jump residuals are at the heart of self-adaptive strategies for solving second-order partial differential equations using finite element approximation methods.

The classical alternative to piecewise approximation is *polynomial interpolation*. This is discussed next.

(Polynomial interpolation)
Given a set of $n+1$ data values $\{(x_k, f_k)\}_{k=0}^{n}$; we want to find the unique polynomial $P : [x_0, x_n] \to \mathbb{R}$ of degree n such that

$$f_k := f(x_k) = P(x_k).$$

The next result extends the error estimate derived for linear interpolation and will be useful later.

Lemma 1.5 *Suppose that $f \in C^{n+1}[a,b]$ and let P be the unique polynomial of degree n that interpolates f at the data values $\{(x_k, f_k)\}_{k=0}^{n}$ where $a = x_0$ and $b = x_n$. There exists at least one point $\xi \in (a,b)$ such that the interpolation error can be expressed as*

$$f(x) - P(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi). \tag{1.21}$$

Proof. The proof follows the structure of the approach taken to quantify the error in linear interpolation. We redefine $e(x) := f(x) - P(x)$ and the function $\varphi_x : [x_0, x_n] \to \mathbb{R}$:

$$\varphi_x(t) := e(t) - e(x)\frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}.$$

Observe that $\varphi_x(t) = 0$ has $n+2$ solutions $\{x_0, x_1, \ldots, x_n, x\}$. Given that P is an nth degree polynomial, if $f \in C^{n+1}[a,b]$ then $e \in C^{n+1}[a,b]$. We denote by $\varphi_x^{(k)}$ the kth derivative of φ_x with respect to t. Applying Rolle's theorem successively $n+1$ times we get

there exist at least $n+1$ points in (a,b) at which $\varphi_x^{(1)} = 0$,

there exist at least n points in (a,b) at which $\varphi_x^{(2)} = 0$,

$\vdots$

there exist at least 2 points in (a,b) at which $\varphi_x^{(n)} = 0$,

there exists at least 1 point in (a,b) at which $\varphi_x^{(n+1)} = 0$.

Let $\xi \in (a,b)$ be the point such that $\varphi_x^{(n+1)}(\xi) = 0$. Now,

$$\varphi_x^{(n+1)}(t) = e^{(n+1)}(t) - e(x)\frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)}. \tag{1.22}$$

As $P^{(n+1)}(t) = 0$ we have $e^{(n+1)}(t) = f^{(n+1)}(t)$. Using the latter result and $\varphi_x^{(n+1)}(\xi) = 0$ in (1.22) we get the result shown in (1.21). ♡

1.2 Cubic Hermite splines

The cubic Hermite spline $S_2(x) \in C^1[a,b]$ is the piecewise polynomial that satisfies

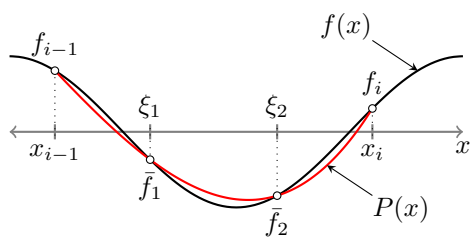
❶ $S_2(x_k) = f_k := f(x_k)$ for $k \in \{0, 1, \ldots, n\}$

❷ $S_2'(x_k) = f_k' := f'(x_k)$ for $k \in \{0, 1, \ldots, n\}$ and

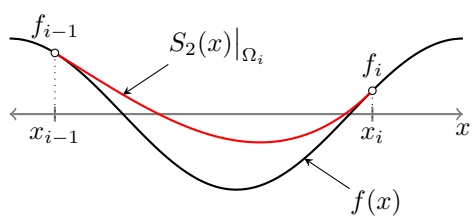
❸ $S_2(x)$ is a cubic polynomial on $[x_{k-1}, x_k]$ for $k \in 1, 2, \ldots, n$.

Note that there are four unknown coefficients and four conditions in every subinterval. It means that each piecewise part of $S_2(x)$ can be determined independently. One approach is to determine each $S_2(x)\big|_{\Omega_k}$ by considering it in the form $a + bx + cx^2 + dx^3$ and solving for the unknowns a, b, c and d. But this approach does not provide geometric insight. We will consider a different approach here that is a more constructive.

Let us focus on the ith subdomain $\Omega_i := [x_{i-1}, x_i]$. Let $P(x)$ be the cubic polynomial interpolating $f(x)$ at the data points $\{x_{i-1}, \xi_1, \xi_2, x_i\}$. Without loss of generality assume that $x_{i-1} \le \xi_1 \le \xi_2 \le x_i$. We will use the notation $\bar{f}_k := f(\xi_k)$ in order to distinguish $\bar{f}_k$ from $f_k := f(x_k)$. When x_{i-1}, ξ_1, ξ_2 and x_i are all distinct, then $P(x)$ will take the following form:



The approach we take to arrive at $S_2(x)\big|_{\Omega_i}$ is to take the limit $\xi_1 \to x_{i-1}$, $\xi_2 \to x_i$ in the expression for $P(x)$. The piece $S_2(x)\big|_{\Omega_i}$ obtained by following this approach and considering the $f(x)$ and $P(x)$ shown in the previous picture, will look like this:



We now examine the transformation of the algebraic expression for $P(x)$.

The expression for $P(x)$ in Lagrange form is

$$\begin{aligned} P(x) = {} & \frac{(x-\xi_1)(x-\xi_2)(x-x_i)}{(x_{i-1}-\xi_1)(x_{i-1}-\xi_2)(x_{i-1}-x_i)} f_{i-1} \\ & + \frac{(x-x_{i-1})(x-\xi_2)(x-x_i)}{(\xi_1-x_{i-1})(\xi_1-\xi_2)(\xi_1-x_i)} \bar{f}_1 \\ & + \frac{(x-x_{i-1})(x-\xi_1)(x-x_i)}{(\xi_2-x_{i-1})(\xi_2-\xi_1)(\xi_2-x_i)} \bar{f}_2 + \frac{(x-x_{i-1})(x-\xi_1)(x-\xi_2)}{(x_i-x_{i-1})(x_i-\xi_1)(x_i-\xi_2)} f_i. \end{aligned}$$

Taking the limit $\xi_1 \to x_{i-1}$, $\xi_2 \to x_i$ in the Lagrange form is not trivial. The Newton form of $P(x)$ is

$$\begin{aligned} P(x) = {} & f[x_{i-1}] + (x-x_{i-1}) f[x_{i-1};\xi_1] \\ & + (x-x_{i-1})(x-\xi_1) f[x_{i-1};\xi_1;\xi_2] \\ & + (x-x_{i-1})(x-\xi_1)(x-\xi_2) f[x_{i-1};\xi_1;\xi_2;x_i]. \end{aligned} \tag{1.23}$$

The coefficients in (1.23) are divided differences of $f(x)$ and they can be defined recursively as

$$f[x_i] := f_i, \quad f[x_i;x_{i+1}] := \frac{f_{i+1}-f_i}{x_{i+1}-x_i},$$

$$f[x_i;x_{i+1};\ldots;x_{i+k-1};x_{i+k}] := \frac{f[x_{i+1};x_{i+2};\ldots;x_{i+k}] - f[x_i;x_{i+1};\ldots;x_{i+k-1}]}{x_{i+k}-x_i}.$$

Further,

$$f[\underbrace{x_i;x_i;\ldots;x_i}_{(k+1)\text{ terms}}] = \frac{1}{k!} f^{(k)}(x_i) = \frac{1}{k!} f_i^{(k)}. \tag{1.24}$$

Now, we derive the transformation of the coefficients in (1.23) when $\xi_1 \to x_{i-1}$ and $\xi_2 \to x_i$. From (1.24) we have

$$\lim_{\xi_1 \to x_{i-1}} f[x_{i-1};\xi_1] = f'_{i-1}, \quad \lim_{\xi_2 \to x_i} f[\xi_2;x_i] = f'_i,$$

$$\lim_{\substack{\xi_1 \to x_{i-1} \\ \xi_2 \to x_i}} f[x_{i-1};\xi_1;\xi_2] = \frac{f[x_{i-1};x_i] - f'_{i-1}}{x_i - x_{i-1}}$$

and

$$\begin{aligned} \lim_{\substack{\xi_1 \to x_{i-1} \\ \xi_2 \to x_i}} f[x_{i-1};\xi_1;\xi_2;x_i] &= f[x_{i-1};x_{i-1};x_i;x_i] \\ &= \frac{f[x_{i-1};x_i;x_i] - f[x_{i-1};x_{i-1};x_i]}{x_i - x_{i-1}} \\ &= \frac{\dfrac{f'_i - f[x_{i-1};x_i]}{x_i - x_{i-1}} - \dfrac{f[x_{i-1};x_i] - f'_{i-1}}{x_i - x_{i-1}}}{x_i - x_{i-1}} \\ &= \frac{f'_{i-1} - 2f[x_{i-1};x_i] + f'_i}{(x_i - x_{i-1})^2}. \end{aligned}$$

Thus, when $\xi_1 \to x_{i-1}$ and $\xi_2 \to x_i$ and substituting $h_i = x_i - x_{i-1}$, the $P(x)$ given in (1.23) transforms to

$$S_2(x)\big|_{\Omega_i} = f_{i-1} + (x - x_{i-1})f'_{i-1} + (x - x_{i-1})^2 \frac{f[x_{i-1}; x_i] - f'_{i-1}}{h_i} + (x - x_{i-1})^2(x - x_i) \frac{f'_{i-1} - 2f[x_{i-1}; x_i] + f'_i}{h_i^2}.$$

Recall that in the subdomain Ω_i, the linear spline basis functions and their derivatives can be written

$$\phi_{i-1}(x)\big|_{\Omega_i} = \frac{x_i - x}{h_i}, \quad \phi_i(x)\big|_{\Omega_i} = \frac{x - x_{i-1}}{h_i}, \quad \phi'_{i-1}(x)\big|_{\Omega_i} = \frac{-1}{h_i}, \quad \phi'_i(x)\big|_{\Omega_i} = \frac{1}{h_i}.$$

Using $\phi_{i-1}(x)\big|_{\Omega_i}$ and $\phi_i(x)\big|_{\Omega_i}$ we can rewrite $S_2(x)\big|_{\Omega_i}$ as

$$S_2(x)\big|_{\Omega_i} = f_{i-1} + h_i\phi_i(x)\big|_{\Omega_i} f'_{i-1} + h_i\phi_i(x)^2\big|_{\Omega_i}(f[x_{i-1}; x_i] - f'_{i-1}) - h_i\phi_{i-1}(x)\big|_{\Omega_i}\phi_i(x)^2\big|_{\Omega_i}\big(f'_{i-1} + f'_i - 2f[x_{i-1}; x_i]\big).$$

Expanding $f[x_{i-1}; x_i]$ in the previous equation and collecting terms that multiply f_{i-1}, f_i, f'_{i-1} and f'_i, we get the form

$$S_2(x)\big|_{\Omega_i} = \widehat{\phi}_{i-1}(x)\Big|_{\Omega_i} f_{i-1} + \widehat{\phi}_i(x)\Big|_{\Omega_i} f_i + \widetilde{\phi}_{i-1}(x)\Big|_{\Omega_i} f'_{i-1} + \widetilde{\phi}_i(x)\Big|_{\Omega_i} f'_i, \tag{1.25}$$

where

$$\widehat{\phi}_i(x)\Big|_{\Omega_i} = \phi_i(x)^2\big|_{\Omega_i}\left(1 + 2\phi_{i-1}(x)\big|_{\Omega_i}\right), \qquad \widehat{\phi}_i(x)\Big|_{\partial\Omega_i} = \begin{cases} 0 & \text{at } x = x_{i-1}, \\ 1 & \text{at } x = x_i, \end{cases} \tag{1.26}$$

$$\widehat{\phi}_{i-1}(x)\Big|_{\Omega_i} = 1 - \widehat{\phi}_i(x)\Big|_{\Omega_i}, \qquad \widehat{\phi}_{i-1}(x)\Big|_{\partial\Omega_i} = \begin{cases} 1 & \text{at } x = x_{i-1}, \\ 0 & \text{at } x = x_i, \end{cases} \tag{1.27}$$

and

$$\widetilde{\phi}_{i-1}(x)\Big|_{\Omega_i} = h_i\phi_{i-1}(x)^2\big|_{\Omega_i}\phi_i(x)\big|_{\Omega_i}, \qquad \widetilde{\phi}_{i-1}(x)\Big|_{\partial\Omega_i} = \begin{cases} 0 & \text{at } x = x_{i-1}, \\ 0 & \text{at } x = x_i, \end{cases} \tag{1.28}$$

$$\widetilde{\phi}_i(x)\Big|_{\Omega_i} = -h_i\phi_{i-1}(x)\big|_{\Omega_i}\phi_i(x)^2\big|_{\Omega_i}, \qquad \widetilde{\phi}_i(x)\Big|_{\partial\Omega_i} = \begin{cases} 0 & \text{at } x = x_{i-1}, \\ 0 & \text{at } x = x_i. \end{cases} \tag{1.29}$$

Observe that $\widehat{\phi}_{i-1}(x)\big|_{\Omega_i}, \widehat{\phi}_i(x)\big|_{\Omega_i}, \widetilde{\phi}_{i-1}(x)\big|_{\Omega_i}$ and $\widetilde{\phi}_i(x)\big|_{\Omega_i}$ are all cubic polynomials. Taking the derivative of $\widehat{\phi}_i(x)\big|_{\Omega_i}$ with respect to x, and using the

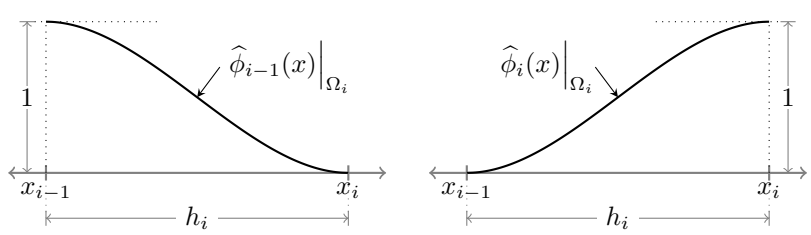
identity $\phi_{i-1}(x)\big|_{\Omega_i}+\phi_i(x)\big|_{\Omega_i}=1$, we find

$$
\begin{aligned}
\widehat{\phi}_i'(x)\Big|_{\Omega_i} &= 2\phi_i(x)\big|_{\Omega_i}\phi_i'(x)\big|_{\Omega_i}\left(1+2\phi_{i-1}(x)\big|_{\Omega_i}\right)+2\phi_i(x)^2\big|_{\Omega_i}\phi_{i-1}'(x)\big|_{\Omega_i} \\
&= \frac{2}{h_i}\phi_i(x)\big|_{\Omega_i}\left(1+2\phi_{i-1}(x)\big|_{\Omega_i}\right)-\frac{2}{h_i}\phi_i(x)^2\big|_{\Omega_i} \\
&= \frac{2}{h_i}\phi_i(x)\big|_{\Omega_i}\left(1+2\phi_{i-1}(x)\big|_{\Omega_i}-\phi_i(x)\big|_{\Omega_i}\right) \\
&= \frac{6}{h_i}\phi_i(x)\big|_{\Omega_i}\phi_{i-1}(x)\big|_{\Omega_i},
\end{aligned}
\tag{1.30}
$$

Note that

$$
\widehat{\phi}_i'(x)\Big|_{\partial\Omega_i}=\begin{cases}0 & \text{at } x=x_{i-1},\\ 0 & \text{at } x=x_i.\end{cases}
\tag{1.31}
$$

From (1.27), it follows that $\widehat{\phi}_{i-1}'(x)\big|_{\Omega_i}=-\widehat{\phi}_i'(x)\big|_{\Omega_i}$. Then, from (1.31), we find that the boundary points of the subdomain Ω_i are *stationary points* of $\widehat{\phi}_{i-1}(x)\big|_{\Omega_i}$ and $\widehat{\phi}_i(x)\big|_{\Omega_i}$. Recalling that a cubic polynomial has at most two stationary points and between those points the curve is monotonic, we can construct plots of $\widehat{\phi}_{i-1}(x)\big|_{\Omega_i}$ and $\widehat{\phi}_i(x)\big|_{\Omega_i}$ showing the salient features. These take the following form:

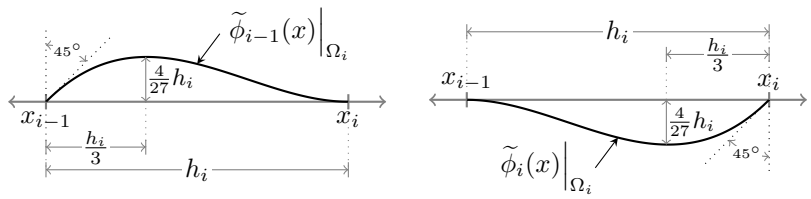


Next, making algebraic manipulations similar to those in (1.31) the derivative of $\widetilde{\phi}_{i-1}(x)\big|_{\Omega_i}$ and $\widetilde{\phi}_i(x)\big|_{\Omega_i}$ can be expressed as

$$
\widetilde{\phi}_{i-1}'(x)\Big|_{\Omega_i}=\phi_{i-1}(x)\big|_{\Omega_i}\left(1-3\phi_i(x)\big|_{\Omega_i}\right),\quad \widetilde{\phi}_{i-1}'(x)\Big|_{\partial\Omega_i}=\begin{cases}1 & \text{at } x=x_{i-1},\\ 0 & \text{at } x=x_i,\end{cases}
\tag{1.32}
$$

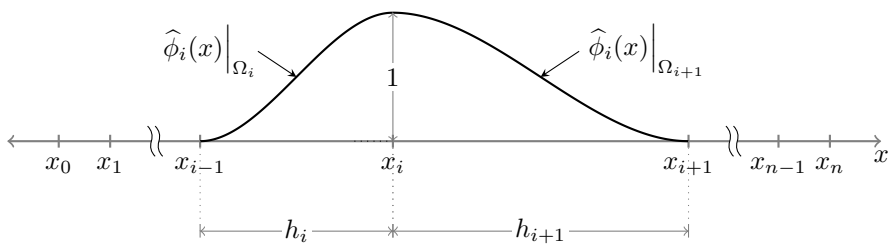
$$
\widetilde{\phi}_i'(x)\Big|_{\Omega_i}=\phi_i(x)\big|_{\Omega_i}\left(1-3\phi_{i-1}(x)\big|_{\Omega_i}\right),\quad \widetilde{\phi}_i'(x)\Big|_{\partial\Omega_i}=\begin{cases}0 & \text{at } x=x_{i-1},\\ 1 & \text{at } x=x_i.\end{cases}
\tag{1.33}
$$

Observe in (1.28) and (1.29) that $\widetilde{\phi}_{i-1}(x)\big|_{\Omega_i}$ and $\widetilde{\phi}_i(x)\big|_{\Omega_i}$ vanish on the boundary of the subdomain Ω_i. Thus, when plotted they take the following form:



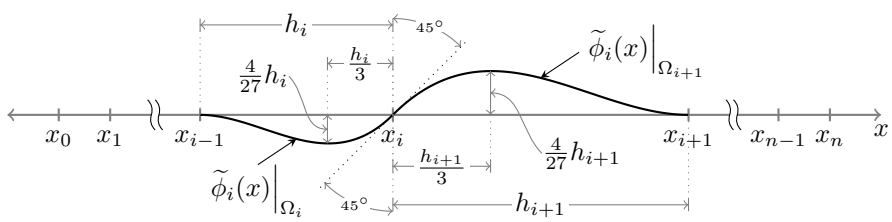
Remark 1.2 In (1.25), the piece of the cubic Hermite spline $S_2(x)\big|_{\Omega_i}$ is written as a linear combination of local basis functions which interpolate both the underlying function values and their derivatives at the boundary points of Ω_i.

Similar to what we have seen for the linear spline, the local interpolation basis functions lead to *global* interpolation basis functions $\widehat{\phi}_i(x)$ and $\widetilde{\phi}_i(x)$ associated to the index i and defined on the whole interval $[a, b]$.



The first global interpolation function is given by the following expression:

$$\widehat{\phi}_i(x) = \begin{cases} 1 & \text{if } x = x_i, \\ \phi_i(x)^2\big|_{\Omega_i}\left(1 + 2\phi_{i-1}(x)\big|_{\Omega_i}\right) & \text{if } x \in (x_{i-1}, x_i), i \neq 0, \\ 1 - \phi_{i+1}(x)^2\big|_{\Omega_{i+1}}\left(1 + 2\phi_i(x)\big|_{\Omega_{i+1}}\right) & \text{if } x \in (x_i, x_{i+1}), i \neq n, \\ 0 & \text{otherwise.} \end{cases}$$



The second global interpolation function is given by the following expression:

$$\widetilde{\phi}_i(x) = \begin{cases} 0 & \text{if } x = x_i, \\ -h_i\phi_{i-1}(x)\big|_{\Omega_i}\phi_i(x)^2\big|_{\Omega_i} & \text{if } x \in (x_{i-1}, x_i), i \neq 0, \\ h_{i+1}\phi_i(x)^2\big|_{\Omega_{i+1}}\phi_{i+1}(x)\big|_{\Omega_{i+1}} & \text{if } x \in (x_i, x_{i+1}), i \neq n, \\ 0 & \text{otherwise.} \end{cases}$$

An immediate observation is that

$$\max_{x\in\Omega} \widehat{\phi}_i(x) = 1 \quad \text{and} \quad \max_{x\in\Omega} \widetilde{\phi}_i(x) = \frac{4}{27} \max\big(h_i, h_{i+1}\big). \tag{1.34}$$

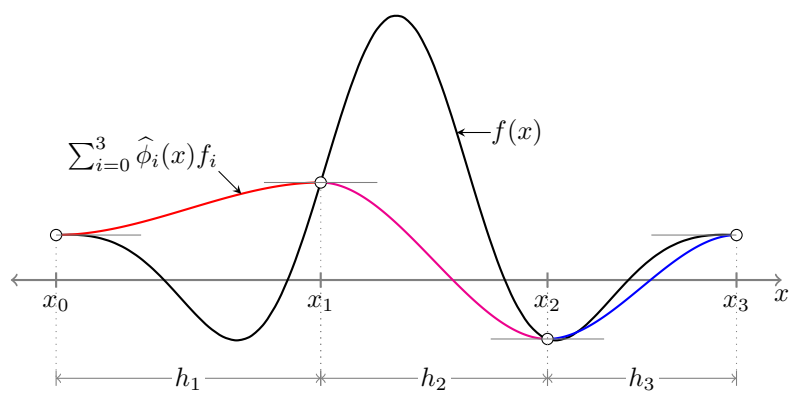
These bounds will be useful later.

In summary, there are two alternative representations of the Hermite spline. These are given by

$$S_2(x) = \sum_{i=0}^{n} \widehat{\phi}_i(x) f_i + \sum_{i=0}^{n} \widetilde{\phi}_i(x) f_i' = \bigwedge_{i=1}^{n} S_2(x)\big|_{\Omega_i}. \tag{1.35}$$

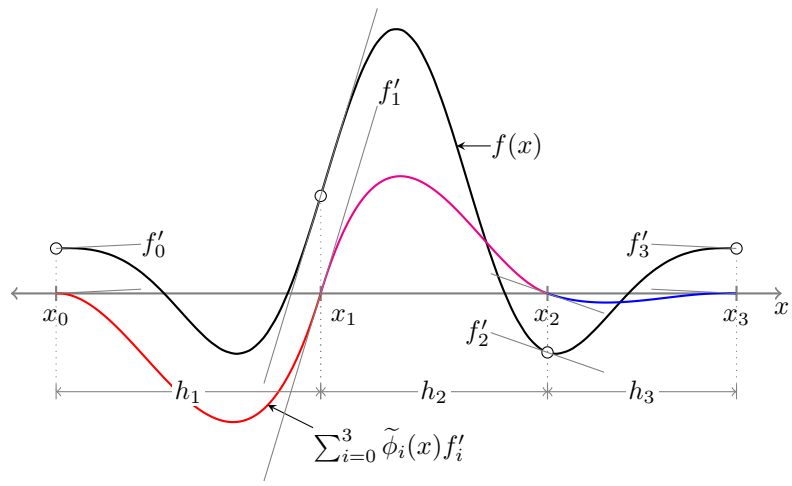
Recall that the symbol $\bigwedge$ means assembly over the individual subdomains.

Considering three subdomains, we can illustrate the geometrical aspects of Hermite splines for a chosen $f(x)$. The nature of the first term in (1.35) implies that the basis functions $\{\widehat{\phi}_i(x)\}_{i=0}^3$ control only the height of the spline at the given data points. That is, linear combinations of $\{\widehat{\phi}_i(x)\}_{i=0}^3$ cannot control the slope (it is always zero) of the spline at the data points.

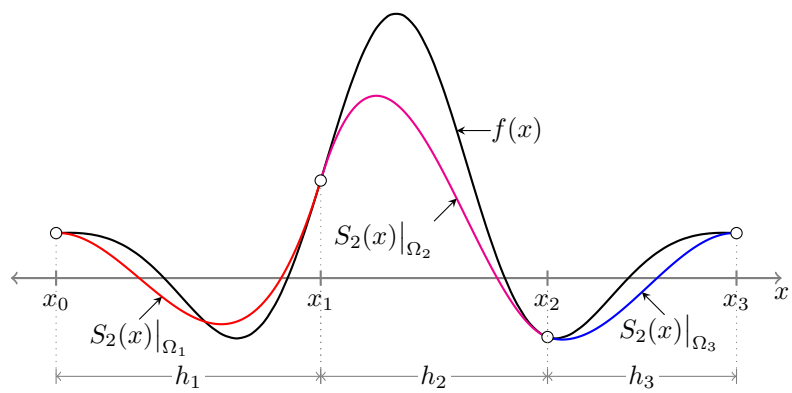


The nature of the second term in (1.35) implies that the bubble[3] basis functions $\{\widetilde{\phi}_i(x)\}_{i=0}^3$ control only the slope of the spline at the given data points. That is, linear combinations of $\{\widetilde{\phi}_i(x)\}_{i=0}^3$ do not affect the height of the spline at the given data points.

[3]Functions that are zero valued at the subdomain boundaries.



Finally, the following picture illustrates the assembly of the piecewise components $S_2(x)\big|_{\Omega_1}$, $S_2(x)\big|_{\Omega_2}$, $S_2(x)\big|_{\Omega_3}$. The global representation is a specific linear combination of $\{\widehat{\phi}_i(x)\}_0^3$ and $\{\widetilde{\phi}_i(x)\}_0^3$.



We can see that, while the spline approximation is smooth, it is not particularly accurate in this case.

The following theorem shows how well S_2 approximates f in general.

Theorem 1.6 *Suppose that $f \in C^4[a,b]$ and let S_2 be the cubic Hermite spline interpolant of f defined in (1.35); then the following error bounds hold:*

$$\|f - S_2\|_{L^\infty(\Omega)} \leq \frac{h^4}{384} \|f^{(4)}\|_{L^\infty(\Omega)}, \tag{1.36}$$

$$\|f' - S_2'\|_{L^\infty(\Omega)} \leq \frac{h^3}{72\sqrt{3}} \|f^{(4)}\|_{L^\infty(\Omega)}, \tag{1.37}$$

where $h = \max\left\{h_k\right\}_{k=1}^n$.

Proof. Considering the cubic polynomial $P(x)$ given in (1.23) and defined in the subdomain Ω_i, we have

$$\big(f(x) - S_2(x)\big)\big|_{\Omega_i} = \lim_{\substack{\xi_1 \to x_{i-1} \\ \xi_2 \to x_i}} f(x)\big|_{\Omega_i} - P(x).$$

Using Lemma 1.5 in the previous equation we find that there exists at least one point $\xi \in \Omega_i$ such that

$$\begin{aligned}\big(f(x) - S_2(x)\big)\big|_{\Omega_i} &= \lim_{\substack{\xi_1 \to x_{i-1} \\ \xi_2 \to x_i}} \frac{(x - x_{i-1})(x - \xi_1)(x - \xi_2)(x - x_i)}{4!} f^{(4)}(\xi) \\ &= \frac{(x - x_{i-1})^2 (x - x_i)^2}{4!} f^{(4)}(\xi).\end{aligned}$$

Taking norms of both sides of the previous equation we get the inequality

$$\|f - S_2\|_{L^\infty(\Omega_i)} \le \frac{\|(x - x_{i-1})^2 (x - x_i)^2\|_{L^\infty(\Omega_i)}}{4!} \|f^{(4)}\|_{L^\infty(\Omega_i)}.$$

Recall that $\|(x - x_{i-1})(x - x_i)\|_{L^\infty(\Omega_i)} = h_i^2/4$ (see the picture before (1.10)). By substituting this result in the previous inequality we get ♣

$$\|f - S_2\|_{L^\infty(\Omega_i)} \le \frac{h_i^4}{384} \|f^{(4)}\|_{L^\infty(\Omega_i)} \le \frac{h^4}{384} \|f^{(4)}\|_{L^\infty(\Omega)}.$$

As the previous inequality should hold for all subdomains, we can substitute Ω_i by $[a, b]$ in the inequality. Thus, we get the first result of this theorem. The proof of the second result proceeds in a similar manner and is left as an exercise for the reader. ♡

Remark 1.3 The shape of the piecewise component $S_2(x)\big|_{\Omega_i}$ depends only on the values $\{f_{i-1}, f_i, f'_{i-1}, f'_i\}$. This locality is an attractive feature of Hermite interpolation: variations in $\{f_k, f'_k\}_{k \notin \{i-1, i\}}$ do not affect $S_2(x)\big|_{\Omega_i}$.

The Hermite spline is not designed to have $C^2[a, b]$ continuity. To investigate this we construct $S_2''(x)\big|_{\Omega_i}$, $S_2''(x)\big|_{\Omega_{i+1}}$ and examine the condition for the continuity of S_2'' at the common boundary point x_i. Thus, differentiating the expressions in (1.30), (1.32), (1.33) and using the identity $\widehat{\phi}_{i-1}(x)\big|_{\Omega_i} + \widehat{\phi}_i(x)\big|_{\Omega_i} = 1$, we find

(1.38)
$$\widehat{\phi}''_{i-1}(x)\Big|_{\Omega_i} = \frac{6}{h_i^2}\left(\phi_i(x)\big|_{\Omega_i} - \phi_{i-1}(x)\big|_{\Omega_i}\right), \qquad \widehat{\phi}''_i(x)\Big|_{\Omega_i} = -\widehat{\phi}''_{i-1}(x)\Big|_{\Omega_i},$$

(1.39)
$$\widetilde{\phi}''_{i-1}(x)\Big|_{\Omega_i} = \frac{1}{h_i}\left(3\phi_i(x)\big|_{\Omega_i} - 3\phi_{i-1}(x)\big|_{\Omega_i} - 1\right), \quad \widetilde{\phi}''_i(x)\Big|_{\Omega_i} = \widetilde{\phi}''_{i-1}(x)\Big|_{\Omega_i} + \frac{2}{h_i}$$

with end values

(1.40)
$$\widehat{\phi}''_{i-1}(x)\Big|_{\partial\Omega_i} = \frac{6}{h_i^2}\begin{cases}-1 & \text{at } x = x_{i-1},\\ 1 & \text{at } x = x_i,\end{cases} \qquad \widehat{\phi}''_{i}(x)\Big|_{\partial\Omega_i} = \frac{6}{h_i^2}\begin{cases}1 & \text{at } x = x_{i-1},\\ -1 & \text{at } x = x_i,\end{cases}$$

(1.41)
$$\widetilde{\phi}''_{i-1}(x)\Big|_{\partial\Omega_i} = \frac{2}{h_i}\begin{cases}-2 & \text{at } x = x_{i-1},\\ 1 & \text{at } x = x_i,\end{cases} \qquad \widetilde{\phi}''_{i}(x)\Big|_{\partial\Omega_i} = \frac{2}{h_i}\begin{cases}-1 & \text{at } x = x_{i-1},\\ 2 & \text{at } x = x_i.\end{cases}$$

Next, differentiating the expressions in (1.35) gives

$$S_2''(x_i)\big|_{\Omega_i} = -\frac{6}{h_i^2}(f_i - f_{i-1}) + \frac{2}{h_i}(f'_{i-1} + 2f'_i). \tag{1.42}$$

Likewise for the subdomain Ω_{i+1} we get

$$S_2''(x_i)\big|_{\Omega_{i+1}} = \frac{6}{h_{i+1}^2}(f_{i+1} - f_i) - \frac{2}{h_{i+1}}(2f'_i + f'_{i+1}). \tag{1.43}$$

Thus, if the continuity condition $S_2''(x_i)\big|_{\Omega_i} = S_2''(x_i)\big|_{\Omega_{i+1}}$ is to hold, then the data $\{f_{i-1}, f_i, f'_{i-1}, f'_i\}$ has to satisfy

$$-\frac{6}{h_i^2}(f_i - f_{i-1}) + \frac{2}{h_i}(f'_{i-1} + 2f'_i) = \frac{6}{h_{i+1}^2}(f_{i+1} - f_i) - \frac{2}{h_{i+1}}(2f'_i + f'_{i+1}),$$

or equivalently

$$\frac{f'_{i-1}}{h_i} + 2\frac{f'_i}{h_i} + 2\frac{f'_i}{h_{i+1}} + \frac{f'_{i+1}}{h_{i+1}} = 3\frac{f[x_{i-1}; x_i]}{h_i} + 3\frac{f[x_i; x_{i+1}]}{h_{i+1}}. \tag{1.44}$$

Since this compatibility condition need not be satisfied for arbitrary data, the cubic Hermite spline $S_2(x) \notin C^2[a, b]$ in general.

The condition (1.44) is the starting point for the construction of cubic splines, for example, *natural splines*, with built-in C^2 continuity. The price to pay for enforcing additional smoothness is that the locality of the approximation is sacrificed. As a result, these ultrasmooth interpolants cannot be used to construct finite element approximations.

In one dimension the difficulty in raising the continuity $C^0 \to C^1 \to C^2$ is relatively modest. "Difficulty" here refers to the construction of the spline and its implementation in a computer program. As we will see later, in two and three dimensions the difficulty level in raising the continuity $C^0(\Omega) \to C^1(\Omega) \to C^2(\Omega)$ is much higher. This is why modest but robust C^0 interpolants and approximations deserve our respect.

References

[1] Tom M. Apostol. *Calculus: One-variable calculus, with an introduction to linear algebra*, volume 1. John Wiley & Sons, 1991. Second edition: ISBN: 0-471-00005-1.